ON THE LARGE SAMPLE PROPERTIES OF CERTAIN NONPARAMETRIC TESTS FOR DISPERSION

關於離勢之某些無母數檢定在大樣本時之特性

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ABSTRACT

Suppose $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ are two independent random samples from populations with continuous distribution functions $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively. We assume that the two populations have a common median $\underset{\sim}{\nu} = (\nu_1,\nu_2)$, which is either known or unknown, and $G_{Y-\nu}(y_1,y_2) = F_{X-\nu}(\theta_1 y_1,\theta_2 y_2)$ for all (y_1,y_2) and for some $\theta_1 > 0, \theta_2 > 0$. In this paper, two nonparametric tests R and R* are suggested to detect differences in variability or dispersion for the two populations. Both tests are shown to be distribution-free and consistent for testing H: $\theta_1 = \theta_2 = 1$ against A: $\min(\theta_1,\theta_2) > 1$ or H: $\theta_1 = \theta_2 = 1$ against A: $\max(\theta_1, \theta_2) < 1$. In addition, the Pitman's Asymptotic Relative Efficiency (ARE) of the nonparametric tests R and R* with respect to the parametric competitors is studied for bivariate normal and bivariate uniform distributions.

摘　要

假定（X_{11}, X_{21}），…（X_{1m}, X_{2m}）及（Y_{11}, Y_{21}），…，（Y_{1n}, Y_{2n}）是分別從具有連續分配函數F_{X_1,X_2}（x_1, x_2）及G_{Y_1,Y_2}（y_1, y_2）之母體抽取出來的兩個獨立隨機樣本。我們假設這兩個母體具有相同之中位數$\underset{\sim}{\nu}$ =（ν_1, ν_2），而這中位數可以是已知，也可以是未知；且存在兩個參數$\theta_1 > 0$及$\theta_2 > 0$，使得對於所有的（y_1, y_2），此式$G_{Y-\nu}$（y_1, y_2）$= F_{X-\nu}$（$\theta_1 y_1, \theta_2 y_2$）皆成立。在這篇論文中，提出了兩種無母數檢定R及R*藉以測出這兩個母體之離勢是否不同。同時證得這兩種檢定都是分配不拘的，以及對於檢定虛

無假設爲H：$\theta_1 = \theta_2 = 1$，對立假設爲A：min（θ_1, θ_2）＞1時；或者慮無假設爲H：$\theta_1 = \theta_2 = 1$，對立假設爲A：max（θ_1, θ_2）＜1時，皆具有一致性。此外，也研究了當母體具有二元常態或二元均等分配時，這兩種無母數檢定與其競爭之有母數檢定之間的皮特曼漸近相對效率。

I. INTRODUCTION

A familiar problem is to test whether two samples have come from identical populations. A frequently considered alternative is that the populations differ in scale. If the observations are univariate and we suppose that the parent populations are governed by a continuous distribution function, then tests proposed by Mood (1954), Sukhatme (1957), Ansari-Bradley (1960), Siegel-Tukey (1960), Klotz (1962), Raghavachari (1965), Fligner (1974), and others are applicable nonparametric analogues of the F-test. In this paper, we are interested in the bivariate case.

Consider a bivariate two-sample problem: Suppose that $(X_{11},X_{21}), \ldots, (X_{1m}, X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ are two independent bivariate random samples from populations with continuous distribution functions $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively such that

$$G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1,y_2) = F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1,\theta_2 y_2) \text{ for all } (y_1,y_2) \text{ and for some } \theta_1 > 0, \theta_2 > 0,$$

where $\underset{\sim}{X} = (X_1,X_2)$, $\underset{\sim}{Y} = (Y_1,Y_2)$, and $\underset{\sim}{\nu} = (\nu_1,\nu_2)$ is the common median. We would like to detect differences in variability or dispersion for the two populations.

Two nonparametric tests R and R* are suggested; the former being applicable when the common median is known, the latter when the common median is unknown. We show that both tests are distribution-free and consistent for testing H: $\theta_1 = \theta_2 = 1$ against $\min(\theta_1,\theta_2) > 1$ or H: $\theta_1 = \theta_2 = 1$ against A: $\max(\theta_1,\theta_2) < 1$. Also, we investigate the Pitman's Asymptotic Relative Efficiency (ARE) of the nonparametric tests R and R* with respect to the parametric competitors for bivariate normal and bivariate uniform distributions.

II. DISTRIBUTION-FREE TEST STATISTICS $R_{m,n}$ and $R^*_{m,n}$

Let $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ be two independent bivariate random samples from populations with continuous distribution functions (d.f.'s) $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively. We assume that F and G differ only in scale, i.e.,

$$G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1,y_2) = F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1,\theta_2 y_2) \quad \text{for all } (y_1,y_2) \text{ and for some } \theta_1 > 0, \theta_2 > 0,$$

where $\underset{\sim}{X} = (X_1,X_2)$, $\underset{\sim}{Y} = (Y_1,Y_2)$, and $\underset{\sim}{\nu} = (\nu_1,\nu_2)$ is the common median. The problem is to detect differences in variability or dispersion for the two populations.

