A REVISED METHOD FOR DETERMINING ALL EXTREME POINTS TO SPAN A CONVEX POLYTOPE UNDER LINEAR INEQUALITIES

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摘 要

線性不等式$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i$， i＝1，2，3，……，m 及 $x_j \geqslant 0$，j＝1，2，3，……，n。求其可行基解（feasi blebasic solution）或非負數解（non-negative solution），此線性不等式可構成一幾何圖形——凸多面體（convex polytope）而其可行基解必在凸多面體頂點上（vertex）或極點（extreme point）上，當變數超過三個即n大於3時無幾何圖形解，本文應用基集合（basic set）及非基集合（non-basic set）為元素（element）對換而根據單純法（Simplex method）計算出全部可行基解而得到所有極點以構成一凸多面體表示其幾何圖形解。

1. Introduction

Consider a system of the linear inqualities in the following general forms:

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \text{ for } i = 1, 2, 3, \ldots m, \tag{1}$$

where $b_i \geqslant 0$ and $x_j \geqslant 0$ for all i and j.

The system (1) can be written by the following matrix form

$$AX \leqslant b, \tag{2}$$

where $A = [a_{ij}]$ is an m by n matrix, $X = (x_1, x_2, x_3, \ldots x_n)^T$ in n-space and $b = (b_1, b_2, b_3, \ldots b_m)^T$ in m-space are non-negative column vectors whose all the components must be non-negative.

In geometry, each one of the system (1) represents a closed hålf hyperplane [1. p165] and the intersection of all the closed half hyperplane together with $x_j \geqslant 0$, for all j, is to span a bounded feasible convex set — convex polytope [1. p172] whose extreme points (vertices) are the basic feasible solutions in algebra. The system (1) of computing all the basic feasible solutions at their corresponding extreme points of the convex polytope has been discussed by the authors, M. E. Dyer and

L. G. Proll [2. p81], M. J. L. Kirby, H. R. Love, and Kanti Swarup [3. p540], T. H. Mattheiss [4. p247] and M. C. Cheng [5. p230, 6. p6].

The non-negative slack variables x_{n+i}, $i = 1, 2, 3 \ldots m$ can be used to add to their corresponding inequalities (1) to obtain the following of linear equalities:

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \tag{3}$$

$x_j \geqslant 0$ and $x_{n+i} \geqslant 0$, for all i and j.

The system (3) can be written by the following matrix form

$$(A + I)X = b, \tag{4}$$

where I is an m-square identity matrix and $X = (x_1, x_2, x_3, \ldots x_n, x_{n+1}, x_{n+2}, \ldots x_{n+m})^T$.

From the system (3), it is obvious that the non-negative variables, $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ are basic and the original variables, $x_1, x_2, x_3, \ldots x_n$ are non-basic and thus, the initial extreme point P_0 can be easily determined by the initial basic feasible solution $b_0 - (b_1, b_2, b_3 \ldots b_n)^T$ and all the non-basic variables must be zero. Then the subsequent extreme points P_k, where k is ranking ascending natural numbers, can be created by the theorems and corollaries in section 3 and the simplex method shown below:

The following simplex matrix can be readily obtained by the matrix form (4)

$$\begin{array}{cccccc|cccccc|c}
x_1 & x_2 & x_3 & \ldots x_k & \ldots x_n & & x_{n+1} & x_{n+2} & x_{n+3} & \ldots x_{n+r} & \ldots x_{n+m} & & b \\
\hline
a_{11} & a_{12} & a_{13} & \ldots a_{1k} & \ldots a_{1n} & & 1 & 0 & 0 & \ldots 0 & \ldots 0 & & b_1 \\
a_{21} & a_{22} & a_{23} & \ldots a_{2k} & \ldots a_{2n} & & 0 & 1 & 0 & \ldots 0 & \ldots 0 & & b_2 \\
a_{31} & a_{32} & a_{33} & \ldots a_{3k} & \ldots a_{3n} & & 0 & 0 & 1 & \ldots 0 & \ldots 0 & & b_3 \\
\cdots & & & & & & \cdots & & & & & & \\
a_{r1} & a_{r2} & a_{r3} & \ldots a_{rk} & \ldots a_{rn} & & 0 & 0 & 0 & \ldots 1 & \ldots 0 & & b_r \\
\cdots & & & & & & \cdots & & & & & & \\
a_{m1} & a_{m2} & a_{m3} & \ldots a_{mk} & \ldots a_{mn} & & 0 & 0 & 0 & \ldots 0 & \ldots 1 & & b_m
\end{array} \tag{5}$$

Each one of non-basic x_j, for all j, can become basic instead of one of the basic x_{n+i}, for all i. Suppose the kth column (x_k become basic) $k \in \{1, 2, 3, \ldots n\}$ is a pivot column, the rth row (x_{n+r} become non-basic), $r \in \{1, 2, 3, \ldots m\}$ is a pivot row and a_{rk} must be greater than zero (or denoted by $r_k = k$, $q_k = r$ and $s_k = n+r$). Hence, the positive ratio

$$\frac{b_r}{a_{rk}} \leqslant \frac{b_i}{a_{ik}}, \; i \neq r, \; i \in M,$$

must be minimum. By use of the elementary row operations [1. p126], the non-basic column vector A_j^1, $j \neq k$ and X_{n+r}^1, $r \neq i$ and the basic feasible solution b^1 can be readily obtained as follows:

The components of A_j^1 are $a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik}$ and $\frac{a_{ij}}{a_{rk}}$,.

the components of X_{n+r}^1 are $-\frac{a_{ik}}{a_{rk}}$ and $\frac{1}{a_{rk}}$,

and $b^1 = (X_{n+1}, x_{n+2}, x_{n+3}, \ldots x_k \ldots x_{n+m})^T$, where $x_k = \frac{b_r}{a_{rk}}$ and

$x_{n+i} = b_i - \frac{b_r}{a_{rk}} a_{ik}$, for all i and j except $i \neq r$ and $j \neq k$,

because b^1 is feasible, $x_{n+i} \geqslant 0$ for all i and j imply $\frac{b_i}{a_{ik}} \geqslant \frac{b_r}{a_{rk}}$ (minimum) [7. p81].

Hence, it is immediatly seen that all the extreme points to span this polytope can be obtained by the iterative simplex method to-gether with theorems and corollaries in section 3.

2. Definitions and Notations

The following useful definitions and notations are throughtout this research:

D_1. Convex polytope. An intersection of a finite number of linear constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, i = 1, 2, 3, \ldots m,$$

is called a convex polytope [1. p172] in n-space in the geometric representation.

