A NOTE ON FINDING ALL THE EXTREME POINTS THAT SPAN AN UNBOUNDED CONVEX POLYTOPE UNDER LINEAR INEQUALITIES

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摘　要

設線性不等式 $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$, $i=1,2,3,\cdots\cdots m$ 及 $x_j \geq 0$，$j = 1,2,3,\cdots\cdots, n$ 求其可行基解（feasible solution，或非負數解（non-negative solution）。則其可行基解必在凸多面體的頂點（Vertex）上或極點 extreme point）上。當變數超過三個即 $n > 3$ 時，無幾何圖形。本文應用基集合（basic set）與非基集合（non-basic set）內元素（element）對換且以單純法（simplex method）計算出全部可行基解而得到所有極點構成一凸多面體表示其幾何圖形解。

1. Introduction

Consider a system of linear inequalities in the following forms:

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i, \ i = 1, 2, 3, \ldots, m, \tag{1}$$

where $b_i \geqslant 0$ and $x_j \geqslant 0$ for all i and j.

The system (1) can be written by the following matrix form

$$AX \geqslant b, \tag{2}$$

where $A = [a_{ij}]$ is an m by n matrix, $X = (x_1, x_2, x_3, \ldots, x_n)^T$ in n-space and $b = (b_1, b_2, b_3, \ldots, b_n)^T$ in m-space are non-negative column vectors whose all the components must be non-negative.

In geometry, each one of the system (1) represents a closed half hyperplanes in n-space [1. P.165] and the intersection of all these closed half hyperplanes together with $x_j \geqslant 0$ for all j is to span a feasible polyhedral convex set (convex polytope), which is infinite inextent or unbounded [1. P.173] and whose extreme points (vertice) are the basic feasible solutions in algebra.

But the system of linear inequalities is in the following forms:

$$\sum_{j=1}^{n} a_{ij}x_j \leqslant b_i, \quad i = 1, 2, 3, \ldots, m, \tag{3}$$

where $b_i \geqslant 0$ and $x_j \geqslant 0$ for all i and j. This system (3) of computing all the basic feasible solutions at their corresponding extreme points of this bounded convex polytope has been discussed by these authors, M. E. Dyer and L. G. Proll [2. P.81], M. J. L. Kirby, H. R. Love and Kanti Swarup [3. P.540], T. H. Matteiss [4. P.427] and M. C. Cheng [5. P.270, 6. P.60]. Similarly, by using this method for solving the system (3), it is clearly seen that an unbounded convex polytope of the system (1) can be also defined.

The non-negative surplus (or slack) variables x_{n+i}, i = 1, 2, 3, . . . , m can be used to substract from their corresponding linear inequalities (1) to obtain the following linear equalities:

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i, \quad i = 1, 2, 3, \ldots, m, \tag{4}$$

where $x_{n+i} \geqslant 0$ and $x_j \geqslant 0$ for all i and j,

$$\text{or} \quad \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=n+1}^{n+m} a_{ij}x_j = b_i, \tag{4'}$$

$$j = n + r, \; a_{ij} = \begin{cases} -1, & i = r, \\ 0, & i \neq r, \end{cases} \quad i \text{ and } r = 1, 2, 3, \ldots, m,$$

where $x_j \geqslant 0$, j = 1, 2, 3, . . . , n, n+1, n+2, n+3, . . . , n+r, . . . , n+m.

It is easily seen that the system (4) or (4′) does not have the initial basic feasible solutions because $x_j = 0$, j = 1, 2, 3, . . . , n, the basic variables $x_{n+i} = -b_i$, i = 1, 2, 3, . . . , m are contradictory in $x_{n+i} \geqslant 0$ and $b_i \geqslant 0$ for all i.

Hence, the non-negative artificial variables $x_{(n+i)'}$ for all i can be used to add to their corresponding system (4) or (4′) to obtain the following forms respectively:

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} + x_{(n+i)'} = b_i, \tag{5}$$

$$\text{or} \quad \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=n+1}^{n+m} a_{ij} x_j + x_{(n+i)'} = b_i, \tag{5'}$$

where $x_j \geqslant 0$, $x_{n+i} \geqslant 0$ and $x_{(n+i)'} \geqslant 0$ for all i and j.

The system (5) or (5′) can be written by the following matrix form:

$$(A - I)X + IX' = b, \tag{6}$$

where I is an m square identity matrix, $X = (x_1, x_2, x_3, \ldots x_n, x_{n+1}, x_{n+2}, \ldots x_{n+m})^T$ and $X' = (x_{(n+1)'}, x_{(n+2)'}, x_{(n+3)'}, \ldots, x_{(n+m)'})^T$.

For the system (5) or (5′), it is obvious that the non-negative artificial variables $x_{(n+i)'}$, $i = 1, 2, 3, \ldots m$, are initial basic and the variables x_j, $j = 1, 2, 3, \ldots n, n+1, n+2, \ldots, n+m$ are non-basic and thus the initial point (or origin) p_0 can be easily determined by the initial basic solution $b_0 = (b_1, b_2, b_3, \ldots, b_m)^T$ when all the non-basic variables are equal to zero. Then the subsequent points p_k, where k is ranking the ascending natural numbers, can be created by the theorems and corollaries in section (3) and the following simplex matrix can be readily obtained by the matrix form (6)

$$\begin{array}{cccccc|cccccccc}
x_1 & x_2 & \ldots & x_k & \ldots & x_n & x_{n+1} & x_{n+2} & \ldots & x_{n+r} & \ldots & x_{n+m} \\
a_{11} & a_{12} & \ldots & a_{1k} & \ldots & a_{1n} & -1 & 0 & \ldots & 0 & \ldots & 0 \\
a_{21} & a_{22} & \ldots & a_{3k} & \ldots & a_{2n} & 0 & -1 & \ldots & 0 & \ldots & 0 \\
a_{31} & a_{32} & \ldots & a_{3k} & \ldots & a_{3n} & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\cdots & & & & & & \cdots & & & & & \\
a_{r1} & a_{r2} & \ldots & a_{rk} & \ldots & a_{rn} & 0 & 0 & \ldots & -1 & \ldots & 0 \\
\cdots & & & & & & \cdots & & & & & \\
a_{m1} & a_{m2} & \ldots & a_{mk} & \ldots & a_{mn} & 0 & 0 & \ldots & 0 & \ldots & -1
\end{array}$$

$$\begin{array}{cccccc|c}
x_{(n+1)'} & x_{(n+2)'} & \ldots & x_{(n+r)'} & \ldots & x_{(n+m)'} & \\
1 & 0 & \ldots & 0 & \ldots & 0 & b_1 \\
0 & 1 & \ldots & 0 & \ldots & 0 & b_2 \\
0 & 0 & \ldots & 0 & \ldots & 0 & b_3 \\
\cdots & & & & & & \\
0 & 0 & & 1 & & 0 & b_r \\
\cdots & & & & & & \\
0 & 0 & & 0 & & 1 & b_m
\end{array} \tag{7}$$

Each one of the non-basic variables $x_j, j \in N_{n+m}$ is able to become basic instead of one of the basic variables $x_{(n+r)'}$, $r \in M$. Suppose the k-th column (x_k become basic), $k \in N_{n+m}$ is a pivot column, the r-th row ($x_{(n+r)'}$ become non-basic) is a pivot row and a_{rk} must be greater than zero (or denoted by $r_k = k$, $q_k = r$ and $s_k = (n+r)'$). Hence, the pivot ratio must be minimum as follows:

$$\frac{b_r}{a_{rk}} \leqslant \frac{b_i}{a_{ik}}, \quad i \neq r \text{ and } i,\ r \in M$$

By using the elementary row operations [1. P.126], the non-basic column vector A_j^1, $x_{(n+r)'}$ and the basic solution b_k^1 together with basic set β_k can be readily obtained as follows:

The components of A_j^1 are $a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik}$ and $\frac{a_{rj}}{a_{rk}}$,

where $j \leqslant n$, $j \neq k$, $i \neq r$ and $j = n+r$, $a_{ij} = \begin{cases} -1, & i = r, \\ 0, & i \neq r, \end{cases}$

the components of $x_{(n+r)'}$ are $-\frac{a_{ik}}{a_{rk}}$ and $\frac{1}{a_{rk}}$, $i \neq r$, i and $r \in M$,

$b_k^1 = (x_{(n+1)'}, x_{(n+2)'}, x_{(n+3)'}, \ldots x_k, \ldots, x_{(n+m)'})^T$,

where $x_k = \frac{b_r}{a_{rk}}$, and $x_{(n+r)'} = b_i - \frac{b_r}{a_{rk}} a_{ik}$, $i \neq r$, i and $r \in M$,

because b_k^1 is non-negative, $x_{(n+r)'} \geqslant 0$ and $x_k \geqslant 0$ imply $\frac{b_r}{a_{rk}} \leqslant \frac{b_i}{a_{ik}}$

[7. P.81] and $\beta_k = \{(n+1)', (n+2)', (n+3)', \ldots k, \ldots, (n+m)'\}$.