If the common median $\underset{\sim}{\nu} = (\nu_1,\nu_2)$ is known, we define $R_{m,n}$ to be the Mann-Whitney (1947) test statistic for the two independent random samples

$$U_1, U_2, \ldots, U_m \text{ and } V_1, V_2, \ldots, V_n$$

i.e., $R_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij}$,

where $D_{ij} = 1$ if $U_i > V_j$ for all $i = 1, 2, \ldots, m$,
$= 0$ otherwise $\quad j = 1, 2, \ldots, n$,

$$U_i = [(X_{1i}-\nu_1)^2+(X_{2i}-\nu_2)^2]^{1/2} \quad \text{for } i = 1, 2, \ldots, m, \text{ and}$$

$$V_j = [(Y_{1j}-\nu_1)^2+(Y_{2j}-\nu_2)^2]^{1/2} \quad \text{for } j = 1, 2, \ldots, n.$$

If the common median $\underset{\sim}{\nu} = (\nu_1,\nu_2)$ is unknown, we define $R^*_{m,n}$ to be the Mann-Whitney test statistic for the two samples,

$$U^*_{1N}, U^*_{2N}, \ldots, U^*_{mN} \text{ and } V^*_{1N}, V^*_{2N}, \ldots, V^*_{nN}$$

i.e., $R^*_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} D^*_{ij}$,

where $D^*_{ji} = 1$ if $U^*_{iN} > V^*_{jN}$ for all $i = 1, 2, \ldots, m$,
$= 0$ otherwise $\quad j = 1, 2, \ldots, n$,

$$U^*_{iN} = [(X_{1i} - M_{1N})^2 + (X_{2i} - M_{2N})^2]^{1/2},$$

$$V^*_{jN} = [(Y_{1j} - M_{1N})^2 + (Y_{2j} - M_{2N})^2]^{1/2},$$

$N = m + n$, and

$\underset{\sim}{M}_N = (M_{1N}, M_{2N})$ is the combined sample median.

A large value of $R_{m,n}$ or $R^*_{m,n}$ would imply that the $\underset{\sim}{X}$'s are more widely dispersed and vice versa.

We now show that the proposed test R has the property that, under H: $\theta_1 = \theta_2 = 1$, the distribution of $R_{m,n}$ does not depend on the underlying population.

Theorem 2.1: Under H: $\theta_1 = \theta_2 = 1$, the test statistic $R_{m,n}$ is completely distribution-free for any underlying bivariate continuous population and its null distribution is the same as that of the Mann-Whitney test statistic.

Proof: Under H: $\theta_1 = \theta_2 = 1$, the two independent random samples (X_{11}, X_{21}), . . . , (X_{1m}, X_{2m}) and (Y_{11}, Y_{21}), . . . , (Y_{1n}, Y_{2n}) come from a common bivariate continuous population. It implies that $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ are two independent random samples from a common univariate continuous population. Hence, $R_{m,n}$, which is defined as the Mann-Whitney test statistic for the two independent random samples $U_i(i=1, \ldots m)$ and $V_j(j=1, \ldots, n)$, is distribution-free and its null distribution is the same as that of the Mann-Whitney test statistic (see { Mann and Whitney (1947)}). This completes the proof.

Note that, under H: $\theta_1 = \theta_2 = 1$, $U^*_{1N}, \ldots, U^*_{mN}, V^*_{1N}, \ldots, V^*_{nN}$ are identically distributed random variables, but they are not independent. However, we can still show that the test statistic $R^*_{m,n}$ is distribution-free.

Theorem 2.2: Under H: $\theta_1 = \theta_2 = 1$, the test statistic $R^*_{m,n}$ is completely distribution free for any underlying bivariate continuous population and its null distribution is the same as that of the Mann-Whitney test statistic.

Proof: Since, under H: $\theta_1 = \theta_2 = 1$, the random vectors $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m}), (Y_{11}, Y_{21}), \ldots, (Y_{1n}, Y_{2n})$ are interchangeable, the vectors $(X_{11} - M_{1N}, X_{21} - M_{2N}), \ldots, (X_{1m} - M_{1N}, X_{2m} - M_{2N}), (Y_{11} - M_{1N}, Y_{21} - M_{2N}), \ldots, (Y_{1n} - M_{1N}, Y_{2n} - M_{2N})$, are interchangeable.

Therefore, each of the N! possible orderings of $(U^*_{1N}, U^*_{2N}, \ldots, U^*_{mN}, V^*_{1N}, V^*_{2N}, \ldots, V^*_{nN})$ is equally probable,

where $U^*_{iN} = [(X_{1i} - M_{1N})^2 + (X_{2i} - M_{2N})^2]^{1/2}$ for $i = 1, \ldots, m$,

$$V^*_{jN} = [(Y_{1j} - M_{1N})^2 + (Y_{2j} - M_{2N})^2]^{1/2} \qquad j = 1, \ldots, n.$$

Hence, the test statistic $R^*_{m,n}$ is completely distribution-free (see { Hájek and Sidák (1967), p. 38}) and its null distribution is the same as that of the Mann-Whitney test statistics (see { Mann and Whitney (1947) }). This concludes the proof.

III. CONSISTENCY OF THE TEST R

Let us define the consistency of a test T. Suppose $\mathfrak{X}^{m+n}$ is the sample space and $\{P_\theta | \theta \in \Omega\}$ is the class of probability measures over $\mathfrak{X}^{m+n}$. Also, for the hypothesis testing problem, H: $\theta \in \omega$ against A: $\theta \in \Omega - \omega$, let $\Phi_{m,n}(X_1, \ldots, X_m, Y_1, \ldots Y_n)$ be a test of size α, which is based on a statistic $T_{m,n}(X_1, \ldots X_m, Y_1, \ldots Y_n)$, and let $P_{\Phi_{m,n}}(\theta)$ be the power of $\Phi_{m,n}$ for $\theta \in \Omega - \omega$. Then the sequence of size α tests $\{\Phi_{m,n}\}$ (or simply the test T) is said to be consistent for $\zeta \subset \Omega - \omega$ if $\lim_{m,n \to \infty} P_{\Phi_{m,n}}(\theta) = 1$ for $\theta \in \zeta$.

We now consider a well-known criteria for consistency. Let $g(\theta)$ be a real-valued parameter defined over Ω such that $g(\theta) = g_o$ if $\theta \in \omega$, $g(\theta) > (<) g_o$ if $\theta \in \zeta$, where g_o is some constant.