D_2. Extreme points. A point is an extreme point (or vertex) of a convex polytope, when it is intersected by n distinct space lines in n-space [1. p172].and also is the corresponding basic feasible solution of the system of linear inequalities (1).

D_3. Subsequent extreme points. P_k and $P_{k'}$ are the two extreme points denoted by two distinct natural numbers k and k′ respectively. $P_{k'}$ is said to be a subsequent extreme point of P_k if $k' > k$.

D_4. Adjacency. The extreme point P_k is said to be adjacent to a subsequent extreme point $P_{k'}$ if a certain element in the non-basic set N_k at P_k enters the basic set β_k to obtain a new basic set equal to the basic set $\beta_{k'}$ at $P_{k'}$ and their basic solutions are also equal.

D_5. Path. A new extreme point is created from the predecessor, called the number of one path because only one element in the non-basic set of the predecessor is instead of one element in its basic set to obtain a new basic set at the new extreme point but computing the number of h paths (or h elements in N_0 entering the β_0) to create a new extreme point must be from the initial extreme point P_0.

N_1. $A = [a_{ij}]$ is an m by n matrix with real entries where $i \in M = \{1, 2, 3, \ldots m\}$ and $j \in N = \{1, 2, 3, \ldots n\}$ denoted by the number of rows (or equations) and the number of columns (or original variables) respectively.

N_2. $A_j = (a_{1j}, a_{2j}, a_{3j}, \ldots a_{mj})^T$ indicates the jth column vector of A and its components are the coefficients of x_j.

N_3. The "T" indicates the transpose of a matrix or a vector.

N_4. P_0 indicates the initial (or original) extreme point of a polytope in n-space.

N_5. $b_0 = (b_1, b_2, b_3, \ldots b_m)^T$ indicates the basic feasible solution at P_0.

N_6. $\beta_0 = \{n+1, n+2, n+3, \ldots n+m\}$ indicates the initial basic set corresponding to the initial basic variables $= (x_{n+1}, x_{n+2}, x_{n+3}, \ldots x_{n+m})$ at P_0.

N_7. N_0 (or N) = 1, 2, 3, . . . n indicates the initial non-basic set corresponding to the initial (or original) non-basic variables $= (x_1, x_2, x_3, \ldots x_n)$ at P_0.

N_8. P_k (or $P_{k'}$) indicates the ranking of an extreme point to span a polytope by the number of h (or h′) paths from P_0, where k (or k′) is ranking the ascending natural numbers.

N_9. b_k^h (or $b_k^{h'}$) indicates the basic feasible solution at P_k (or $P_{k'}$).

N_{10}. β_k (or $\beta_{k'}$), containing m elements, indicates the basic set corresponding b_k^h (or $b_k^{h'}$) at P_k (or $P_{k'}$).

N_{11}. $N_k = \alpha_k \cup H$ (or $N_{k'} = \alpha_{k'} \cup H'$), containing n elements, indicates the non-basic set at P_k (or $P_{k'}$), where H (or H′) contains the number of h (or h′) elements from β_0 to become the non-basic set and α_k (or $\alpha_{k'}$) contains the remaining the number of n-h (or n-h′) elements because the number of h (or h′) elements in N_0 enters β_k (or $\beta_{k'}$) to become basic and hence $\alpha_k \cap H = \phi$ (empty set) (or $\alpha_{k'} \cap H' = \phi$).

N_{12}. A_j^h (or $A_j^{h'}$) indicates the jth column vector corresponding α_k (or $\alpha_{k'}$) at P_k (or $P_{k'}$).

N_{13}. X_{n+i}^h (or $X_{n+i}^{h'}$) indicates the (n+i)th the non-basic column vector corresponding H (or H′) at P_k (or $P_{k'}$).

N_{14}. r_k indicates the pivot column in the simplex method entering the predecessor basic set of P_k to get β_k.

N_{15}. q_k indicates the pivot row in the simplex method leaving the predecessor basic set of P_k.

N_{16}. s_k indicates the corresponding column vector leaving the predecessor basic set of P_k.

N_{17}. $P(k) = k$ indicates the number of stages at P_k (define $P(0) = 0$).

N_{18}. $h(k) = h$ indicates the total number of solid lines in any path from P_0 to P_k (define $h(0) = 0$).

3. Theoritical Development

By use of the simplex method, whether the new extreme point can be created, if each element of the non-basic set enters the basic set. Firstly, the adjacency test must be used and thus, the following theorems and corollaries must be defined:

(1) **Theorem 1.** Let the two extreme points P_k and $P_{k'}$ of the system (3) can be created by the distinct one path ($h = 1$ and $h' = 1$) from P_0 with the following properties:

1) $P_{k'}$ is a subsequent extreme point of P_k, where $k' > k$.
2) There is only one element of the set β_k different from the set $\beta_{k'}$.
3) There is only one element of the set α_k in N_k different from the set $\alpha_{k'}$ in $N_{k'}$ but the set H is equal to H′ where H contains only one element and H′ too.

Then P_k is said to be adjacent to the subsequent exteme point $P_{k'}$.

Proof. Suppose P_k is created from P_0 by one ($h = 1$) path denoted by

$$r_k = 1, \quad q_k = 2 \text{ and } s_k = n+2.$$

By use of the simplex method, the basic feasible solution b_k^1, the basic set β_k, the non-basic set N_k and the non-basic column vectors A_j^1 and X_{n+r}^1 corresponding N_k can be readily computed as follows:

$$b_k^1 = (x_{n+1}, x_1, x_{n+3}, \ldots x_{n+m}) \text{ where } x_1 = \frac{b_2}{a_{21}},\ x_{n+r} = b_r - \frac{b_2}{a_{21}} a_{r1},$$

for $r \in M$ but $r \neq 2$,

$$\beta_k = \{n+1, 1, n+3, \ldots n+m\},$$

$$N_k = \alpha_k \cup H, \text{ where } \alpha_k = \{2, 3, 4, \ldots n\} \text{ and } H = \{n+2\},$$

the components of A_j^1 are $\frac{a_{2j}}{a_{21}}$ and $a_{1j} - \frac{a_{2j}}{a_{21}} a_{r1}$, for $r \neq 2$, $j \neq 1$, the components of X^1_{n+2} are $\frac{1}{a_{21}}$ and $\frac{-a_{r1}}{a_{21}}$ for $r \neq 2$.