Hence, it is immediately seen that all the points can be obtained by the iterative simplex method together with theorems and corollaries in section 3.

2. Definitions and Notations

The following useful definitions and notations suffice this note:

D_1. Points. By using the simplex method a point (denoted by p_k in section 5 in figure) can be created from p_0 by the number of h paths corresponding to its basic solutions for the system (1) and it is intersected by n distinct space lines in n-space.

D_2. Extreme points. A point (D_1) is called an extreme point (denoted by P_k in section 5 in figure) when it has to be created by the number of $h(k) = h = m$ (N_{24}) paths and then the basic solution at this extreme point must be feasible (satisfies all the linear inequalities of the system (1)).

D_3. Unbounded Convex Polytope. All the extreme points (D_2) must span an infinite feasible polyhedral convex set called an unbounded convex polytope.

D_4. Subsequent points. p_k and $p_{k'}$ are the two points (D_1) denoted by the two distinct natural numbers k and k'. $p_{k'}$ is said to be a subsequent point p_k, if $k' > k$.

D_5. Adjacency. The point p_k is said to be adjacent to a subsequent point $p_{k'}$, if a certain element in the non-basic set N_k at p_k enters the basic β_k to obtain a new basic set equal to the basic set $\beta_{k'}$ and their basic solutions are also equal.

D_6. Paths. A new point is created from the predecessor, called the number of one path because only one element in the non-basic set of the predecessor is instead of one element in its basic set to obtain a new basic set at the new point but computing the number of h paths (or h elements in N_0 entering in β_0) to creat a new point must be from the initial point (origin) p_0.

N_1. $A = [a_{ij}]$ is an m by n matrix with real entries, $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots n$ denoted by the number of rows (or equations) and the number of columns (or original variables) respectively.

N_2. $x_1, x_2, x_3, \ldots, x_n$ are the original variables.

N_3. $x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+r}, \ldots, x_{n+m}$ are the surplus variables.

N_4. $x_{(n+1)'}, x_{(n+2)'}, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'}$ are the artificial variable.

N_5. $[A-I] = [a_{ij}]$ is an m by (n+m) matrix $j = 1, 2, 3, \ldots, k, \ldots, n, n+1, n+2, n+3, \ldots, n+r, \ldots, n+m$, $j = n+r$, $a_{ij} = \begin{cases} -1, & i=r \\ 0, & i \neq r \end{cases}$ and i and $r = 1, 2, 3, \ldots, m$, where I is an m square identity matrix.

N_6. The number of linear inequalities (or equations) can be denoted by a set $M = \{1, 2, 3, \ldots, m\}$ and i and $r \in M$.

N_7. The number of original variables and surplus variables can be denoted by a set $N_{n+m} = \{1, 2, 3, \ldots, k, \ldots, n, n+1, n+2, n+3, \ldots, n+r, \ldots, n+m\}$ (or a set N_0) and j and $k \in N_{n+m}$.

N_8. $A_j = (a_{ij}, a_{2j}, a_{3j}, \ldots, a_{mj})^T$ indicates the j-th column vector of the matrix [A − I] and its components are the coefficients of x_j, $j \in N_{n+m}$ (N_5).

N_9. The "T" indicates the transpose of a matrix or a vector.

N_{10}. p_0 indicates the initial point or origin.

N_{11}. $b_0 = (b_1, b_2, b_3, \ldots, b_r, \ldots, b_m)^T$ indicates the initial basic solution at p_0.

N_{12}. $\beta_0 = \{(n+1)', (n+2)', (n+3)', \ldots, (n+r)', \ldots, (n+m)'\}$ indicates the initial basic set corresponding to the initial basic variables (or artificial variables) $x_{(n+1)'}$, $x_{(n+2)'}, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'}$ at p_0.

N_{13}. $N_0 = \{1, 2, 3, \ldots, k, \ldots, n, n+1, n+2, n+3, \ldots, n+r, \ldots, n+m\}$ indicates the initial non-basic set corresponding to the initial non-basic variables $x_1, x_2, x_3, \ldots, x_k, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+r}, \ldots, x_{n+m}$ at p_0.

N_{14}. p_k (or $p_{k'}$) indicates the ranking of a point created by the number of h (or h') paths from p_0 where k (or k') is ranking the ascending natural numbers.

N_{15}. b_k^h (or $b_{k'}^{h'}$) indicates the basic solution at p_k (or $p_{k'}$).

N_{16}. β_k (or $\beta_{k'}$) containing m elements indicates the basic set corresponding to b_k^h (or $b_{k'}^{h'}$) at p_k (or $p_{k'}$).

N_{17}. $N_k = \alpha_k \cup H$ (or $N_{k'} = \alpha_{k'} \cup H'$) containing n+m elements indicates the non-basic set at p_k (or $p_{k'}$) where H (or H') contains the number of h (or h') elements from β_0 to become the non-basic set and α_k (or $\alpha_{k'}$) contains the remaining number of n+m−h (or n+m−h') elements because the number of h (or h') paths in N_0 enters β_k (or $\beta_{k'}$) to become basic, hence $\alpha_k \cap H = \phi$ (empty set) (or $\alpha_{k'} \cap H' = \phi$).

N_{18}. A_j^h (or $A_j^{h'}$) indicates the j-th column vector, $j \in N_{n+m}$ (N_5) corresponding to α_k (or $\alpha_{k'}$) at p_k (or $p_{k'}$).

N_{19}. $x_{(n+r)'}^h$ (or $x_{(n+r)'}^{h'}$) indicates the $(n+r)'$-th column vector, $r \in M$ corresponding to H (or H') at p_k (or $p_{k'}$).

N_{20}. r_k indicates the pivot column in the simplex method entering the predecessor basic set of p_k to get β_k.

N_{21}. q_k indicates the pivot row in the simplex method leaving the predecessor basic set of p_k.

N_{22}. s_k indicates the corresponding column vector leaving the predecessor set of p_k.

N_{23}. $P(k) = k$ indicates the number of stages at p_k (define $P(0) = 0$).

N_{24}. $h(k) = h$ indicates the total number of solid lines in any path from p_0 to p_k (define $h(0) = 0$).

3. Theoretical Development

By using the simplex method, whether the new point (or extreme point) can be created depends upon the condition that each element of the non-basic set enters basic set. Firstly, the adjacency test must be used and thus, the following theorems and corollaries must be defined:

(1) Theorem 1. Let the two points p_k and $p_{k'}$ of the system (6) can be created from p_0 by the distinct one path (h=1 and h'=1) with the following properties:

(1) $p_{k'}$ is a subsequent point of p_k when $k' > k$.

(2) there is only one element of the set β_k different from the set $\beta_{k'}$.

(3) there is only one element of the set α_k in N_k different from the set $\alpha_{k'}$ in $N_{k'}$ but the set H is equal to the set H' where H contains only one element and H' too.

Then p_k is said to be adjacent to the subsequent point $p_{k'}$.