Theorem 3.1:

If $T_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ is a real-valued statistic defined over $\mathfrak{X}^{m+n}$ for each m and each n, and if for all $\theta \in \Omega$, $T_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n) \xrightarrow[m,n \to \infty]{P} g(\theta)$ provided this convergence is uniform for $\theta \in \omega$, then the sequence of tests $\Phi_{m,n}$ of size α,

$$\Phi_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n) = 1$$
$$\quad \text{if } T_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n) - g_o > (<)\, c_{m,n},$$
$$\Phi_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n) = 0$$
$$\quad \text{if } T_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n) - g_o < (>)\, c_{m,n},$$

is consistent for ζ.

Proof: See {Fraser (1957), p. 267}.

Since $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m})$ is a random sample from a population with a continuous d.f., $U_1, \ldots, U_m$ constitutes a random sample from a population with a univariate continuous d.f., say S(u). Similarly, $V_1, \ldots, V_n$ constitutes a random sample from a population with a univariate continuous d.f., say T(v). In order to prove the consistency of the test R, we establish the following theorem concerning S(u).

Theorem 3.2: Under the assumption

$$G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1, y_2) = F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1, \theta_2 y_2) \quad \text{for all } (y_1, y_2) \text{ and for some } \theta_1 > 0, \theta_2 > 0,$$

where $\underset{\sim}{\nu} = (\nu_1, \nu_2)$ is the true common median, we have

(1) $T(u) \geqslant S([\min(\theta_1, \theta_2)]\, u)$ for all u,

(2) $T(u) \leqslant S([\max(\theta_1, \theta_2)]\, u)$ for all u, and

(3) $P(U < 0) = P(V < 0) = 0$.

Proof: (1) Since $G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1, y_2) = P(Y_1 - \nu_1 \leqslant y_1, Y_2 - \nu_2 \leqslant y_2)$,

$$F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1, \theta_2 y_2) = P(X_1 - \nu_1 \leqslant \theta_1 y_1, X_2 - \nu_2 \leqslant \theta_2 y_2)$$

$$= P((X_1 - \nu_1)/\theta_1 \leqslant y_1, (X_2 - \nu_2)/\theta_2 \leqslant y_2),$$

and the assumption $G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1, y_2) = F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1, \theta_2 y_2)$ for all

(y_1, y_2) and for some $\theta_1 > 0$, $\theta_2 > 0$, it follows that

$(Y_1 - \nu_1, Y_2 - \nu_2)$ and $((X_1 - \nu_1)/\theta_1, (X_2 - \nu_2)/\theta_2)$ have the same distribution.

Thus, $T(u) = P(V \leqslant u)$

$$= P([(Y_1 - \nu_1)^2 + (Y_2 - \nu_2)^2]^{1/2} \leqslant u)$$

$$= P(\{[(X_1 - \nu_1)/\theta_1]^2 + [(X_2 - \nu_2)/\theta_2]^2\}^{1/2} \leqslant u),$$

since $(Y_1 - \nu_1, Y_2 - \nu_2)$ and $((X_1 - \nu_1)/\theta_1, (X_2 - \nu_2)/\theta_2)$ have the same distribution,

$$\geqslant P(\{[(X_1 - \nu_1)/\min(\theta_1, \theta_2)]^2 + [(X_2 - \nu_2)/\min(\theta_1, \theta_2)]^2\}^{1/2} \leqslant u)$$

$$= P([(X_1 - \nu_1)^2 + (X_2 - \nu_2)^2]^{1/2} \leqslant [\min(\theta_1, \theta_2)]\, u)$$

$$= P(U \leqslant [\min(\theta_1, \theta_2)]\, u)$$

$$= S([\min(\theta_1, \theta_2)]\, u) \quad \text{for all u.}$$

Therefore, $T(u) \geqslant S([\min(\theta_1, \theta_2)]\, u)$ for all u.

(2) The proof of $T(u) \leqslant S([\max(\theta_1, \theta_2)]\, u)$ for all u is similar to (1).

(3) Since $U = [(X_1 - \nu_1)^2 + (X_2 - \nu_2)^2]^{1/2} \geqslant 0$ and

$V = [(Y_1 - \nu_1)^2 + (Y_2 - \nu_2)^2]^{1/2} \geqslant 0$, then $P(U < 0) = P(V < 0) = 0$, as was to be proved. This completes the proof of Theorem 3.2.

Lemma 3.1:

For any two random variables X and Y, we have

$P(|X+Y| \geqslant \varepsilon) \leqslant P(|X| \geqslant \varepsilon/2) + P(|Y| \geqslant \varepsilon/2)$ for every $\varepsilon > 0$.

Proof: See {Tucker (1967), p. 102}.

Theorem 3.3:

For every $\theta_1 > 0$, $\theta_2 > 0$, we have

(1) $E(R_{m,n}/mn) = P(U > V)$,

where $U = [(X_1-\nu_1)^2 + (X_2-\nu_2)^2]^{1/2}$ and $V = [(Y_1-\nu_1)^2 + (Y_2-\nu_2)^2]^{1/2}$,

and

(2) $\lim_{m,n \to \infty} \text{Var}(R_{m,n}/mn) = 0$.

Proof: Since $R_{m,n}$ is the Mann-Whitney test statistic for the two independent random samples U_i (i=1, . . . , m) and V_j (j=1, . . . , n) from populations with univariate continuous d.f.'s S(u) and T(v) respectively, (1) and (2) follow immediately (see { Gibbons (1971), p. 141-142).

Now, we are ready to prove the consistency of the test R.