Suppose $P_{k'}$ is created by one ($h' = 1$) path denoted by

$r_{k'} = 3$, $q_{k'} = 2$ and $s_{k'} = n+2$.

Similarly, we have

$b^1_{k'} = (x_{n+1}, x_3, x_{n+3}, \ldots x_{n+m})$ where $x_3 = \frac{b_2}{a_{23}}$, $x_{n+r} = b_r - \frac{b_2}{a_{23}} a_{r3}$, for $r \neq 2$,

$\beta_{k'} = \{n+1, 3, n+3, \ldots n+m\}$,

$N_{k'} = \alpha_{k'} \cup H'$ where $\alpha_{k'} = \{1, 2, 4, \ldots n\}$ and $H' = \{n+2\}$,

the components of $A^{1'}_j$ are $\frac{a_{2j}}{a_{23}}$ and $a_{rj} - \frac{a_{2j}}{a_{23}} a_{r3}$, for $r \neq 2$, $j \neq 3$,

the components of $X^{1'}_{n+2}$ are $\frac{1}{a_{23}}$ and $\frac{-a_{r3}}{a_{23}}$ for $r \neq 2$.

By use of the properties 2 and 3, $H = H'$, it is obvious that the element "3" in α_k is instead of the element "1" in β_k to become basic and by use of the simplex method, the new basic set $\{n+1, 3, n+3, \ldots n+m\}$ and the new basic solution $(x_{n+1}, x_3, x_{n+3}, \ldots x_{n+m})$ can be immediately obtained, where

$$x_3 = \frac{b_2}{a_{21}} \Big/ \frac{b_{23}}{a_{21}} = \frac{b_2}{a_{23}},$$

$$x_{n+r} = [(b_r - \frac{b_2}{a_{23}} a_{r3}) - (a_{r3} - \frac{a_{23}}{a_{21}} a_{r1})(\frac{b_2}{a_{21}} / \frac{a_{23}}{a_{21}})]$$

$$= b_r - \frac{b_2}{a_{23}} a_{r3}, \text{ for } r \neq 2..$$

It is easily seen that the new basic set and the basic feasible solution are exactly equal to $\beta_{k'}$ and $b^1_{k'}$ respectively.

Hence, P_k is said to be adjacent to $P_{k'}$.

(2) **Corollary** 1. Let the two extreme points P_k and $P_{k'}$ of the system (3) be created from P_0 by the number of $h = h' = 1$ paths with the following properties:

1) $P_{k'}$ is a subsequent extreme point of P_k, where $k' > k$.
2) There is only one element of the set β_k different from the set $\beta_{k'}$.
3) There is only one element of the set α_k in N_k different from the set $\alpha_{k'}$ in $N_{k'}$ but the set H is equal to the set H′ where H and H′ contain the same number

of h elements.

Then P_k is said to be adjacent to the subsequent extreme point $P_{k'}$.

(3) **Theorem 2.** Let the two extreme points P_k and $P_{k'}$ of the system (3) be created by the number of $h = h' = 2$ paths with the following properties:

1) $P_{k'}$ is a subsequent extreme point P_k, where $k' > k$.
2) There is only one element of the set β_k different from the set $\beta_{k'}$.
3) There is only one element of the set H in N_k different from the set H′ in $N_{k'}$ but the set α_k is equal to the set $\alpha_{k'}$.

Then P_k is said to be adjacent to the subsequent extreme point $P_{k'}$.

Proof. Suppose P_k is created from P_0 by the first path

$$r_k^1 = 2,\ q_k^1 = 1 \text{ and } s_k^1 = n+1,$$

and the second path

$$r_k^1 = 3,\ q_k^2 = 3 \text{ and } s_k^2 = n+3.$$

By use of the simplex method, the basic feasible solution b_k^2, the basic set β_k, the non-basic set N_k and the non-basic column vectors A_j^2, X_{n+1}^2 and X_{n+3}^2 can be readily computed as follows:

$$b_k^2 = (x_2, x_{n+2}, x_3, x_{n+4}, \dots x_{n+m}),\ \text{where } x_2 = \frac{b_1 a_{33} - b_3 a_{13}}{a_{12} a_{33} - a_{32} a_{13}},$$

$$x_3 = \frac{b_3 a_{12} - b_1 a_{32}}{a_{33} a_{12} - a_{13} a_{32}} \text{ and}$$

$$x_{n+r} = b_r - \frac{a_{r2} a_{33} - a_{32} a_{r3}}{a_{12} a_{33} - a_{32} a_{13}} b_1 - \frac{a_{12} a_{r3} - a_{r2} a_{13}}{a_{12} a_{33} - a_{32} a_{13}} b_3 \text{ for } r \neq 1, 3,$$

$$\beta_k = \{2, n+2, 3, n+4, \dots n+m\},$$

$$N_k = \alpha_k \cup H, \text{ where } \alpha_k = \{1, 4, 5, \dots n\} \text{ and } H = \{n+1, n+3\},$$

the components of A_j^2 are $\dfrac{a_{1j} a_{33} - a_{13} a_{3j}}{a_{12} a_{33} - a_{13} a_{32}}$, $\dfrac{a_{12} a_{3j} - a_{1j} a_{32}}{a_{12} a_{33} - a_{13} a_{32}}$ and

$$a_{rj} - \frac{a_{r2} a_{33} - a_{32} a_{r3}}{a_{12} a_{33} - a_{32} a_{13}} a_{1j} - \frac{a_{12} a_{r2} - a_{r2} a_{13}}{a_{12} a_{33} - a_{32} a_{13}} a_{3j} \text{ for } r \neq 1, 3,\ j \neq 2, 3,$$

the components of x^2_{n+1} are $\frac{a_{33}}{a_{12}a_{33} - a_{13}a_{32}}$, $\frac{-a_{32}}{a_{12}a_{33} - a_{13}a_{32}}$ and

$-\frac{a_{r2}a_{33} - a_{r3}a_{32}}{a_{12}a_{33} - a_{13}a_{32}}$ for $r \neq 1, 3$,

the components of x^2_{n+3} are $\frac{-a_{13}}{a_{12}a_{33} - a_{13}a_{32}}$, $\frac{a_{12}}{a_{12}a_{33} - a_{13}a_{32}}$ and

$-\frac{a_{r3}a_{12} - a_{r2}a_{13}}{a_{12}a_{33} - a_{32}a_{13}}$ for $r \neq 1, 3$.

Suppose $P_{k'}$ is created from P_0 by the first path

$r^1_{k'} = 3$, $q^1_{k'} = 2$ and $s^2_{k'} = n+2$,

and the second path

$r^2_{k'} = 2$, $q^2_{k'} = 3$ and $s^2_{k'} = n+3$.