Proof. Suppose p_k is created from p_0 by h=1 path denoted by

$$r_k = 1,\ q_k = 2 \text{ and } s_k = (n+2)'.$$

By using the simplex method, the basic solution b_k^1, the basic set β_k, the non-basic set N_k and the non-basic column vectors A_j^1, $j \in N_0, j \neq 1$ (N_5) and $x^1_{(n+2)'}$ corresponding to N_k can be readily computed as follows:

$$b_k^1 = (x_{(n+1)'}, x_1, x_{(n+3)'}, \dots, x_{(n+r)'}, \dots, x_{(n+m)'})^T,$$

where $x_1 = \frac{b_2}{a_{21}}$ and $x_{(n+r)'} = b_r - \frac{b_2}{a_{21}} a_{r1}$, for $r \in M$, but $r \neq 2$,

$$\beta_k = \{(n+1)', 1, (n+3)', \dots, (n+r)', \dots, (n+m)'\},$$

$$N_k = \alpha_k \cup H \text{ where } \alpha_k = \{2, 3, 4, \dots, n, n+1, n+2, n+3, \dots, n+r, \dots, n+m\} \text{ and } H = \{(n+2)'\},$$

the components of A_j^1 are $\frac{a_{2j}}{a_{21}}$ and $a_{rj} - \frac{a_{2j}}{a_{21}} a_{r1}$ where $r \in M$, $j \neq 1$,

and $j \in N_0$ (according to N_5),

and the components of $x_{(n+2)'}$ are $\frac{1}{a_{21}}$ and $-\frac{a_{r1}}{a_{21}}$, $r \in M$ but $r \neq 2$.

Supoose $p_{k'}$ is created by h'=1 path denoted by

$$r_{k'} = 3,\ q_{k'} = 2 \text{ and } s_{k'} = (n+2)'$$

Similarly, we have

$b^1_{k'} = (x_{(n+1)'}, x_3, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'})^T$

where $x_3 = \frac{b_2}{a_{23}}$ and $x_{(n+r)'} = b_r - \frac{b_2}{a_{23}} a_{r3}$, for $r \in M$, but $r \neq 2$,

$\beta_{k'} = \{(n+1)', 3, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\}$

$N_{k'} = \alpha_{k'} \cup H$ where $\alpha_{k'} = \{1, 2, 4, \ldots, n, n+1, n+2, n+3, \ldots, n+r, \ldots$ $n+m\}$ and $H' = \{(n+2)'\}$,

the components of $A^{1'}_j$ are $\frac{a_{2j}}{a_{23}}$ and $a_{rj} - \frac{a_{2j}}{a_{23}} a_{r3}$, $r \neq 2$ and $j \neq 3$, and $j \in N_0$ (according to N_5),

and the components of $x_{(n+2)'}$ are $\frac{1}{a_{23}}$ and $-\frac{a_{r3}}{a_{23}}$, $r \in M$, but $r \neq 2$.

By using the properties (2) and (3) H=H′, it is obvious that the element "3" in α_k instead of the element "1" in β_k to become basic and by the use of the simplex method, the new basic set $\{(n+1)', 3, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\}$ and the new basic solution $(x_{(n+1)'}, x_3, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)})$ can be immediately obtained,

where $x_3 = \frac{b_2}{a_{21}} / \frac{a_{23}}{a_{21}} = \frac{b_2}{a_{23}}$

$$x_{(n+r)'} = [(b_r - \frac{b_2}{a_{23}} a_{r3}) - (a_{r3} - \frac{a_{23}}{a_{21}} a_{r1})] (\frac{b_2}{a_{21}} / \frac{a_{23}}{a_{21}}) = b_r - \frac{b_2}{a_{23}} a_{r3}, \quad r \neq 2.$$

It is easily seen that the new basic set and its basic solution are exactly equal to $\beta_{k'}$ and $b^1_{k'}$ respectively.

Hence, p_k is said to be adjacent to $p_{k'}$.

(2) **Corollary 1.** Let the two points p_k and $p_{k'}$ of the system (6) can be created from p_0 by the number of two paths (h=2 and h′=2) or more than two with the following properties:

(1) $p_{k'}$ is a subsequent point of p_k when $k' > k$.

(2) there is only one element of the set β_k different from the set $\beta_{k'}$.

(3) there is only one element of the set α_k in N_k different from the set $\alpha_{k'}$ in $N_{k'}$ but the set H is equal to H′ where H contains only two elements or more than two and H′ too.

Then p_k is said to be adjacent to the subsequent point $p_{k'}$.

(3) Theorem 2. Let the point p_k be created from p_0 by the number of h=2 paths and the point $p_{k'}$ by h′=1 path with the following properties:

(1) p_k is a subsequent point of $p_{k'}$ when $k > k'$.

(2) there is only one element of the set β_k different from the set $\beta_{k'}$.

(3) the set α_k is a subset of $\alpha_{k'}$ denoted by $\alpha_k \subset \alpha_{k'}$ but $\alpha_{k'} - \alpha_k$ contains only one element belonging to $\alpha_{k'}$ and the set H′ is a subset of H denoted by $H' \subset H$ but $H - H'$ contains only one element belonging to H.

Then $p_{k'}$ is said to be adjacent subsequent point p_k.

Proof. Suppose p_k is created from p_0 by the first path

$$r_k^1 = 2,\ q_k^1 = 1 \text{ and } s_k^1 = (n+1)'$$

and the second path

$$r_k^2 = 3,\ q_k^2 = 2 \text{ and } s_k^2 = (n+2)'.$$

By using the simplex method, the basic solution b_k^2, the basic set β_k, the non-basic set N_k and the column vectors A_j^2, $x_{(n+1)'}^2$ and $x_{(n+2)'}^2$ can be readily computed respectively as follows:

$$b_k^2 = (x_2, x_3, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'})^T$$

where $x_2 = \dfrac{b_1 a_{23} - b_2 a_{13}}{a_{12}a_{23} - a_{22}a_{13}}$, $x_3 = \dfrac{b_2 a_{12} - b_1 a_{22}}{a_{12}a_{23} - a_{22}a_{13}}$,

$$x_{(n+r)'} = b_r - \frac{a_{r2}a_{23} - a_{r2}a_{22}}{a_{12}a_{23} - a_{22}a_{13}} b_1 - \frac{a_{r3}a_{12} - a_{r2}a_{13}}{a_{12}a_{23} - a_{22}a_{13}} b_2,$$

for $r \in M$ but $r \neq 1, 2$,

$\beta_k = \{2, 3, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\}$,

$N_k = \alpha_k \cup H$ where $\alpha_k = \{1, 4, \ldots, n, n+1, n+2, n+3, \ldots, n+r, \ldots, n+m\}$

and $H = \{(n+1)', (n+2)'\}$,

the components of A_j^2 are $\dfrac{a_{1j}a_{23} - a_{2j}a_{13}}{a_{12}a_{23} - a_{22}a_{13}}$, $\dfrac{a_{2j}a_{12} - a_{1j}a_{22}}{a_{12}a_{23} - a_{22}a_{13}}$,

and $a_{rj} - \dfrac{a_{r2}a_{23} - a_{r3}a_{22}}{a_{12}a_{23} - a_{22}a_{13}} a_{1j} - \dfrac{a_{r3}a_{12} - a_{r2}a_{13}}{a_{12}a_{23} - a_{22}a_{13}} a_{2j}$,

$j \neq 2, 3$ and $j \in N_{n+m}$ (according to N_5)

the components of $x^2_{(n+1)'}$ are $\frac{a_{23}}{a_{12}a_{23}-a_{22}a_{13}}$, $\frac{-a_{12}}{a_{12}a_{23}-a_{22}a_{13}}$,

and $-\frac{a_{r2}a_{23}-a_{r3}a_{22}}{a_{12}a_{23}-a_{22}a_{13}}$,

and the components of $x^2_{(n+2)'}$ are $\frac{-a_{13}}{a_{12}a_{23}-a_{22}a_{13}}$, $\frac{a_{12}}{a_{12}a_{23}-a_{22}a_{13}}$

and $-\frac{a_{r3}a_{12}-a_{r2}a_{13}}{a_{12}a_{23}-a_{22}a_{13}}$ for $r \in M$ but $r \neq 1, 2$.

Suppose $p_{k'}$ is created by $h'=1$ path as theorem 1.