Theorem 3.4:

Let $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ be two independent bivariate random samples from populations with continuous d.f.'s $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively, such that $G_{\underline{Y}-\underline{\nu}}(y_1,y_2) = F_{\underline{X}-\underline{\nu}}(\theta_1 y_1, \theta_2 y_2)$ for all (y_1,y_2) and for some $\theta_1 > 0$, $\theta_2 > 0$,
where $\underline{X} = (X_1,X_2)$, $\underline{Y} = (Y_1,Y_2)$ and $\underline{\nu} = (\nu_1,\nu_2)$ is the known common median. Then the test R is consistent in the following cases:

Subclass of Alternatives	Rejection Region
A: $\min(\theta_1,\theta_2) > 1$	$R_{m,n} - mn/2 > d_1$
A: $\max(\theta_1,\theta_2) < 1$	$R_{m,n} - mn/2 < d_2$

Proof: Let us define $\Omega = \{(\theta_1,\theta_2) | \theta_1 > 0, \theta_2 > 0\}$, $\omega = \{(1,1)\}$,

$g_o = 1/2$ and $g(\theta_1,\theta_2) = P(U > V)$ for $(\theta_1,\theta_2) \in \Omega$,

where $U = [(X_1-\nu_1)^2 + (X_2-\nu_2)^2]^{1/2}$ and

$V = [(Y_1-\nu_1)^2 + (Y_2-\nu_2)^2]^{1/2}$.

Thus, $g(\theta_1,\theta_2) = P(U > V)$

$= \int_{-\infty}^{\infty} P(V < u | U = u) dS(u)$, where S(u) is the d.f. of U,

$= \int_{-\infty}^{\infty} P(V < u) dS(u)$, since U and V are independent,

$$= \int_0^\infty P(V < u)dS(u) \text{ by Theorem 3.2(3),}$$
$$= \int_0^\infty T(u)dS(u), \text{ where } T(u) \text{ is the d.f. of } V.$$

By Theorem 3.2(1) and (2), $g(\theta_1,\theta_2)$ distinguishes between H and A in the following manner:

$$g(\theta_1,\theta_2) = \int_0^\infty T(u)dS(u) = \int_0^\infty S(u)dS(u) = \tfrac{1}{2} = g_0 \quad \text{if } (\theta_1,\theta_2) \in \omega,$$
$$g(\theta_1,\theta_2) = \int_0^\infty T(u)dS(u) \geqslant \int_0^\infty S([\min(\theta_1,\theta_2)]\, u)dS(u)$$
$$> \int_0^\infty S(u)dS(u) = \tfrac{1}{2} = g_0 \quad \text{if } \min(\theta_1,\theta_2) > 1, \text{ and}$$
$$g(\theta_1,\theta_2) = \int_0^\infty T(u)dS(u) \leqslant \int_0^\infty S([\max(\theta_1,\theta_2)]\, u)dS(u)$$
$$< \int_0^\infty S(u)dS(u) = \tfrac{1}{2} = g_0 \quad \text{if } \max(\theta_1,\theta_2) < 1.$$

To show the consistency of the test R, it is sufficient to show that $R_{m,n}/mn \xrightarrow[m,n \to \infty]{P} g(\theta_1,\theta_2)$ for all $(\theta_1,\theta_2) \in \Omega$ and its convergence is uniform for $(\theta_1,\theta_2) \in \omega$. Let $\varepsilon > 0$ be given and $(\theta_1,\theta_2) \in \Omega$. Then

$$P(|R_{m,n}/mn - g(\theta_1,\theta_2)| \geqslant \varepsilon)$$
$$= P(|\,[R_{m,n}/mn - E(R_{m,n}/mn)] + [E(R_{m,n}/mn) - g(\theta_1,\theta_2)]\,| \geqslant \varepsilon)$$
$$\leqslant P(|R_{m,n}/mn - E(R_{m,n}/mn)| \geqslant \varepsilon/2)$$
$$+ P(|E(R_{m,n}/mn) - g(\theta_1,\theta_2)| \geqslant \varepsilon/2) \text{ by Lemma 3.1,}$$
$$\leqslant [\mathrm{Var}(R_{m,n}/mn)/(\varepsilon/2)^2] + 0 \text{ by Chebyshev's inequality and Theorem 3.3(1),}$$
$$= \mathrm{Var}(R_{m,n}/mn)/(\varepsilon/2)^2 \longrightarrow 0 \text{ as } m,n \longrightarrow \infty \text{ by Theorem 3.3(2).}$$

Therefore, we have $R_{m,n}/mn \xrightarrow[m,n \to \infty]{P} g(\theta_1,\theta_2)$ for all (θ_1,θ_2) and, of course, this convergence is uniform for $(\theta_1,\theta_2) \in \omega = \{(1,1)\}$.
By Theorem 3.1, we complete the proof.

IV. CONSISTENCY OF THE TEST R*

Let us consider the case when the common median $\underset{\sim}{\nu} = (\nu_1, \nu_2)$ is unknown. In that case, we can define the test statistic by using the combined sample median $\underset{\sim}{M}_N = (M_{1N}, M_{2N})$ instead of $\underset{\sim}{\nu} = (\nu_1, \nu_2)$.

Lemma 4.1: If $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots$ are two sequences of random variables, if $X_N \xrightarrow[N \to \infty]{P} X$ and $Y_N \xrightarrow[N \to \infty]{P} Y$, and if f is a measurable function defined over $E^{(2)}$ such that $P((X,Y) \in \text{Cont } f) = 1$, where Cont f is a set of points in $E^{(2)}$ at which f is continuous, then $f(X_N, Y_N) \xrightarrow[N \to \infty]{P} f(X,Y)$.

Proof: See {Tucker (1967), p. 104}.

Lemma 4.2: Assume the marginals of F and G are increasing in some neighborhoods about their medians. For every $\theta_1 > 0, \theta_2 > 0$, we have

(1) $U^*_{iN} \xrightarrow[m,n \to \infty]{P} U_i$ for every fixed positive integer i, and

(2) $V^*_{jN} \xrightarrow[m,n \to \infty]{P} V_j$ for every fixed positive integer j.