Similarly, we have

$b^2_{k'} = (x_{n+1}, x_2, x_3, x_{n+4}, \ldots x_{n+m})$, where $x_2 = \frac{b_3a_{23} - b_2a_{33}}{a_{32}a_{23} - a_{22}a_{33}}$,

$x_3 = \frac{b_2a_{32} - b_3a_{22}}{a_{32}a_{23} - a_{22}a_{33}}$ and

$x_{n+r} = b_r - \frac{a_{32}a_{r3} - a_{r2}a_{33}}{a_{32}a_{23} - a_{22}a_{33}} b_2 - \frac{a_{r2}a_{23} - a_{r3}a_{22}}{a_{32}a_{23} - a_{33}a_{22}} b_3$, for $r \neq 2, 3$,

$\beta_{k'} = \{n+1, 2, 3, n+4, \ldots n+m\}$,

$N_{k'} = \alpha_{k'} \cup H'$, where $\alpha_{k'} = \{1, 4, 5, \ldots n\}$ and $H' = \{n+2, n+3\}$,

the components of $A^{2'}_j$ are $\frac{a_{2j}a_{32} - a_{22}a_{3j}}{a_{23}a_{32} - a_{22}a_{33}}$, $\frac{a_{3j}a_{23} - a_{2j}a_{33}}{a_{32}a_{23} - a_{22}a_{33}}$ and

$a_{rj} - \frac{a_{r3}a_{32} - a_{33}a_{r2}}{a_{23}a_{32} - a_{33}a_{22}} a_{2j} - \frac{a_{r2}a_{23} - a_{22}a_{r3}}{a_{32}a_{23} - a_{22}a_{33}} a_{3j}$, for $r \neq 2, 3$, $j \neq 2, 3$,

the components of $x^{2'}_{n+2}$ are $\frac{a_{32}}{a_{23}a_{32} - a_{22}a_{33}}$, $\frac{-a_{33}}{a_{23}a_{32} - a_{22}a_{33}}$ and

$-\frac{a_{r3}a_{32} - a_{33}a_{r2}}{a_{23}a_{32} - a_{33}a_{22}}$ for $r \neq 2, 3$,

the components of $x^{2'}_{n+3}$ are $\frac{-a_{22}}{a_{23}a_{32} - a_{22}a_{33}}$, $\frac{a_{23}}{a_{23}a_{32} - a_{22}a_{33}}$ and

$-\frac{a_{r2}a_{23} - a_{r3}a_{22}}{a_{32}a_{23} - a_{33}a_{22}}$ for $r \neq 2, 3$.

By use of the properties 2 and 3, $\alpha_k = \alpha_{k'}$ $H = \{n+1, n+3\}$ $H' = \{n+2, n+3\}$. It is obvious that the element "n+1" in H is instead of the element "n+2" in β_k to become basic and by use of the simplex method, the new basic set $\{2, n+1, 3, n+4, \ldots n+m\}$, and the new basic solution $(x_1, x_{n+1}, x_3, x_{n+4}, \ldots x_{n+m})$ can be immediately obtained,

$$\text{where } x_2 = \frac{b_3 a_{23} - b_2 a_{33}}{a_{32} a_{23} - a_{22} a_{33}}, \quad x_3 = \frac{b_2 a_{32} - b_3 a_{22}}{a_{32} a_{23} - a_{33} a_{22}}$$

$$\text{and } x_{n+r} = b_r - \frac{a_{r3} a_{32} - a_{33} a_{r2}}{a_{23} a_{32} - a_{33} a_{22}} b_2 - \frac{a_{r2} a_{23} - a_{22} a_{r3}}{a_{32} a_{23} - a_{22} a_{33}} b_3, \text{ for } r \neq 2, 3.$$

It is easily seen that the new basic set and the new basic solution are exactly equal to $\beta_{k'}$ and $b^2_{k'}$ respectively.

Hence, P_k is said to be adjacent to $P_{k'}$.

(4) **Corollary 2.** Let the two extreme points P_k and $P_{k'}$ be created from P_0 by the number of $h = h' > 2$ paths with the following properties:

1) $P_{k'}$ is a subsequent extreme point P_k, where $k' > k$,
2) there is only one element of the set β_k different from the set $\beta_{k'}$,
3) there is only one element of the set H in N_k different from the set H' in $N_{k'}$ but the set α_k in N_k is equal to the set $\alpha_{k'}$ in $N_{k'}$.

Then P_k is said to be adjacent to the subsequent extreme point $P_{k'}$.

(5) **Theorem 3.** Let the extreme point P_k be created from P_0 by the number of $h = 2$ paths and the extreme point $P_{k'}$ from P_0 by $h' = 1$ path with the following porperties:

1) P_k is a subsequent extreme point $P_{k'}$, where $k > k'$,
2) there is only one element of the set β_k different from the set $\beta_{k'}$,
3) there set α_k is a subset of $\alpha_{k'}$, denoted by $\alpha_k \subset \alpha_{k'}$ but $\alpha_{k'} - \alpha_k$ contains only one element belonging to $\alpha_{k'}$ and the set H' is a subset of H denoted by $H' \subset H$ but H–H' contains only one element belonging to H.

Then $P_{k'}$ is said to be adjacent to the subsequent extreme point P_k.

Proof. Suppose P_k is created by the number of $h = 2$ paths as theorem 2. Suppose $P_{k'}$ is created by the number of $h' = 1$ path denoted by

$$r^1_{k'} = 3, \; q^1_{k'} = 3 \text{ and } s^1_{k'} = n+3.$$

Similarly, the basic feasible solution $b^1_{k'}$, the basic set $\beta_{k'}$, the non-basic set $N_{k'}$

and the non-basic column vectors $A_j^{1'}$ and $x_{n+3}^{1'}$ can be easily computed as follows:

$$b_{k'}^1 = (x_{n+1}, x_{n+2}, x_3, x_{n+4}, \ldots x_{n+m}), \text{ where } x_3 = \frac{b_3}{a_{33}} \text{ and}$$

$$b_{n+r} = b_r - \frac{b_3}{a_{33}} a_{r3}, \text{ for } r \neq 3,$$

$$\beta_{k'} = \{n+1, n+2, 3, n+4, \ldots n+m\},$$

$$N_{k'} = \alpha_{k'} \cup H', \text{ where } \alpha_{k'} = \{1, 2, 4, \ldots n\} \text{ and } H' = \{n+3\},$$

the components of $A_j^{1'}$ are $\frac{a_{3j}}{a_{33}}$ and $a_{rj} - \frac{a_{3j}}{a_{33}} a_{r3}$, for $r \neq 3$, $j \neq 3$,

the components of $x_{n+3}^{1'}$ are $\frac{1}{a_{33}}$ and $-\frac{a_{r3}}{a_{33}}$, for $r \neq 3$.