By using the properties (2) and (3), $\alpha_{k'} - \alpha_k = \{2\}$, $H - H' = \{(n+1)'\}$, it is obvious that the element "2" of the set $\alpha_{k'}$ in $N_{k'}$ is instead of the element "$(n+1)'$" in $\beta_{k'}$ to become basic and by the use of the simplex method, the new basic set $\{2, 3, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\}$ and the basic solution $(x_2, x_3, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'})$ can be immediately obtained

where $x_2 = \frac{b_1 a_{23} - b_2 a_{13}}{a_{12}a_{23}-a_{22}a_{13}}$, $x_3 = \frac{b_2 a_{12} - b_1 a_{22}}{a_{12}a_{23}-a_{22}a_{13}}$,

and $x_{(n+r)'} = b_r - \frac{a_{r2}a_{23}-a_{r3}a_{22}}{a_{12}a_{23}-a_{22}a_{13}} b_1 - \frac{a_{r3}a_{12}-a_{r2}a_{13}}{a_{12}a_{23}-a_{22}a_{13}} b_2$ for $r \in M$ but $r \neq 1, 2$.

It is easily seen that the new basic set and the new basic solution are exactly equal to β_k and b^2_k respectively.

Hence, $p_{k'}$ is said to be adjacent to the subsequent point p_k.

(4) **Corollary 2.** Let the point p_k be created from p_0 by the number of three paths ($h=3$) or more than three and the point $p_{k'}$ by the number of two paths ($h'=2$) or more than two with the following properties:

(1) p_k is a subsequent point of $p_{k'}$ when $k > k'$.

(2) there is only one element of the set β_k different from the set $\beta_{k'}$.

(3) the set α_k is a subset of $\alpha_{k'}$ denoted by $\alpha_k \subset \alpha_{k'}$ but $\alpha_{k'} - \alpha_k$ contains only one element belonging to $\alpha_{k'}$ and the set H' is a subset of H denoted by $H' \subset H$ but $H - H'$ contains only one element belonging to H.

Then $p_{k'}$ is said to be adjacent to the subsequent point p_k.

(5) **Theorem 3.** Let the three points be created from p_0 by a number of paths, whose basic sets have the following properties:

(1) The difference is only in one element between each pair of basic sets.

(2) There is no difference when one basic set contains the element $(n+r)$ while the other contains $(n+r)'$ for $r \in M$.

Then the three points are said to be collinear.

Proof. Let p_k be created from p_0 by the number of $h=2$ paths whose corresponding basic set β_k and the basic solution b_k^2 as in proof of the theorem 2 are shown below respectively:

$$\beta_k = \{2, 3, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\},$$

and $b_k^2 = (x_2, x_3, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'})^T$

where $x_2 = \dfrac{b_1 a_{23} - b_2 a_{13}}{a_{12} a_{23} - a_{22} a_{13}}, \quad x_3 = \dfrac{b_2 a_{12} - b_1 a_{22}}{a_{12} a_{23} - a_{22} a_{13}},$

and $x_{(n+r)'} = b_r - \dfrac{a_{r2} a_{23} - a_{r3} a_{22}}{a_{12} a_{23} - a_{22} a_{13}} b_1 - \dfrac{a_{r3} a_{12} - a_{r2} a_{13}}{a_{12} a_{23} - a_{22} a_{13}} b_2$

for $r \in M$ but $r \neq 1, 2$.

Suppose $p_{k'}$ is created from p_0 by the number of $h=2$ paths, the first path

$$r_{k''}^1 = 3, \quad q_{k'}^1 = 2 \text{ and } s_{k'}^1 = (n+2)'$$

and the second path

$$r_{k'}^2 = n+1, \quad q_{k'}^2 = 1 \text{ and } s_{k'}^2 = (n+1)'.$$

Similarly, its basic set $\beta_{k'}$ and basic solution $b_{k'}^2$ as in the proof of the theorem 2 are shown below respectively:

$$\beta_{k'} = \{(n+1), 3, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\},$$

$$b_{k'}^2 = (x_{n+1}, x_3, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'})^T$$

where $x_{n+1} = \dfrac{b_2 a_{12} - b_1 a_{22}}{a_{23}}, \quad x_3 = \dfrac{b_2}{a_{23}}$

and $x_{(n+r)'} = b_r - \dfrac{b_2}{a_{23}} a_{r2}$, for $r \in M$ but $r \neq 1, 2$.

Suppose $p_{k''}$ is created from p_0 by the number of $h=1$ path

$$r_{k''}^1 = 2, \quad q_{k''}^1 = 2 \text{ and } s_{k''}^1 = (n+2)'$$

Similarly, its basic set $\beta_{k''}$ and basic solution $b^1_{k''}$ as in the proof of the theorem 1 are shown below respectively:

$$\beta_{k''} = \{(n+1)', 2, (n+3)', \ldots, (n+r)', \ldots, (n+m)'\},$$

$$b^1_{k''} = (x_{(n+1)'}, x_2, x_{(n+3)'}, \ldots, x_{(n+r)'}, \ldots, x_{(n+m)'})^T$$

where $x_2 = \frac{b_2}{a_{22}}$ and $x_{(n+r)'} = b_r - \frac{b_2}{a_{22}} a_{r2}$ for $r \in M$ but $r \neq 2$.

Having the two given properties and the basic solutions b^2_k, $b^2_{k'}$ and $b^1_{k''}$ corresponding to their basic sets β_k, $\beta_{k'}$ and $\beta_{k''}$ respectively, the components forms (or coordinates) of the position vectors (or points) p_k, $p_{k'}$ and $p_{k''}$ in n-space can be obtained as follows:

$p_k = (x_1, x_2, x_3, \ldots, x_n)^T$, where $x_2 = \frac{b_1 a_{23} - b_2 a_{13}}{a_{12} a_{23} - a_{22} a_{13}}$,

$x_3 = \frac{b_2 a_{12} - b_1 a_{22}}{a_{12} a_{23} - a_{22} a_{13}}$ and the others are zero,

$p_{k'} = (x_1, x_2, x_3, \ldots, x_n)^T$, where $x_3 = \frac{b_2}{a_{23}}$ and the others are zero,

and $p_{k''} = (x_1, x_2, x_3, \ldots, x_n)^T$, where $x_2 = \frac{b_2}{a_{22}}$ and the others are zero.

The terminals of these three distinct position vectors p_k, $p_{k'}$ and $p_{k''}$ with a common origin (from p_0) in n-space are collinear, then there exists a real number t such that $0 < t < 1$ [1. P. 176] and

$$p_k = t p_{k'} + (1 - t) p_{k''}.$$

Hence, we have

$$(0, \frac{b_1 a_{23} - b_2 a_{13}}{a_{12} a_{23} - a_{22} a_{13}}, \frac{b_2 a_{12} - b_1 a_{22}}{a_{12} a_{23} - a_{22} a_{13}}, \ldots, 0)^T$$

$$= t(0, 0, \frac{b_2}{a_{23}}, \ldots,)^T + (1-t)(0, \frac{b_2}{a_{22}}, \ldots, 0)^T$$

or
$$\frac{b_1 a_{23} - b_2 a_{13}}{a_{12} a_{23} - a_{22} a_{13}} = t(0) + (1-t)\frac{b_2}{a_{22}}$$

and
$$\frac{b_2 a_{12} - b_1 a_{22}}{a_{12} a_{23} - a_{22} a_{13}} = t\frac{b_2}{a_{23}} + (1-t)(0)$$

Simplifying, we find

$$t = \frac{a_{23}(b_2 a_{12} - b_1 a_{22})}{b_2(a_{12}a_{23} - a_{22}a_{13})} .$$

Hence, these three points p_k, $p_{k'}$ and $p_{k''}$ are said to be collinear.

4. Algorithmic Steps

In the simplex method for computing the basic solutions, the iterative steps consists of selecting an element from the non-basic set to enter the basic set to become basic and an element from the basic set to leave the basic set to become non-basic, hence, the following iterative steps can be designed:

Step 1. Initially, the original point p_0 can be readily determined by its initial basic set β_0 and non-basic set N_0 from the given problem.