Proof: (1) Let i be a fixed positive integer. By the definition,

$U^*_{iN} = [(X_{1i} - M_{1N})^2 + (X_{2i} - M_{2N})^2]^{1/2}$,

where $\underset{\sim}{M}_N = (M_{1N}, M_{2N})$ is the combined sample median.

The assumption that the marginals of F and G are increasing in some neighborhood about their medians insures that $\sqrt{N}(M_{1N}-\nu_1)$ and $\sqrt{N}(M_{2N}-\nu_2)$ are bounded in probability as $N \to \infty$ (see {Fligner (1974)}). Hence $M_{1N} \xrightarrow[m,n \to \infty]{P} \nu_1$ and $M_{2N} \xrightarrow[m,n \to \infty]{P} \nu_2$.

By Lemma 4.1, we have

$U^*_{iN} \xrightarrow[m,n \to \infty]{P} U_i$ for every fixed positive integer i.

(2) The proof is similar to (1).

Lemma 4.3: If $X, X_1, X_2, \ldots$ is a sequence of random variables, and if $X_N \xrightarrow[N \to \infty]{P} X$, then $X_N \xrightarrow[N \to \infty]{L} X$.

Proof: See {Tucker (1967), p. 105}.

Define $\pi_N^* = P(U_{iN}^* > V_{jN}^*)$, $\pi_o = P(U_i > V_j)$,

$a_N^* = P(V_{jN}^* < U_{iN}^* \cap V_{kN}^* < U_{iN}^*)$, $a = P(V_j < U_i \cap V_k < U_i)$

for $j \neq k$,

$b_N^* = P(U_{iN}^* > V_{jN}^* \cap U_{hN}^* > V_{jN}^*)$, $b = P(U_i > V_j \cap U_h > V_j)$

for $i \neq h$,

$c_N^* = P(U_{iN}^* > V_{jN}^* \cap U_{hN}^* > V_{kN}^*)$, $c = P(U_i > V_j \cap U_h > V_k)$

for $i \neq h$, and $j \neq k$.

Lemma 4.4: Under the same assumption as in Lemma 4.2, we have

(1) $\lim\limits_{m,n \to \infty} \pi_N^* = \pi_o$,

(2) $\lim\limits_{m,n \to \infty} a_N^* = a$,

(3) $\lim\limits_{m,n \to \infty} b_N^* = b$, and

(4) $\lim\limits_{m,n \to \infty} c_N^* = c$,

for every $\theta_1 > 0, \theta_2 > 0$.

Proof: (1) $\pi_N^* = P(U_{iN}^* > V_{jN}^*)$

$= P(U_{iN}^* - V_{jN}^* > 0)$

$\pi_o = P(U_i > V_j)$

$= P(U_i - V_j > 0)$

To show that $\lim\limits_{m,n \to \infty} \pi_N^* = \pi_o$, it is sufficient to show that

$U_{iN}^* - V_{jN}^* \xrightarrow[m,n \to \infty]{L} U_i - V_j$.

By Lemma 4.2, we have $U_{iN}^* \xrightarrow[m,n \to \infty]{P} U_i$ and $V_{jN}^* \xrightarrow[m,n \to \infty]{P} V_j$.

Thus, $U_{iN}^* - V_{jN}^* \xrightarrow[m,n \to \infty]{P} U_i - V_j$ by Lemma 4.1.

Accordingly, $U_{iN}^* - V_{jN}^* \xrightarrow[m,n \to \infty]{L} U_i - V_j$ by Lemma 4.3.

This completes the proof of (1): $\lim_{m,n \to \infty} \pi_N^* = \pi_o$.

(2) $a_N^* = P(V_{jN}^* < U_{iN}^* \cap V_{kN}^* < U_{iN}^*)$

$= P(\max(V_{jN}^*, V_{kN}^*) < U_{iN}^*)$

$= P(\max(V_{jN}^*, V_{kN}^*) - U_{iN}^* < 0)$.

$a \;= P(V_j < U_i \cap V_k < U_i)$

$= P(\max(V_j, V_k) < U_i)$

$= P(\max(V_j, V_k) - U_i < 0)$

To show that $\lim_{m,n \to \infty} a_N^* = a$, it is sufficient to show that

$\max(V_{jN}^*, V_{kN}^*) - U_{iN}^* \xrightarrow[m,n \to \infty]{L} \max(V_j, V_k) - U_i$.

By Lemma 4.2, we have $V_{jN}^* \xrightarrow[m,n \to \infty]{P} V_j$, $V_{kN}^* \xrightarrow[m,n \to \infty]{P} V_k$, and

$U_{iN}^* \xrightarrow[m,n \to \infty]{P} U_i$. Thus,

$\max(V_{jN}^*, V_{kN}^*) - U_{iN}^* \xrightarrow[m,n \to \infty]{P} \max(V_j, V_k) - U_i$ by Lemma 4.1.

Accordingly,

$\max(V_{jN}^*, V_{kN}^*) - U_{iN}^* \xrightarrow[m,n \to \infty]{L} \max(V_j, V_k) - U_i$ by Lemma 4.3.

This completes the proof of (2): $\lim_{m,n \to \infty} a_N^* = a$.

(3) $b_N^* = P(U_{iN}^* > V_{jN}^* \cap U_{hN}^* > V_{jN}^*)$

$= P(\min(U_{iN}^*, U_{hN}^*) > V_{jN}^*)$

$= P(\min(U_{iN}^*, U_{hN}^*) - V_{jN}^* > 0)$.

$b \;= P(U_i > V_j \cap U_h > V_j)$

$= P(\min(U_i, U_h) > V_j)$

$= P(\min(U_i, U_h) - V_j > 0)$.