By use of the properties 2 and 3, $\alpha_{k'} - \alpha_k = \{2\}$, $H - H' = \{n+1\}$, it is obvious that the element "2" of the set $\alpha_{k'}$ in $N_{k'}$ is instead of the element "n+1" in $\beta_{k'}$ to become basic and by use of the simplex method, the new basic set $\{2, n+2, 3, n+4, \ldots n+m\}$ and the basic feasible solution $(x_2, x_{n+2}, x_3, x_{n+4}, \ldots x_{n+m})$ can be immediately obtained,

$$\text{where } x_2 = \frac{b_1 a_{33} - b_3 a_{13}}{a_{12} a_{33} - a_{32} a_{13}}, \quad x_3 = \frac{b_3 a_{12} - b_1 a_{32}}{a_{12} a_{33} - a_{13} a_{32}} \text{ and}$$

$$x_{n+r} = b_r - \frac{a_{33} a_{r2} - a_{32} a_{r3}}{a_{12} a_{33} - a_{32} a_{13}} b_1 - \frac{a_{12} a_{r3} - a_{r2} a_{13}}{a_{12} a_{33} - a_{32} a_{13}} b_3, \text{ for } r \neq 1, 3.$$

It is easily seen that the new basic set and the new basic feasible solution are exactly equal to β_k and b_k^2 respectively.

Hence, $P_{k'}$ is said to be adjacent to P_k.

(6) **Corollary 3.** Let the extreme point P_k be created from P_0 by the number of $h+1$, $h > 2$, paths and $P_{k'}$ by the number of $h' = h$ paths with the following properties:

1) P_k is a subsequent extreme point $P_{k'}$, where $k > k'$,
2) there is only one element of the set β_k different from the set $\beta_{k'}$,
3) there set α_k is a subset of $\alpha_{k'}$ denoted by $\alpha_k \subset \alpha_{k'}$ but $\alpha_{k'} - \alpha_k$ contains only one element belonging to $\alpha_{k'}$ and the set H' is a subset of H, denoted by $H' \subset H$ but $H - H'$ contains only one element belonging to H.

Then $P_{k'}$ is said to be adjacent to the subsequent extreme point P_k.

4. Algorithmic Steps

In the simplex method for computing the basic feasible solution, the iterative steps consists of selecting an element of the non-basic set to enter the basic and an element of the basic set to leave the basic to become non-basic. Hence, the following iterative steps can be designed:

Step 1. Initially, the original extreme point P_0 can be readily determined by its initial basic set β_0 and the non-basic set N_0 from the given problem.

Step 2. By use of the simplex method, each element of the ranking in ascending order of the non-basic set N_0 at P_0 must enter the initial basic set β_0 to be created a number of extreme points P_k, $k \in N_0$, ranking ascending order, where P_0 and P_k are joined by a solid line to span the convex polytope.

Step 3. Whether P_k is adjacent to the subsequent extreme point, if the theorems and corollaries in section 3 can be used. If it is, the two adjacent extreme points are joined by a broken line and if not, go to the next step.

Step 4. By use of the simplex method, each element (except adjacency in step 3) ranking in ascending order of the non-basic set at P_k must enter its basic set and the new extreme point can be immediately created by the second path and then they joined by a solid line to span the convex polytope.

After these steps, the new extreme points can be created by the third path, fourth path and so on and by use of the iterative step 3 and step 4, until all the extreme points are intersected by n distinct lines including solid and broken in n-space. Hence, this polytope can be easily constructed.

5. Numerical Example

Given the following system of linear inequalities, find all the basic feasible solutions:

$$\begin{aligned}
3x_1 - x_2 + x_3 + 2x_4 &\leqslant 8 \\
x_1 + 2x_2 + 4x_3 - x_4 &\leqslant 6 \\
2x_1 + 3x_2 - 3x_3 + x_4 &\leqslant 10 \\
x_1 + x_3 + x_4 &\leqslant 7 \\
x_j \geqslant 0,\ j = 1, 2, 3, 4,
\end{aligned}$$

where m = 4, n = 4 and $i \in M = \{1, 2, 3, 4\}$ and $j \in N = \{1, 2, 3, 4\}$.

The slack variables $x_{n+i} \geqslant 0$, $i \in M$, add to the corresponding linear inequalities respectively, we have the following linear equalities:

$$\begin{aligned} 3x_1 - x_2 + x_3 + 2x_4 + x_5 &= 8 \\ x_1 + 2x_2 + 4x_3 - x_4 + x_6 &= 6 \\ 2x_1 + 3x_2 - 3x_3 + x_4 + x_7 &= 10 \\ x_1 + x_3 + x_4 + x_8 &= 7 \end{aligned}$$

$x_j \geqslant 0$, $j \in N$ and $x_{4+i} \geqslant 0$, $i \in M$.

By use of the simplex method, all the basic feasible solutions at the corresponding extreme points can be computed by the following stages respectively:

1) Stage 0. $P(0) = 0$, $h(0) = 0$. Initially, the extreme point P_0 is determined by $b_0 = (8,6,10,7)^T$, $\beta_0 = \{5,6,7,8\}$ and $N_0 = \{1,2,3,4\}$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\ 3 & -1 & 1 & 2 & 1 & 0 & 0 & 0 & 8 \\ 1 & 2 & 4 & -1 & 0 & 1 & 0 & 0 & 6 \\ 2 & 3 & -3 & 1 & 0 & 0 & 1 & 0 & 10 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 7 \end{array}.$$

Matrix P(0)

Hence, create new extreme points P_1, P_2, P_3, P_4 for j = 1, 2, 3, 4 respectively

P_1: $r_1 = 1$ $q_1 = 1$ $s_1 = 5$ $\beta_1 = \{1,6,7,8\}$ $N_1 = \alpha_1 \cup H = \{2,3,4\} \cup \{5\}$.

P_2: $r_2 = 2$ $q_2 = 2$ $s_2 = 6$ $\beta_2 = \{5,2,7,8\}$ $N_2 = \alpha_2 \cup H = \{1,3,4\} \cup \{6\}$.

