國立政治大學九十 七 學年度研究所模士班入學考試命題紙

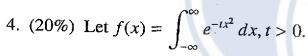
考試科目

所列應用數學系明考試時間 3月15日

- 1. (20%) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, f(1) = 1, f'(x) < 0 for all $x \in (-\infty, 1)$ and f'(x) > 0 for all $x \in (1, \infty)$.
 - (a) Show that $f(x) \ge 1$ for all $x \in \mathbb{R}$.
 - (b) Evaluate f'(1).
- 2. (20%) Let $f(x) = x^2 \cos x$.
 - (a) Find the Maclaurin series of $\cos x$.
 - (b) Evaluate $f^{(10)}(0)$ and $f^{(11)}(0)$.
- 3. (20%) Let $P, Q: \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable functions and C be the unit circle $x^2 +$ $y^2 = 1$ in the counterclockwise direction. Evaluate the line integral $\oint_C P(x, y) dx + Q(x, y) dy$ under the following assumptions.

(a)
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$
 on \mathbb{R}^2 .

(b)
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x^2 + y^2$$
 on \mathbb{R}^2 .



- (a) Evaluate f(1).
- (b) Find f'(t) explicitly for t > 0.
- 5. (10%) Show that the shortest distance from a point (x_0, y_0, z_0) to a plane ax + by + cz + d = 0 is $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.
- 6. (10%) Show that every continuous real-valued function on a closed bounded interval [a, b] is Riemann integrable.

備

随 交

題委員:

- 命題紙使用説明:1.試題將用原件印製,敬請使用黑色墨水正楷書寫或打字(紅色不能製版請勿使用)。
 - 2. 書寫時請勿超出格外,以免印製不清。
 - 3.試題由郵寄遞者請以掛號寄出,以免遺失而示愼重。

Please show all your work.

- 1. Define $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.
 - (a) Let $P^{-1}AP = D$ be a diagonal matrix. Prove that $e^A = Pe^D P^{-1}$. (10%)

(b) Let
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
. Compute e^A . (7%)

- 2. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), A, B are matrices.
 - (a) If $T,U:V \to W$ are both linear and agree on a basis for V, then T=U.
 - (b) If $m = \dim(V)$ and $n = \dim(W)$, β, γ are ordered basis of V and W, respectively,

and T is a linear transformation, then $[T]^{\gamma}_{\beta}$ is an $m \times n$ matrix.

- (c) $A^2 = I \Rightarrow A = I$ or A = -I.
- (d) AB = I implies that A and B are invertible.
- (e) Let T be a linear operator on a finite-dimensional vector space V. Let β and α be ordered basis of V, and let Q be the change of coordinate matrix that changes α -coordinates into β -coordinates. Then $[T]_{\beta} = Q[T]_{\alpha}Q^{-1}$. (20%)

3. Let
$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

- (a) Find the characteristic polynomial of A. (6%)
- (b) Find a Jordan canonical form J and an invertible matrix Q such that $J = Q^{-1}AQ$. (10%)
- 4. A matrix $M \in M_{n \times n}(C)$ is called skew-symmetric if M' = -M.

Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even? (15%)

- 5. (a) Let V = P(R) with the inner product $\langle f, g \rangle = \int_{1}^{1} f(t)g(t)dt$. Use Gram-Schmidt process to obtain an orthonormal basis for $P_2(R)$ from the standard basis $\{1, x, x^2\}$. (10%) (b) Let $V = P_3(R)$ with the inner product $\langle f, g \rangle = \int_{1}^{1} f(t)g(t)dt$. Compute the orthogonal projection of $f(x) = x^3$ on $P_2(R)$. (7%)
- 6. Let F be a field that is not of characteristic 2. Define $W_1 = \{A \in M_{n \times n} : A_{ij} = 0 \text{ whenever } i \leq j\}$ and W_2 to be the set of all symmetric $n \times n$ matrices with entries from F. Both W_1 and W_2 are subspaces of $M_{n \times n}(F)$. Prove that $M_{n \times n}(F) = W_1 \oplus W_2$. (15%)