Step 2. By using the simplex method, each possible element of the ranking in ascending order of the non-basic set N_0 at p_0 must enter the initial basic set β_0 to be created by a number of points p_k, where k is ranking the natural numbers and p_0 and p_k are joined by a solid line to span the unbound convex polytope.

Step 3. Whether p_k is adjacent to the subsequent point depends upon the condition that the theorems and corollaries in section 3 can be used. If it is, the two adjacent points are joined by a broken line and if not, go to the next step.

Step 4. By using the simplex method, each possible element (except adjacency in step 3) ranking in ascending order of the non-basic set at p_k must enter its basic set and the new point (or points) can be immediately created by the second path and they are joined by a solid line to span the unbounded convex polytope.

After these steps, the new point (or points) can be created by the third path, fourth path until m-th path (m is equal to the number of linear inequalities (1)) and by the use of iterative step 3 and step 4. Hence, by the use of the definition D_2, this unbounded convex polytope can be easily constructed by all the extreme points.

5. Numerical Example

Given the following system of linear inequalities (constraints), find all the basic feasible solutions to span an infinite feasible polyhedral convex set (unbounded convex polytope).

$$\begin{aligned}
2x_1 + 4x_2 + 5x_3 + x_4 &\geqslant 10 \\
3x_1 + x_2 + 7x_3 + 2x_4 &\geqslant 2 \\
5x_1 + 2x_2 + x_3 + 6x_4 &\geqslant 15 \\
x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ x_4 &\geqslant 0,
\end{aligned} \tag{8}$$

where m = 3 (number of linear inequalities), the set $M = \{1, 2, 3\}$, and n = 4 (number of original variables).

The non-negative surplus variables, $x_{n+i} \geqslant 0$, $i \in M$ can be used to subtract from their corresponding linear inequalities (8) respectively, we have the following linear equalities:

$$\begin{aligned}
2x_1 + 4x_2 + 5x_3 + x_4 - x_5 \qquad\qquad &= 10 \\
3x_1 + x_2 + 7x_3 + 2x_4 \qquad - x_6 \qquad &= 2 \\
5x_1 + 2x_2 + x_3 + 6x_4 \qquad\qquad - x_7 &= 15 \\
x_j \geqslant 0,\ j \in N_{n+m}(N_0) = \{1, 2, 3, 4, 5, 6, 7\}&.
\end{aligned} \tag{9}$$

The non-negative artificial variables, $x_{(n+i)'} \geqslant 0$, $i \in M$, can be used to add to their corresponding linear equalities (9), we have the following linear equalities:

$$\begin{aligned}
2x_1 + 4x_2 + 5x_3 + x_4 - x_5 \qquad\qquad + x_{5'} \qquad\qquad &= 10 \\
3x_1 + x_2 + 7x_3 + 2x_4 \qquad - x_6 \qquad\qquad + x_{6'} \qquad &= 0 \\
5x_1 + 2x_2 + x_3 + 6x_4 \qquad\qquad - x_7 \qquad\qquad + x_{7'} &= 15 \\
x_j \geqslant 0,\ j \in N_{n+m} \text{ and } x_{(n+i)'} \geqslant 0,\ i \in M&.
\end{aligned} \tag{10}$$

By using the simplex method, all the basic solutions (feasible) at their corresponding points (extreme points) can be computed by the following stages respectively:

1) Stage 0. $P(0) = 0$, $h(0) = 0$. Initially, the original point p_0 can be immediately determined by $b_0 = (10, 2, 15)^T$, $\beta_0 = \{5', 6', 7'\}$, and $N_0 = \{1, 2, 3, 4, 5, 6, 7\}$ denoted by the following matrix $P(0)$:

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \hline 2 & 4 & 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 10 \\ 3 & 1 & 7 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & 2 \\ 5 & 2 & 1 & 6 & 0 & 0 & -1 & 0 & 0 & 1 & 15 \end{array}$$

Matrix P(0)

Hence, creat the new points p_1, p_2, p_3 and p_4 for $j = 1, 2, 3, 4 \in N_0$ repectively

$p_1: r_1=1,\ q_1=2,\ s_1=6',\ \beta_1=\{5',1,7'\},\ N_1=\alpha_1 \cup H= \{2,3,4,5,6,7\} \cup \{6'\}.$
$p_2: r_2=2,\ q_2=2,\ s_2=6',\ \beta_2=\{5',2,7'\},\ N_2=\alpha_2 \cup H= \{1,3,4,5,6,7\} \cup \{6'\}.$
$p_3: r_3=3,\ q_3=2,\ s_3=6',\ \beta_3=\{5',3,7'\},\ N_3=\alpha_3 \cup H= \{1,2,4,5,6,7\} \cup \{6'\}.$
$p_4: r_4=4,\ q_4=2,\ s_4=6',\ \beta_4=\{5',4,7'\},\ N_4=\alpha_4 \cup H= \{1,2,3,5,6,7\} \cup \{6'\}.$

2) Stage 1. $P(1) = 1$, $h(1) = 1$. We have $b_1^1 = (\frac{26}{3}, \frac{2}{3}, \frac{35}{3})^T$, $\beta_1 = \{5', 1, 7'\}$, $N_1 = \alpha_1 \cup H = \{2,3,4,5,6,7\} \cup \{6'\}$ and A_2^1, A_3^1, A_4^1, A_6^1 and $x_{6'}^1$ denoted by the following matrix P(1):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \hline 0 & \frac{10}{3} & \frac{1}{3} & \frac{-1}{3} & -1 & \frac{2}{3} & 0 & 1 & \frac{-2}{3} & 0 & \frac{26}{3} \\ 1 & \frac{1}{3} & \frac{7}{3} & \frac{2}{3} & 0 & \frac{-1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{-32}{3} & \frac{8}{3} & 0 & \frac{5}{3} & -1 & 0 & \frac{-5}{3} & 1 & \frac{35}{3} \end{array}$$

Matrix P(1)

Adjacency test indicates p_1 is adjacent to p_2, p_3, and p_4 (Theorem 1). Hence, creat a new point p_5 for $j = 6$.

$p_5: r_5 = 6,\ q_5 = 3,\ s_5 = 7',\ \beta_5 = \{5', 1, 6\},\ N_5 = \alpha_5 \cup H = \{2,3,4,5,7\} \cup \{6',7'\}.$

3) Stage 2. $P(2) = 2$, $h(2) = 1$. We have $b_2^1 = (2, 2, 11)^T$, $\beta_2 = \{5', 2, 7'\}$, $N_2 = \alpha_2 \cup H = \{1,3,4,5,6,7\} \cup \{6'\}$ and A_1^1, A_3^1, A_4^1, A_6^1 and $x_{6'}^1$ denoted by the following matrix P(2):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \hline -10 & 0 & -23 & -7 & -1 & 4 & 0 & 1 & -4 & 0 & 2 \\ 3 & 1 & 7 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & 2 \\ -1 & 0 & -13 & 2 & 0 & 2 & -1 & 0 & -2 & 0 & 11 \end{array}$$

Matrix P(2)

Adjacency test indicates p_2 is adjacent to p_3 and p_4 (Theorem 1). Hence, creat a new point p_7 for j = 6.

p_6: $r_6 = 6$, $q_6 = 1$, $s_6 = 5'$, $\beta_6 = \{6, 2, 7'\}$, $N_6 = \alpha_6 \cup H = \{1,3,4,5,7\} \cup \{6', 5'\}$.