P_3: $r_3 = 2$ $q_3 = 2$ $s_3 = 6$ $\beta_3 = \{5,3,7,8\}$ $N_3 = \alpha_3 \cup H = \{1,2,4\} \cup \{6\}$.

P_4: $r_4 = 1$ $q_4 = 1$ $s_4 = 5$ $\beta_4 = \{4,6,7,8\}$ $N_4 = \alpha_4 \cup H = \{1,2,3\} \cup \{5\}$.

2) Stage 1. $P(1) = 1$, $h(1) = 1$. We have $b_1^1 = (\frac{8}{3}, \frac{10}{3}, \frac{14}{3}, \frac{13}{3})^T$, $\beta_1 = \{1,6,7,8\}$, $N_1 = \alpha_1 \cup H = \{2,3,4\} \cup \{5\}$, $A_2^1, A_3^1, A_4^1, X_5^1$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\ 1 & \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{8}{3} \\ 0 & \frac{7}{3} & \frac{11}{3} & \frac{-5}{3} & \frac{-1}{3} & 1 & 0 & 0 & \frac{10}{3} \\ 0 & \frac{11}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-2}{3} & 0 & 1 & 0 & \frac{14}{3} \end{array}$$

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \Big| & \frac{-1}{3} & 0 & 0 & 1 & \Big| & \frac{13}{3} \end{bmatrix}$$

Matrix P(1)

Adjacency test indicates P_1 is adjacent to P_4 (theorem 1). Hence, create new extreme points P_5, P_6 for $j = 2, 3$ respectively.

$P_5:\ r_5 = 2\ \ q_5 = 3\ \ s_5 = 7\ \ \beta_5 = \{1,6,2,8\}\ \ N_5 = \alpha_5 \cup H = \{3,4\} \cup \{5,7\}.$

$P_6:\ r_6 = 3\ \ q_6 = 2\ \ s_6 = 6\ \ \beta_6 = \{1,3,7,8\}\ \ N_6 = \alpha_6 \cup H = \{2,4\} \cup \{5,6\}.$

3) Stage 2. $P(2) = 2$, $h(2) = 1$. We have $b_2^1 = (11, 3, 1, 7)^T$, $\beta_2 = \{5,2,7,8\}$, $N_2 = \alpha_2 \cup H = \{1,3,4\} \cup \{6\}$, $A_1^1, A_3^1, A_4^1, X_6^1$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\ \frac{7}{2} & 0 & 3 & \frac{3}{2} & 1 & \frac{1}{2} & 0 & 0 & 11 \\ \frac{1}{2} & 1 & 2 & \frac{-1}{2} & 0 & \frac{1}{2} & 0 & 0 & 3 \\ \frac{1}{2} & 0 & -9 & \frac{5}{2} & 0 & \frac{-1}{2} & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 7 \end{array}$$

Matrix P(2)

Adjacency test indicates P_2 is adjacent to P_3 (theorem 1). Hence, create a new extreme points P_7, P_8 for $j = 1, 4$ respectively.

$P_7:\ r_7 = 1\ \ q_7 = 3\ \ s_7 = 7\ \ \beta_7 = \{5,2,1,8\}\ \ N_7 = \alpha_7 \cup H = \{3,4\} \cup \{6,7\}.$

$P_8:\ r_8 = 4\ \ q_8 = 3\ \ s_8 = 7\ \ \beta_8 = \{5,2,4,8\}\ \ N_8 = \alpha_8 \cup H = \{1,3\} \cup \{6,7\}.$

4) Stege 3. $P(3) = 3$, $h(3) = 1$. We have $b_3^1 = (\frac{26}{4}, \frac{6}{4}, \frac{58}{4}, \frac{22}{4})^T$, $\beta_3 = \{5,3,7,8\}$, $N_3 = \alpha_3 \cup H = \{1,2,4\} \cup \{6\}$, $A_1^1, A_2^1, A_4^1, X_6^1$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\ \frac{11}{4} & \frac{-6}{4} & 0 & \frac{9}{4} & 1 & \frac{-1}{4} & 0 & 0 & \frac{26}{4} \\ \frac{1}{4} & \frac{2}{4} & 1 & \frac{-1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{6}{4} \\ \frac{11}{4} & \frac{18}{4} & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 1 & 0 & \frac{58}{4} \\ \frac{3}{4} & \frac{-2}{4} & 0 & \frac{5}{4} & 0 & \frac{-1}{4} & 0 & 1 & \frac{22}{4} \end{array}$$

Matrix P(3)

Adjacency test indicates P_3 is adjacent to P_6 (theorem 3). Hence, create new extreme point P_9 for $j = 4$.

P_9: $r_9 = 4$ $q_9 = 1$ $s_9 = 5$ $\beta_9 = \{4,3,7,8\}$ $N_9 = \alpha_9 \cup H = \{1,2\} \cup \{6,5\}$.

5) Stage 4. $P(4) = 4$, $h(4) = 1$. We have $b_4^1 = (4, 10, 6, 3)^T$, $\beta_4 = \{4,6,7,8\}$, $N_4 = \alpha_4 \cup H = \{1,2,3\} \cup \{5\}$. $A_1^1, A_2^1, A_3^1, X_5^1$ denoted by the following matrix:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
$\frac{3}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	4
$\frac{5}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	0	$\frac{1}{2}$	1	0	0	10
$\frac{1}{2}$	$\frac{7}{2}$	$\frac{-7}{2}$	0	$\frac{-1}{2}$	0	1	0	6
$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	0	0	1	3

Matrix P(4)

Adjacency test indicates P_4 is adjacent to P_9 (theorem 3). Hence, create a new extreme point P_{10} for $j = 2$.

P_{10}: $r_{10} = 2$ $q_{10} = 3$ $s_{10} = 7$ $\beta_{10} = \{4,6,2,8\}$ $N_{10} = \alpha_{10} \cup H = \{1,3\} \cup \{5,7\}$.

6) Stage 5. $P(5) = 5$, $h(5) = 2$. We have $b_5^2 = (\frac{34}{11}, \frac{4}{11}, \frac{14}{11}, \frac{43}{11})^T$, $\beta_5 = \{1,6,2,8\}$, $N_5 = \alpha_5 \cup H = \{3,4\} \cup \{5,7\}$, $A_3^2, A_4^2, X_5^2, X_7^2$ denoted by the following matrix:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
1	0	0	$\frac{7}{11}$	$\frac{3}{11}$	0	$\frac{1}{11}$	0	$\frac{34}{11}$
0	0	$\frac{66}{11}$	$\frac{6}{11}$	$\frac{1}{11}$	1	$\frac{-7}{11}$	0	$\frac{4}{11}$
0	1	$\frac{-1}{11}$	$\frac{-1}{11}$	$\frac{-2}{11}$	0	$\frac{3}{11}$	0	$\frac{14}{11}$
0	0	$\frac{11}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	0	$\frac{-1}{11}$	1	$\frac{43}{11}$

Matrix P(5)

Adjacency test indicates P_5 is adjacent to P_7 (theorem 2). Hence, create a new extreme point P_{11} for $j = 3$.