To show that $\lim_{m,n \to \infty} b_N^* = b$, it is sufficient to show that

$\min(U^*_{iN}, U^*_{hN}) - V^*_{jN} \xrightarrow[m,n \to \infty]{L} \min(U_i, U_h) - V_j$.

By Lemma 4.2, we have $U^*_{iN} \xrightarrow[m,n \to \infty]{P} U_i$, $U^*_{hN} \xrightarrow[m,n \to \infty]{P} U_h$, and

$V^*_{jN} \xrightarrow[m,n \to \infty]{P} V_j$. Thus,

$\min(U^*_{iN}, U^*_{hN}) - V^*_{jN} \xrightarrow[m,n \to \infty]{P} \min(U_i, U_h) - V_j$ by Lemma 4.1.

Accordingly, $\min(U^*_{iN}, U^*_{hN}) - V^*_{jN} \xrightarrow[m,n \to \infty]{L} \min(U_i, U_h) - V_j$ by Lemma 4.3.

This completes the proof of (3): $\lim_{m,n \to \infty} b^*_N = b$.

(4) $c^*_N = P(U^*_{iN} > V^*_{jN} \cap U^*_{hN} > V^*_{kN})$

$= P(U^*_{iN} - V^*_{jN} > 0 \cap U^*_{hN} - V^*_{kN} > 0)$

$= P(\min(U^*_{iN} - V^*_{jN}, U^*_{hN} - V^*_{kN}) > 0)$

$c = P(U_i > V_j \cap U_h > V_k)$

$= P(U_i - V_j > 0 \cap U_h - V_k > 0)$

$= P(\min(U_i - V_j, U_h - V_k) > 0)$

To show that $\lim_{m,n \to \infty} c^*_N = c$, it is sufficient to show that

$\min(U^*_{iN} - V^*_{jN}, U^*_{hN} - V^*_{kN}) \xrightarrow[m,n \to \infty]{L} \min(U_i - V_j, U_h - V_k)$.

By Lemma 4.2, we have $U^*_{iN} \xrightarrow[m,n \to \infty]{P} U_i$, $V^*_{jN} \xrightarrow[m,n \to \infty]{P} V_j$,

$U^*_{hN} \xrightarrow[m,n \to \infty]{P} U_h$, and $V^*_{kN} \xrightarrow[m,n \to \infty]{P} V_k$.

Hence, $\min(U^*_{iN} - U^*_{hN}, V^*_{jN} - V^*_{kN}) \xrightarrow[m,n \to \infty]{P} \min(U_i - U_h, V_j - V_k)$ by Lemma 4.1.

Accordingly, $\min(U^*_{iN} - U^*_{hN}, V^*_{jN} - V^*_{kN}) \xrightarrow[m,n \to \infty]{L} \min(U_i - U_h, V_j - V_k)$ by Lemma 4.3.

This completes the proof of (4): $\lim_{m,n\to\infty} c_N^* = c.$

Therefore Lemma 4.4 is proved.

Theorem 4.1: Under the same assumption as in Lemma 4.2, we have

(1) $\lim_{m,n\to\infty} E(R_{m,n}^*/mn) = \pi_o$, and

(2) $\lim_{m,n\to\infty} Var(R_{m,n}^*/mn) = 0,$

for every $\theta_1 > 0, \theta_2 > 0$.

Proof: (1) Since $E(R_{m,n}^*/mn) = E(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij}^*/mn)$

$$= (1/mn)(\sum_{i=1}^{m}\sum_{j=1}^{n} E(D_{ij}^*))$$

$$= (1/mn)(mnE(D_{ij}^*))$$

$$= E(D_{ij}^*)$$

$$= P(U_{iN}^* > V_{jN}^*)$$

$$= \pi_N^*,$$

we have $\lim_{m,n\to\infty} E(R_{m,n}^*/mn) = \lim_{m,n\to\infty} \pi_N^* = \pi_o$ by Lemma 4.4(1).

(2) Since $Var(R_{m,n}^*/mn)$

$$= (1/m^2n^2)Var(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij}^*)$$

$$= (1/m^2n^2)[\sum_{i=1}^{m}\sum_{j=1}^{n} Var(D_{ij}^*) + \sum_{i=1}^{m}\sum\sum_{1<j\neq k<n} Cov(D_{ij}^*, D_{ik}^*)$$

$$+ \sum_{j=1}^{n}\sum\sum_{1<i\neq h<m} Cov(D_{ij}^*, D_{hj}^*) + \sum\sum_{1<i\neq h<m}\sum\sum_{1<j\neq k<n} Cov$$

$$(D_{ij}^*, D_{hk}^*)]$$

$$= (1/m^2n^2)[mn(\pi_N^* - \pi_N^{*2}) + mn(n-1)(a_N^* - \pi_N^{*2})$$

$$+ nm(m-1)(b_N^* - \pi_N^{*2}) + mn(m-1)(n-1)(c_N^* - \pi_N^{*2})]$$

$$= (1/mn)[(\pi_N^* - \pi_N^{*2}) + (n-1)(a_N^* - \pi_N^{*2})$$

$$+ (m-1)(b_N^* - \pi_N^{*2}) + (m-1)(n-1)(c_N^* - \pi_N^{*2})]$$

and $\quad \mathrm{Var}(R_{m,n}/mn)$

$$= (1/m^2n^2)\mathrm{Var}(\sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij})$$

$$= (1/mn)[(\pi_o - \pi_o^2) + (n-1)(a - \pi_o^2) + (m-1)$$

$$\cdot (b - \pi_o^2) + (m-1)(n-1)(c - \pi_o^2)],$$

we have $\lim_{m,n\to\infty} [\mathrm{Var}(R^*_{m,n}/mn) - \mathrm{Var}(R_{m,n}/mn)] = 0$ by Lemma 4.4.