4) Stage 3. P(3) = 3, h(3) = 1. We have $b_3^1 = (\frac{60}{7}, \frac{2}{7}, \frac{103}{7})^T$, $\beta_3 = \{5', 3, 7'\}$, $N_3 = \alpha_3 \cup H = \{1,2,4,5,6,7\} \cup \{6'\}$ and A_1^1, A_2^1, A_4^1, A_6^1 and $x_{6'}^1$ denoted by the following matrix P(3):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
\frac{-1}{7} & \frac{23}{7} & 0 & \frac{-3}{7} & -1 & \frac{5}{7} & 0 & 1 & \frac{-5}{7} & 0 & \frac{60}{7} \\
\frac{3}{7} & \frac{1}{7} & 1 & \frac{2}{7} & 0 & \frac{-1}{7} & 0 & 0 & \frac{1}{7} & 0 & \frac{2}{7} \\
\frac{32}{7} & \frac{13}{7} & 0 & \frac{40}{7} & 0 & \frac{1}{7} & -1 & 0 & \frac{-1}{7} & 1 & \frac{103}{7}
\end{array}
$$

Matrix P(3)

Adjacency test indicates p_3 is adjacent to p_4 (Theorem 1). Hence, creat a new point p_7 for j = 6.

p_7: $r_7 = 6$, $q_7 = 1$, $s_7 = 5'$, $\beta_7 = \{6, 3, 7'\}$, $N_7 = \alpha_7 \cup H = \{1,2,4,5,7\} \cup \{6', 5'\}$.

5) Stage 4. P(4) = 4, h(4) = 1. We have $b_4^1 = (9, 1, 9)^T$, $\beta_4 = \{5', 4, 7'\}$, $N_4 = \alpha_4 \cup H = \{1,2,3,5,6,7\} \cup \{6'\}$ and A_1^1, A_2^1, A_3^1, A_6^1 and $x_{6'}^1$ denoted by the following matrix P(4):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
\frac{1}{2} & \frac{7}{2} & \frac{3}{2} & 0 & -1 & \frac{1}{2} & 0 & 1 & \frac{-1}{2} & 0 & 9 \\
\frac{3}{2} & \frac{1}{2} & \frac{7}{2} & 1 & 0 & \frac{-1}{2} & 0 & 0 & \frac{1}{2} & 0 & 1 \\
-4 & -1 & -20 & 0 & 0 & 3 & -1 & 0 & -3 & 1 & 9
\end{array}
$$

Matrix P(4)

Adjacency test indicates p_4 is not adjacent to its subsequent points. Hence, creat a new point p_8 for j = 6.

p_8: $r_8 = 6$, $q_8 = 3$, $s_8 = 7'$, $\beta_8 = \{5', 4, 6\}$, $N_8 = \alpha_8 \cup H = \{1,2,3,5,7\} \cup \{6', 7'\}$.

6) Stage 5. P(5) = 5, h(5) = 2. We have $h_5^2 = (4, 3, 7)^T$, $\beta_5 = \{5', 1, 6\}$, $N_5 = \alpha_5 \cup H = \{2,3,4,6,7\} \cup \{6', 7'\}$ and A_2^2, A_3^2, A_4^2, A_7^2 and $x_{7'}^2$ denoted by the following matrix P(5):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
0 & \frac{16}{5} & \frac{23}{5} & \frac{-7}{5} & -1 & 0 & \frac{2}{5} & 1 & 0 & \frac{-2}{5} & 4 \\
1 & \frac{2}{5} & \frac{1}{5} & \frac{6}{5} & 0 & 0 & \frac{-1}{5} & 0 & 0 & \frac{1}{5} & 3 \\
0 & \frac{1}{5} & \frac{-32}{5} & \frac{8}{5} & 0 & 1 & \frac{-3}{5} & 0 & -1 & \frac{3}{5} & 7
\end{array}
$$

Matrix P(5)

Adjacency test indicate p_5 is adjacent to p_8 (corollary 1). Hence, creat the new points p_9, p_{10}, and p_{11} for $j = 2, 3, 7$ respectively.

$p_9 : r_9 = 2,\ q_9 = 1,\ s_9 = 5',\ \beta_9 = \{2,1,6\},\ N_9 = \alpha_9 \cup H = \{3,4,5,7\} \cup \{6',7',5'\}.$
$p_{10} : r_{10} = 3,\ q_{10} = 1,\ s_{10} = 5',\ \beta_{10} = \{3,1,6\},\ N_{10} = \alpha_{10} \cup H = \{2,4,5,7\} \cup \{6',7',5'\}.$
$p_{11} : r_{11} = 7,\ q_{11} = 1,\ s_{11} = 5',\ \beta_{11} = \{7,1,6\},\ N_{11} = \alpha_{11} \cup H = \{2,3,4,5\} \cup \{6',7',5'\}.$

7) Stage 6. $P(6) = 6$, $h(6) = 2$. We have $b_6^2 = (\frac{2}{4}, \frac{10}{4}, \frac{40}{4})^T$, $\beta_6 = \{6, 2, 7'\}$, $N_6 = \alpha_6 \cup H = \{1,3,4,5,7\} \cup \{6', 5'\}$ and A_1^2, A_3^2, A_4^2, A_5^2 and $x_{5'}^2$ denoted by the following matrix P(6):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
\frac{-10}{4} & 0 & \frac{-23}{4} & \frac{-7}{4} & \frac{-1}{4} & 1 & 0 & \frac{1}{4} & -1 & 0 & \frac{2}{4} \\
\frac{2}{4} & 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{-1}{4} & 0 & 0 & \frac{10}{4} \\
\frac{16}{4} & 0 & \frac{-6}{4} & \frac{22}{4} & \frac{2}{4} & 0 & -1 & \frac{-2}{4} & 0 & 1 & \frac{40}{4}
\end{array}
$$

Matrix P(6)

Adjacency test indicates p_6 is adjacent to p_7 and p_9 (corollary 1, 2). Hence, creat the new points p_{12} and p_{13} for $j = 4, 5$ respectively:

$p_{12} : r_{12} = 4,\ q_{12} = 3,\ s_{12} = 7',\ \beta_{12} = \{6,2,4\},\ N_{12} = \alpha_{12} \cup H = \{1,3,5,7\} \cup \{6',5',7'\}.$
$p_{13} : r_{13} = 5,\ q_{13} = 3,\ s_{13} = 7',\ \beta_{13} = \{6,2,5\},\ N_{13} = \alpha_{13} \cup H = \{1,3,4,7\} \cup \{6',5',7'\}.$

8) Stage 7. $P(7) = 7$, $h(7) = 2$. We have $b_7^2 = (12, 2, 13)^T$, $\beta_7 = \{6, 3, 7'\}$, $N_7 = \alpha_7 \cup H = \{1, 2, 4, 5, 7\} \cup \{6', 5'\}$ and A_1^2, A_2^2, A_4^2, A_5^2 and $x_{5'}^2$ denoted by the following matrix P(7):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
\frac{-1}{5} & \frac{23}{5} & 0 & \frac{-3}{5} & \frac{-7}{5} & 0 & 0 & \frac{7}{5} & -1 & 0 & 12 \\
\frac{2}{5} & \frac{4}{5} & 1 & \frac{1}{5} & \frac{-1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & 2
\end{array}
$$

$$\left. \begin{array}{cccc|ccc|ccc|c} \frac{23}{5} & \frac{6}{5} & 0 & \frac{29}{5} & \frac{1}{5} & 0 & -1 & \frac{-1}{5} & 0 & 1 & 13 \end{array} \right]$$

Matrix P(7)

Adjacency test indicates p_7 is adjacent to p_{10} (corollary 2). Hence, creat the new points p_{14} and p_{15} for j = 4,5 respectively.

p_{14}: $r_{14}=4$, $q_{14}=3$, $s_{14}=7'$, $\beta_{14}=\{6,3,4\}$, $N_{14}=\alpha_{14}\cup H=\{1,2,5,7\}\cup\{6',5',7'\}$.
p_{15}: $r_{15}=5$, $q_{15}=3$, $s_{15}=7'$, $\beta_{15}=\{6,3,5\}$, $N_{15}=\alpha_{15}\cup H=\{1,2,4,7\}\cup\{6',5',7'\}$.