P_{11}: $r_{11} = 3$ $q_{11} = 2$ $s_{11} = 6$ $\beta_{11} = \{1,3,2,8\}$ $N_{11} = \alpha_{11} \cup H = \{4\} \cup \{5,6,7\}$.

7) Stage 6. $P(6) = 6$, $h(6) = 2$. We have $b_6^2 = (\frac{26}{11}, \frac{10}{11}, 8, \frac{41}{11})^T$, $\beta_6 = \{1,3,7,8\}$, $N_6 = \alpha_6 \cup H = \{2,4\} \cup \{5,6\}$, $A_2^2, A_4^2, X_5^2, X_6^2$ denoted by the following matrix:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
1	$\frac{-6}{11}$	0	$\frac{9}{11}$	$\frac{4}{11}$	$\frac{-1}{11}$	0	0	$\frac{26}{11}$

$$\left[\begin{array}{cccc|cccc|c} 0 & \frac{7}{11} & 1 & \frac{-5}{11} & \frac{-1}{11} & \frac{3}{11} & 0 & 0 & \frac{10}{11} \\ 0 & 6 & 0 & -2 & -1 & 1 & 1 & 0 & 8 \\ 0 & \frac{-1}{11} & 0 & \frac{7}{11} & \frac{-3}{11} & \frac{-2}{11} & 0 & 1 & \frac{41}{11} \end{array}\right]$$

Matrix P(6)

Adajcency test indicates P_6 is adjacent to P_9 (corollary 1) and to P_{11} (theorem 3). Hence, no extreme point can be created.

8) Stage 7. P(7) = 7, h(7) = 2. We have $b_7^2 = (4, 2, 2, 5)^T$, $\beta_7 = \{5,2,1,8\}$, $N_7 = \alpha_7 \cup H = \{3,4\} \cup \{6,7\}$, $A_3^2, A_4^2, X_6^2, X_7^2$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\ \hline 0 & 0 & 66 & -16 & 1 & 11 & -7 & 0 & 4 \\ 0 & 1 & 11 & -3 & 0 & 2 & -1 & 0 & 2 \\ 1 & 0 & -18 & 5 & 0 & -3 & 2 & 0 & 2 \\ 0 & 0 & 19 & -4 & 0 & 3 & -2 & 1 & 5 \end{array}$$

Matrix P(7)

Adjacency test indicates P_7 is adjacent to P_8 (corollary 1) and to P_{11} corollary 3). Hence, no extreme point can be created.

9) Stage 8. P(8) = 8, h(8) = 2. We have $b_8^2 = (\frac{52}{5}, \frac{16}{5}, \frac{2}{5}, \frac{33}{5})^T$, $\beta_8 = \{5,2,4,8\}$, $N_8 = \alpha_8 \cup H = \{1,3\} \cup \{6,7\}$, $A_1^2, A_3^2, X_6^2, X_7^2$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\ \hline \frac{16}{5} & 0 & \frac{42}{5} & 0 & 1 & \frac{7}{5} & \frac{-3}{5} & 0 & \frac{50}{5} \\ \frac{3}{5} & 1 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & \frac{16}{5} \\ \frac{1}{5} & 0 & \frac{-18}{5} & 1 & 0 & \frac{-3}{5} & \frac{2}{5} & 0 & \frac{2}{5} \\ \frac{4}{5} & 0 & \frac{23}{5} & 0 & 0 & \frac{1}{5} & \frac{-2}{5} & 1 & \frac{33}{5} \end{array}$$

Matrix P(8)

Adjacency test indicates P_8 is adjacent to P_{10} (theorem 2). Hence, create a new extreme point P_{12} for j = 3.

P_{12}: $r_{12} = 3$ $q_{12} = 1$ $s_{12} = 5$ $\beta_{12} = \{3,4,2,8\}$ $N_{12} = \alpha_{12} \cup H = \{1\} \cup \{6,7,5\}$.

10) Stage 9. P(9) = 9, h(9) = 2. We have $b_9^2 = (\frac{26}{9}, \frac{20}{9}, \frac{124}{9}, \frac{17}{9})^T$, $\beta_9 = \{4,3,7,8\}$, $N_9 = \alpha_9 \cup H = \{1,2\} \cup \{5,6\}$, $A_1^2, A_2^2, X_5^2, X_6^2$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\
\frac{11}{9} & \frac{-6}{9} & 0 & 1 & \frac{4}{9} & \frac{-1}{9} & 0 & 0 & \frac{26}{9} \\
\frac{5}{9} & \frac{3}{9} & 1 & 0 & \frac{1}{9} & \frac{2}{9} & 0 & 0 & \frac{20}{9} \\
\frac{22}{9} & \frac{42}{9} & 0 & 0 & \frac{-1}{9} & \frac{7}{9} & 1 & 0 & \frac{124}{9} \\
\frac{-7}{9} & \frac{3}{9} & 0 & 0 & \frac{-5}{9} & \frac{1}{9} & 0 & 1 & \frac{17}{9}
\end{array}$$

Matrix P(9)

Adjacency test indicates P_9 is adjacent to P_{12} (corollary 3). Hence, no extreme point can be created

11) Stage 10. P(10) = 10, h(10) = 2. We have $b_{10}^2 = (\frac{34}{7}, \frac{52}{7}, \frac{12}{7}, \frac{15}{7})^T$, $\beta_{10} = \{4,6,2,8\}$, $N_{10} = \alpha_{10} \cup H = \{1,3\} \cup \{5,7\}$, $A_1^2, A_3^2, X_5^2, X_7^2$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\
\frac{11}{7} & 0 & 0 & 1 & \frac{3}{7} & 0 & \frac{1}{7} & 0 & \frac{34}{7} \\
\frac{16}{7} & 0 & 6 & 0 & \frac{5}{7} & 1 & \frac{-3}{7} & 0 & \frac{52}{7} \\
\frac{1}{7} & 1 & -1 & 0 & \frac{-1}{7} & 0 & \frac{2}{7} & 0 & \frac{12}{7} \\
\frac{-4}{7} & 0 & 1 & 0 & \frac{-3}{7} & 0 & \frac{-1}{7} & 1 & \frac{15}{7}
\end{array}$$

Matrix P(10)

Adjacency test indicates P_{10} is adjacent to P_{12} (corollary 3). Hence, no extreme point can be created.