By Theorem 3.3(2), $\lim_{m,n\to\infty} \mathrm{Var}(R_{m,n}/mn) = 0$ implies that

$\lim_{m,n\to\infty} \mathrm{Var}(R^*_{m,n}/mn) = 0.$

Thus, the theorem is proved.

We are now in a position to prove the consistency of the test R^*.

Theorem 4.2: Let $(X_{11},X_{21}),\ldots,(X_{1m},X_{2m})$ and $(Y_{11},Y_{21}),\ldots,(Y_{1n},Y_{2n})$ be two independent bivariate random samples from populations with continuous d.f.'s $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively, such that $G_{\underset{\sim}{Y}-\underset{\sim}{\nu}}(y_1,y_2) = F_{\underset{\sim}{X}-\underset{\sim}{\nu}}(\theta_1 y_1, \theta_2 y_2)$ for all (y_1,y_2) and for some $\theta_1 > 0, \theta_2 > 0$, where $\underset{\sim}{X} = (X_1,X_2)$, $\underset{\sim}{Y} = (Y_1,Y_2)$ and $\underset{\sim}{\nu} = (\nu_1,\nu_2)$ is the unknown common median. Assume the marginals of F and G are increasing in some neighborhoods about their medians. Then the test R^* is consistent in the following cases:

Subclass of Alternatives	Rejection Region
A: $\min(\theta_1,\theta_2) > 1$	$R^*_{m,n} - mn/2 > d_1^*$
A: $\max(\theta_1,\theta_2) < 1$	$R^*_{m,n} - mn/2 < d_2^*$

Proof: Define $\Omega = \{(\theta_1,\theta_2) \mid \theta_1 > 0, \theta_2 > 0\}$, $\omega = \{(1,1)\}$, $g_o = ½$,

and $g(\theta_1,\theta_2) = P(U > V)$ for $(\theta_1,\theta_2) \in \Omega$,

where $U = [(X_1 - \nu_1)^2 + (X_2 - \nu_2)^2]^{½}$ and

$$V = [(Y_1 - \nu_1)^2 + (Y_2 - \nu_2)^2]^{½}$$

In the proof of Theorem 3.4, we have obtained that

$$g(\theta_1,\theta_2) = g_o \text{ if } (\theta_1,\theta_2) \in \omega,$$
$$> g_o \text{ if } \min(\theta_1,\theta_2) > 1,$$
$$< g_o \text{ if } \max(\theta_1,\theta_2) < 1.$$

To show the consistency of the test R^*, it is sufficient to show that $R^*_{m,n}/mn \xrightarrow[m,n\to\infty]{P} g(\theta_1,\theta_2)$ for all $(\theta_1,\theta_2) \in \Omega$ and this convergence is uniform for $(\theta_1,\theta_2) \in \omega$. Let $\varepsilon > 0$ be given and $(\theta_1,\theta_2) \in \Omega$. Then

$P(|R^*_{m,n}/mn - g(\theta_1,\theta_2)| \geqslant \varepsilon)$

$= P(|[R^*_{m,n}/mn - E(R^*_{m,n}/mn)] + [E(R^*_{m,n}/mn) - g(\theta_1,\theta_2)]| \geqslant \varepsilon)$

$\leqslant P(|R^*_{m,n}/mn - E(R^*_{m,n}/mn)| \geqslant \varepsilon/2)$

$+ P(|E(R^*_{m,n}/mn) - g(\theta_1,\theta_2)| \geqslant \varepsilon/2)$ by Lemma 3.1.

Since $P(|R^*_{m,n}/mn - E(R^*_{m,n}/mn)| \geqslant \varepsilon/2) \leqslant Var(R^*_{m,n}/mn)/(\varepsilon/2)^2 \to 0$ as $m,n \to \infty$ by Chebyshev's inequality and Theorem 4.1(2); and $\lim_{m,n\to\infty} P(|E(R^*_{m,n}/mn) - g(\theta_1,\theta_2)| \geqslant \varepsilon/2) = 0$ by Theorem 4.1(1), we have $\lim_{m,n\to\infty} P(|R^*_{m,n}/mn - g(\theta_1,\theta_2)| \geqslant \varepsilon) = 0$.

Therefore, $R^*_{m,n}/mn \xrightarrow[m,n\to\infty]{P} g(\theta_1,\theta_2)$ for all $(\theta_1,\theta_2) \in \Omega$, and, of course, this convergence is uniform for $(\theta_1,\theta_2) \in \omega = \{(1,1)\}$. By Theorem 3.1, we complete the proof.

V. EFFICIENCY OF THE TESTS R AND R^*

First, let us consider the bivariate normal two-sample scale-model as follows:

Suppose $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ are two independent random samples from $N_2((\mu_1,\mu_2), \begin{pmatrix} a & \rho_1 a \\ \rho_1 a & a \end{pmatrix})$ and $N_2((\eta_1,\eta_2), \begin{pmatrix} b & \rho_2 b \\ \rho_2 b & b \end{pmatrix})$, respectively, where ρ_1 and ρ_2 are the known correlation coefficients, $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ and $\underset{\sim}{\eta} = (\eta_1,\eta_2)$ are unknown means, a and b are unknown scale parameters. Then $\Omega = \{(a,b,\mu_1,\mu_2,\eta_1,\eta_2) | 0 < a,b < \infty, -\infty < \mu_1,\mu_2,\eta_1,\eta_2 < \infty\}$. The hypothesis H': $a = b$, $\underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified, is to be tested against A': $a \neq b$,