9) Stage 8. P(8) = 8, h(8) = 2. We have $b_8^2=(\frac{45}{6},\frac{15}{6},\frac{18}{6})^T$, $\beta_8=\{5',4,6\}$, $N_8=\alpha_8\cup H=\{1,2,3,5,7\}\cup\{6',7'\}$ and A_1^2, A_2^2, A_3^2, A_7^2 and $x_{7'}^2$ denoted by the following matrix P(8):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \frac{7}{6} & \frac{22}{6} & \frac{29}{6} & 0 & -1 & 0 & \frac{1}{6} & 1 & 0 & \frac{-1}{6} & \frac{46}{6} \\ \frac{5}{6} & \frac{2}{6} & \frac{1}{6} & 1 & 0 & 0 & \frac{-1}{6} & 0 & 0 & \frac{1}{6} & \frac{15}{6} \\ \frac{-8}{6} & \frac{-7}{6} & \frac{-40}{6} & 0 & 0 & 1 & \frac{-2}{6} & 0 & -1 & \frac{2}{6} & \frac{18}{6} \end{array}$$

Matrix P(8)

Adjacency test indicates p_8 is adjacent to p_{12} and p_{14} (corollary 2). Hence creat a new point p_{16} for j = 7.

p_{16}: $r_{16}=7$, $q_{16}=1$, $s_{16}=5'$, $\beta_{16}=\{7,4,6\}$, $N_{16}=\alpha_{16}\cup H=\{1,2,3,5\}\cup\{6',7',5'\}$.

10) Stage 9. P(9) = 9, h(9) = 3. We have $b_9^3=(\frac{20}{16},\frac{40}{16},\frac{108}{16})^T$, $\beta_9=\{2,1,6\}$, $N_9=\alpha_9\cup H=\{3,4,5,7\}\cup\{6',7',5'\}$ and A_3^3, A_4^3, A_5^3, A_7^3, $x_{5'}^3$ and $x_{7'}^3$ denoted by the following matrix P(9):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ 0 & 1 & \frac{23}{16} & \frac{-7}{16} & \frac{-5}{16} & 0 & \frac{2}{16} & \frac{5}{16} & 0 & \frac{-2}{16} & \frac{20}{16} \\ 1 & 0 & \frac{-6}{16} & \frac{22}{16} & \frac{2}{16} & 0 & \frac{-4}{16} & \frac{-2}{16} & 0 & \frac{4}{16} & \frac{40}{16} \\ 0 & 0 & \frac{-107}{16} & \frac{27}{16} & \frac{1}{16} & 1 & \frac{-10}{16} & \frac{-1}{16} & -1 & \frac{10}{16} & \frac{108}{16} \end{array}$$

Matrix P(9)

Adjacency test indicates p_9 is adjacent to p_{10}, p_{11}, p_{12}, and p_{13} (corollary 1). Hence, no point can be created.

11) Stage 10. $P(10) = 10, h(10) = 3$. We have $b_{10}^3 = (\frac{22}{23}, \frac{65}{23}, \frac{289}{23})^T$, $\beta_{10} = \{3, 1, 6\}$, $N_{10} = \alpha_{10} \cup H = \{2, 4, 5, 7\} \cup \{6', 7', 5'\}$ and A_2^3, A_4^3, A_5^3, A_7^3, $x_{5'}^3$ and $x_{7'}^3$ denoted by the following matrix P(10):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \hline 0 & \frac{11}{23} & 1 & \frac{-7}{23} & \frac{-5}{23} & 0 & \frac{2}{23} & \frac{-5}{23} & 0 & \frac{-2}{23} & \frac{22}{23} \\ 1 & \frac{6}{23} & 0 & \frac{29}{23} & \frac{1}{23} & 0 & \frac{-5}{23} & \frac{-1}{23} & 0 & \frac{5}{23} & \frac{65}{23} \\ 0 & \frac{107}{23} & 0 & \frac{-8}{23} & \frac{-32}{23} & 1 & \frac{-1}{23} & \frac{32}{23} & -1 & \frac{1}{23} & \frac{289}{23} \end{array}$$

Matrix P(10)

Adjacency test indicates p_{10} is adjacent to p_{11}, p_{14} and p_{15} (corollary 2). Hence, no point can be created.

12) Stage 11. $P(11) = 11, h(11) = 3$. We have $b_{11}^3 = (10, 5, 13)^T$, $\beta_{11} = \{7, 1, 6\}$, $N_{11} = \alpha_{11} \cup H = \{2, 3, 4, 5\} \cup \{6', 7', 5'\}$ and A_2^3, A_3^3, A_4^3, A_5^3, and $x_{5'}^3$ denoted by the following matrix P(11):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \hline 0 & \frac{16}{2} & \frac{23}{2} & \frac{-7}{2} & \frac{-5}{2} & 0 & 1 & \frac{5}{2} & 0 & -1 & 10 \\ 1 & \frac{4}{2} & \frac{5}{2} & \frac{1}{2} & \frac{-1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 5 \\ 0 & \frac{5}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{2} & 1 & 0 & \frac{3}{2} & -1 & 0 & 13 \end{array}$$

Matrix P(11)

Adjacency test indicates p_{11} is adjacent to p_{16} (corollary 2). Hence, no point can be created.

13) Stage 12. $P(12) = 12, h(12) = 3$. We have $b_{12}^3 = (\frac{81}{22}, \frac{45}{22}, \frac{40}{22})^T$, $\beta_{12} = \{6, 2, 4\}$, $N_{12} = \alpha_{12} \cup H = \{1, 3, 5, 7\} \cup \{6', 5', 7'\}$ and A_1^3, A_3^3, A_5^3, A_7^3, $x_{5'}^3$ and $x_{7'}^3$ denoted by the following matrix P(12):

$$\begin{array}{cccc|ccc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\ \hline \frac{-27}{22} & 0 & \frac{-137}{22} & 0 & \frac{-2}{22} & 1 & \frac{-7}{22} & \frac{2}{22} & -1 & \frac{7}{22} & \frac{81}{22} \\ \frac{2}{22} & 1 & \frac{29}{22} & 0 & \frac{-6}{22} & 0 & \frac{1}{22} & \frac{6}{22} & 0 & \frac{-1}{22} & \frac{45}{22} \\ \frac{16}{22} & 0 & \frac{-6}{22} & 1 & \frac{2}{22} & 0 & \frac{-4}{22} & \frac{-2}{22} & 0 & \frac{4}{22} & \frac{40}{22} \end{array}$$

Matrix P(12)

Adjacency test indicates p_{12} is adjacent to p_{13}, p_{14}, and p_{16} (corollary 2). Hence, no point can be created.

14) Stage 13. P(13) = 13, h(13) = 3. We have $b_{13}^3 = (\frac{11}{2}, \frac{15}{2}, 20)^T$, $\beta_{13} = \{6, 2, 5\}$, $N_{13} = \alpha_{13} \cup H = \{1, 3, 4, 7\} \cup \{6', 5', 7'\}$ and A_1^3, A_3^3, A_4^3, A_7^3 and $x_{7'}^3$ denoted by the following matrix P(13):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
\frac{-1}{2} & 0 & \frac{-13}{2} & \frac{2}{2} & 0 & 1 & \frac{-1}{2} & 0 & -1 & \frac{1}{2} & \frac{11}{2} \\
\frac{5}{2} & 1 & \frac{1}{2} & \frac{6}{2} & 0 & 0 & \frac{-1}{2} & 0 & 0 & \frac{1}{2} & \frac{15}{2} \\
8 & 0 & -3 & 11 & 1 & 0 & -2 & -1 & 0 & 2 & 20
\end{array}
$$

Matrix P(13)

Adjacency test indicates p_{13} is adjacent to p_{15} (corollary 1). Hence, no point can be created.

15) Stage 14. P(14) = 14, h(14) = 3. We have $b_{14}^3 = (\frac{387}{29}, \frac{45}{29}, \frac{65}{29})^T$, $\beta_{14} = \{6, 3, 4\}$, $N_{14} = \alpha_{14} \cup H = \{1, 2, 5, 7\} \cup \{6', 5', 7'\}$ and A_1^3, A_2^3, A_5^3, A_7^3, $x_{5'}^3$ and $x_{7'}^3$ denoted by the following matrix P(14):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
\frac{8}{29} & \frac{137}{29} & 0 & 0 & \frac{-40}{29} & 1 & \frac{-3}{29} & \frac{40}{29} & -1 & \frac{3}{29} & \frac{387}{29} \\
\frac{7}{29} & \frac{22}{29} & 1 & 0 & \frac{-60}{29} & 0 & \frac{1}{29} & \frac{60}{29} & 0 & \frac{-1}{29} & \frac{45}{29} \\
\frac{23}{29} & \frac{6}{29} & 0 & 1 & \frac{1}{29} & 0 & \frac{-5}{29} & \frac{-1}{29} & 0 & \frac{5}{29} & \frac{65}{29}
\end{array}
$$

Matrix P(14)

Adjacency test indicates p_{14} is adjacent to p_{15} and p_{16} (corollary 2). Hence, no point can be created.