12) Stage 11. P(11) = 11, h(11) = 3. We have $b_{11}^3 = (\frac{34}{11}, \frac{4}{66}, \frac{88}{66}, \frac{254}{66})^T$, $\beta_{11} = \{1,3,2,8\}$, $N_{11} = \alpha_{11} \cup H = \{4\} \cup \{5,7,6\}$, $A_4^3, X_5^3, X_6^3, X_7^3$ denoted by the following matrix:

$$\begin{array}{cccc|cccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \\
1 & 0 & 0 & \frac{7}{11} & \frac{3}{11} & 0 & \frac{1}{11} & 0 & \frac{34}{11} \\
0 & 0 & 1 & \frac{-16}{66} & \frac{1}{66} & \frac{11}{66} & \frac{-7}{66} & 0 & \frac{4}{66} \\
0 & 1 & 0 & \frac{-22}{66} & \frac{-11}{66} & \frac{1}{6} & \frac{11}{66} & 0 & \frac{88}{66}
\end{array}$$

$$\left[\begin{array}{cccc|cccc|c} 0 & 0 & 0 & \frac{40}{66} & \frac{17}{66} & \frac{-1}{6} & \frac{1}{66} & 1 & \frac{254}{66} \end{array}\right]$$

Matrix P(11)

Adjacency test indicates P_{11} is adjacent to P_{12} (corollary 1). Hence, no extreme point can be created.

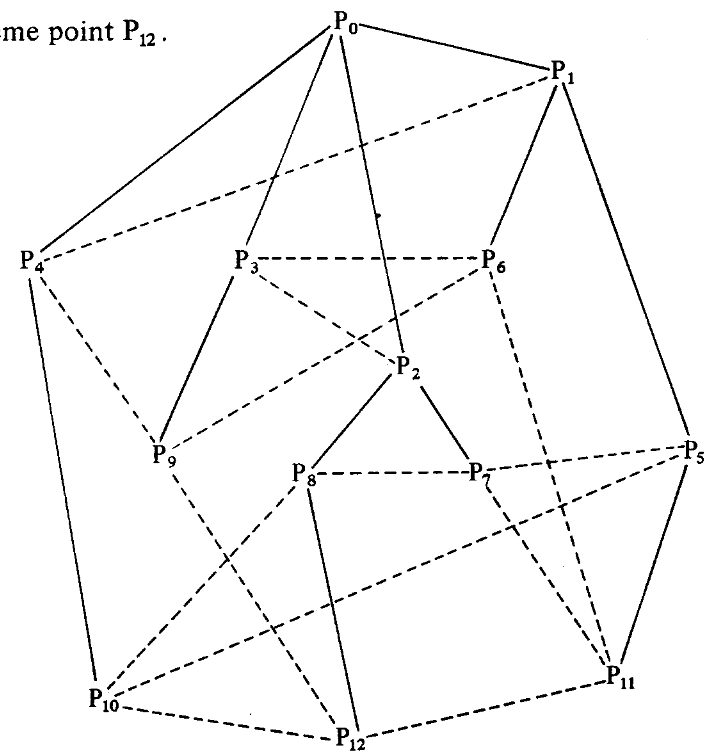
13) Stage 12. P(12) = 12, h(12) = 3. We have $b_{12}^3 = (\frac{52}{42}, \frac{124}{42}, \frac{204}{42}, \frac{38}{42})^T$, $\beta_{12} = \{3,2,4,8\}$, $N_{12} = \alpha_{12} \cup H = \{1\} \cup \{6,7,5\}$, $A_1^3, X_5^3, X_6^3, X_7^3$ denoted by the following matrix:

$$\begin{array}{c} \begin{array}{cccc|cccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b \end{array} \\ \left[\begin{array}{cccc|cccc|c} \frac{16}{42} & 0 & 1 & 0 & \frac{5}{42} & \frac{7}{42} & \frac{-3}{42} & 0 & \frac{52}{42} \\ \frac{22}{42} & 1 & 0 & 0 & \frac{-1}{42} & \frac{7}{42} & \frac{9}{42} & 0 & \frac{124}{42} \\ \frac{66}{42} & 0 & 0 & 1 & \frac{18}{42} & 0 & \frac{6}{42} & 0 & \frac{204}{42} \\ \frac{-40}{42} & 0 & 0 & 0 & \frac{-23}{42} & \frac{-7}{42} & \frac{-3}{42} & 1 & \frac{38}{42} \end{array}\right] \end{array}$$

Matrix P(12)

End of this stage at the extreme point P_{12}.

Hence, it is clear that all the basic feasible solutions are obtained and their corresponding extreme points are also created to span the following convex polytope (geometric graph) whose each extreme point is intersected by n=4 lines including solid and broken in 4-space.



Convex Plytope

6. Remarks

1. It is clear that all the extreme points to span a convex polytope of the geometric representation are simply and easily determined by use of the theorems and corollaries in section 3.
2. Obviously, when the original variables are greater than three, $n > 3$, it is difficult to find the geometric representation and hence, this method for determining all the basic feasible solutions at their corresponding extreme points is better than the corner point method [8, p.8].
3. This method can not be applied for computing the optimal solution for a linear programming problem, although its optimal solution must occure at the extreme points of its linear constraints to span a bounded convex polytope [1. p178]. Obviously, it does not need all the extreme points to obtain its optimal solution, and therefore the following exmple will illustrate how to compute its optimal solution:

   Example. Maximize $f(x_1, x_2, x_3, x_4) = 5x_1 + 10x_2 + 6x_3 + 2x_4$, subject to the linear constraints as shown in numerical example in section 5.

   Its optimal solution can be readily computed by use of only four extreme points, P_0, P_2, P_8 and P_{12} as shown in four matrices P(0), P(2), P(8), and P(12) respectively in section 5, and hence, we obtain its opimal (maximum) solution of $f(x_1, x_2, x_3, x_4)$ equal to $\frac{1960}{42}$ at the extreme point P_{12}, whose basic feasible solution $x_1 = 0$, $x_2 = \frac{124}{42}$, $x_3 = \frac{52}{42}$ and $x_4 = \frac{204}{42}$.

7. References

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