$\underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified. Then $\omega = \{ (a,b,\mu_1,\mu_2,\eta_1,\eta_2 | 0 < a = b < \infty, -\infty < \mu_1,\mu_2,\eta_1, \eta_2 < \infty \}$. The likelihood ratio test for testing H': $a = b, \underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified, against A': $a \neq b, \underset{\sim}{\mu}$ and $\underset{\sim}{\eta}$ unspecified, can be based on $F^*_{m,n}$(see { Liu(1982) }), where

$$F^*_{m,n} = \frac{\sum_{i=1}^{m} [(X_{1i}-\bar{X}_1)^2-2\rho_1(X_{1i}-\bar{X}_1)(X_{2i}-\bar{X}_2)+(X_{2i}-\bar{X}_2)^2]/[(1-\rho_1^2)(m-1)]}{\sum_{j=1}^{n} [(Y_{1j}-\bar{Y}_1)^2-2\rho_2(Y_{1j}-\bar{Y}_1)(Y_{2j}-\bar{Y}_2)+(Y_{2j}-\bar{Y}_2)^2]/[(1-\rho_2^2)(n-1)]} .$$

Next, consider the bivariate uniform two-sample scale-model as follows:

Suppose $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ are two independent random samples from bivariate uniform populations with p.d.f.'s

$$f(x_1,x_2) = 1/(\pi c_1^2), \qquad 0 \leqslant (x_1-\mu_1)^2 + (x_2-\mu_2)^2 \leqslant c_1^2,$$

and

$$g(y_1,y_2) = 1/(\pi c_2^2), \qquad 0 \leqslant (y_1-\mu_1)^2 + (y_2-\mu_2)^2 \leqslant c_2^2,$$

where $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ is the known common mean, c_1 and c_2 are unknown scale parameters. Set θ c_1/c_2. Then the problem is to test the hypothesis H': $\theta = 1$ against either one- or two-sided alternatives. Since the common correlation coefficient ρ becomes 0 in this model, the test statistic $F^*_{m,n}$ used in the normal-theory model is reduced to be $\dfrac{\sum_{i=1}^{m} [(X_{1i} - \bar{X}_1)^2 + (X_{2i} - \bar{X}_2)^2]/(m-1)}{\sum_{j=1}^{n} [(Y_{1j} - \bar{Y}_1)^2 + (Y_{2j} - \bar{Y}_2)^2]/(n-1)}$. In order to simplify our computation for ARE, we would like to compare the nonparametric test statistic $R_{m,n}$ with the test statistic $F^{**}_{m,n} = \dfrac{\sum_{i=1}^{m} [(X_{1i} - \mu_1)^2 + (X_{2i} - \mu_2)^2]/m}{\sum_{j=1}^{n} [(Y_{1j} - \mu_1)^2 + (Y_{2j} - \mu_2)^2]/n}$ if the common mean (μ_1,μ_2) is known.

By the fact that $R_{m,n}$, $R^*_{m,n}$, $W_{m,n}$, and $W^*_{m,n}$ have the same limiting distribution under both H' and A' if the underlying populations satisfy some regular conditions (see { Liu(1981) }), the tests R, R*, W, and W* have the same ARE with respect to the test F* (or F**) (see { Liu(1982) }). We now summerize the results in the following theorems.

Theorem 5.1: Let $(X_{11},X_{21}), \ldots, (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ be two independent random samples from bivariate normal populations with p.d.f.'s

$$f(x_1,x_2) = \frac{1}{2\pi a\sqrt{1-\rho^2}} \exp\left\{-\frac{[(x_1-\mu_1)^2-2\rho(x_1-\mu_1)(x_2-\mu_2)+(x_2-\mu_2)^2]}{2a(1-\rho^2)}\right\},$$
$$-\infty<x_1,x_2<\infty,$$

and
$$g(y_1,y_2) = \frac{1}{2\pi b\sqrt{1-\rho^2}} \exp\left\{-\frac{[(y_1-\mu_1)^2-2\rho(y_1-\mu_1)(y_2-\mu_2)+(y_2-\mu_2)^2]}{2b(1-\rho^2)}\right\}$$
$$-\infty<y_1,y_2<\infty$$

respectively, where ρ is the known common correlation coefficient, $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ is the common mean (either known or unknown), a and b are unknown scale parameters. Set $\theta = (a/b)^{1/2}$. For the hypothesis testing problem H': $\theta = 1$ against either one- or two-sided alternatives, we have

$$ARE(R,F^*) = ARE(R^*,F^*) = \frac{3(1-\rho^2)^2}{\pi^2}\left[\int_0^{2\pi}\frac{(2-\rho\sin t)}{[(2-\rho\sin t)^2-\rho^2]^{3/2}}dt\right]^2.$$

Particularly, if $\rho = 0$ then $ARE(R,F^*) = ARE(R^*,F^*) = 3/4$.

Theorem 5.2: Let $(X_{11},X_{21}),\ldots,(X_{1m},X_{2m})$ and $(Y_{11},Y_{21}),\ldots,(Y_{1n},Y_{2n})$ be two independent random samples from bivariate uniform populations with p.d.f.'s

$$f(x_1,x_2) = 1/(\pi c_1^2), \qquad 0 \leqslant (x_1-\mu_1)^2 + (x_2-\mu_2)^2 \leqslant c_1^2$$
and
$$g(y_1,y_2) = 1/(\pi c_2^2), \qquad 0 \leqslant (y_1-\mu_1)^2 + (y_2-\mu_2)^2 \leqslant c_2^2$$

respectively, where $\underset{\sim}{\mu} = (\mu_1,\mu_2)$ is the known common mean, c_1 and c_2 are unknown scale parameters. Set $\theta = c_1/c_2$. For the hypothesis testing problem H': $\theta = 1$ against either one- or two-sided alternatives, we have $ARE(R,F^{**}) = ARE(R^*,F^{**}) \doteq 1$.

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