16) Stage 15. P(15) = 15, h(15) = 3. We have $b_{15}^3 = (103, 15, 65)^T$, $\beta_{15} = \{6, 3, 5\}$, $N_{15} = \alpha_{15} \cup H = \{1, 2, 4, 7\} \cup \{6', 5', 7'\}$ and A_1^3, A_2^3, A_4^3, A_7^3 and $x_{7'}^3$ denoted by the following matrix P(15):

$$
\begin{array}{cccc|ccc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{5'} & x_{6'} & x_{7'} & \\
32 & 13 & 0 & 40 & 0 & 1 & -7 & 0 & -1 & 7 & 103 \\
5 & 2 & 1 & 6 & 0 & 0 & -1 & 0 & 0 & 1 & 15
\end{array}
$$

23	6	0	29	1	0	-5	-1	0	5	65

Matrix P(15)

Adjacency test indicates p_{15} is not adjacent to its subsequent point. Hence, no point can be created.

17) Stage 16. P(16) = 16, h(16) = 3. We have $b_{16}^3 = (45, 10, 15)^T$, $\beta_{16} = \{7, 4, 6\}$, $N_{16} = \alpha_{16} \cup H = \{1, 2, 3, 5\} \cup \{6', 7', 5'\}$ and A_1^3, A_2^3, A_3^3, A_5^3 and $x_{5'}^3$ denoted by the following matrix P(16):

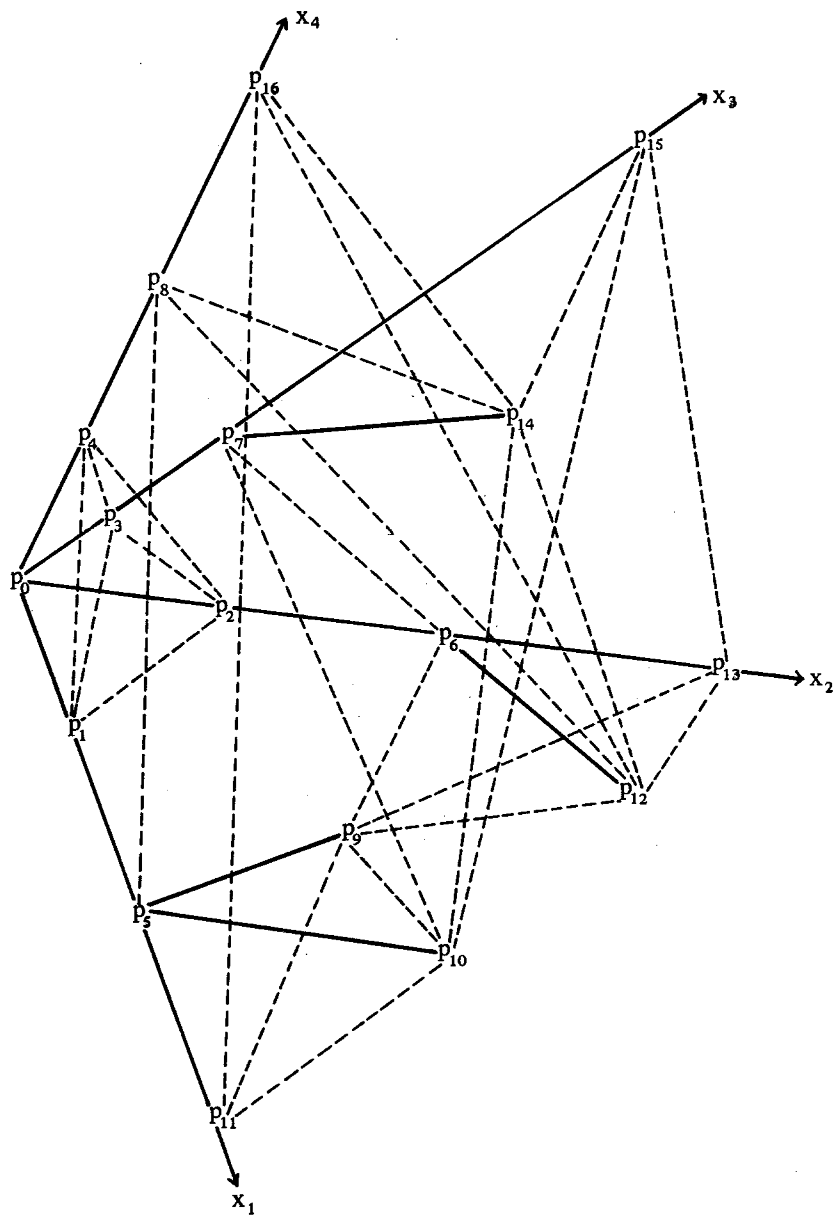
x_1	x_2	x_3	x_4	x_5	x_6	x_7	$x_{5'}$	$x_{6'}$	$x_{7'}$	
7	22	29	0	-6	0	1	6	0	-1	45
2	4	5	1	-1	0	0	1	0	0	10
1	7	3	0	-2	0	0	2	-1	0	15

Matrix P(16)

End of this stage at the point p_{16}.

It is clearly seen that all the extreme points P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15} and P_{16} can be readily obtained by the definition D_2, h(k) = 3, k = 9, 10, 11, 12, 13, 14, 15 and 16 respectively and the basic solutions at their extreme points have to be feasible because they must satisfy all the linear inequalities (1). Hence, the eight extreme points P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15} and P_{16} can span the following unbounded convex polytope (geometric graph) whose each one of these extreme points must be intersected the 4-space (n=4) lines including solid and broken in 4-space such that the extreme points P_9, P_{10}, P_{12} and P_{14} can be shown by the use of the theorem 3 as follows:

P_6, P_9, P_{11} are collinear at the extreme point P_9 and P_5, P_9, P_{13} are also.
P_7, P_{10}, P_{11} are collinear at the extreme point P_{10} and P_5, P_{10}, P_{15} are also.
P_8, P_{12}, P_{13} are collinear at the extreme point P_{12} and P_6, P_{12}, P_{16} are also.
P_9, P_{14}, P_{16} are collinear at the extreme point P_{14} and P_8, P_{14}, P_{15} are also.



Unbounded Convex Polytope

6. Remarks

1) It is clearly seen that all the extreme points to span an unbounded convex polytope of geometric graph are simply and easily determined by the use of the theorems and corollaries in section 3.
2) Obviously, the number of original variables are grater than three $n > 3$, it is difficult to find the geometric graph and hence, this method for finding all the basic feasible solutions at their corresponding extreme points is better than corner point method [8. P.8].
3) This method can not be used to compute the optimal solution for the linear programming problem (minimize objective linear function) although its optimal solution must occur at the extreme points of the linear constraints (or linear inequalities of the system (1)) to span an unbounded convex polytope [1. P.178]. Obviously, it does not need all the extreme points and therefore the following example will illustrate how to compute its optimal solution.

   Example. Minimize $f(x_1, x_2, x_3, x_4) = 3x_1 + 2x_2 + x_3 + 4x_4$, subject to the linear constraints (8) as shown in the numerical example in section 5.

   By the use of either big-M method or two-phase method, its optimal (minimum) solution can be readily computed by using only seven Matrices P(3), P(4), P(2), P(6), P(12), P(14) and P(10) respectively in section 5, and hence, we can obtain its optimal (minimum) solution of $f(x_1, x_2, x_3, x_4)$ equal to $\frac{215}{23}$ at the extreme point P_{10} whose basic feasible solution is $x_1 = \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$ and $x_4 = 0$.

7. References

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