國立政治大學　104　學年度碩士班招生考試試題

考試科目	微積分 81111, 81161	所別	應用數學系 8111	考試時間	3月1日(星期日)第一節

1. Evaluate the limits.

(a) (6%) $\lim_{x\to 0} \dfrac{\sin^3(3x)(1-\cos(2x))}{x\tan^4(5x)}$.　(b) (6%) $\lim_{x\to\infty}(x - x^2\ln(\dfrac{1+x}{x}))$.

2. Evaluate the integrals.

(a) (8%) $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$.

(b) (8%) $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx$.

(c) (8%) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

(d) (8%) $\displaystyle\int_0^1\int_y^1 \sin(x^2)\,dx\,dy$.

3. Determine if each series converges or diverges.

(a) (8%) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$.　(b) (8%) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$.

4. (10%) Evaluate the function $\varphi(t)$ defined by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\cos(xt)\,dx.$$

5. (10%) Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$.

6. (10%) Evaluate the line integral

$$\oint_C (3y - e^{\sin^3(2x)})\,dx + (7x + \sqrt{y^4+3})\,dy,$$

where C is the circle $x^2+y^2=1$.

7. (10%) Suppose that the function $f:[a,b]\to\mathbb{R}$ is continuously differentiable and one to one on the interval $[a,b]$. Prove that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a).$$

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考試科目	線性代數 81112、81162	所別	應用數學系 8111	考試時間	3月1日(日)第二節

1. (20%) Let a, b be two distinct real numbers. Let

$$A = \begin{bmatrix} a^3 & a^2 & a & 1 \\ 3a^2 & 2a & 1 & 0 \\ b^3 & b^2 & b & 1 \\ 3b^2 & 2b & 1 & 0 \end{bmatrix}.$$

Determine all possible values of rank(A).

2. (20%) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & 7 \\ 1 & 0 & 3 \end{bmatrix}.$$

(a) Determine if A is diagonalizable or not.

(b) Find matrices Q and D such that D is either a diagonal matrix or a Jordan block matrix, satisfying

$$D = Q^{-1}AQ.$$

3. (20%) Let V, W be two vector spaces over a field $\mathbb{F}$, and let $T: V \to W$ be a linear transformation of T. Suppose U is a subspace of W. Is $T^{-1}(U)$ a subspace of V? Justify your answer.

4. (20%) Let $P_3(\mathbb{R})$ be the set of all real polynomials of degrees at most 3. We define $\langle \cdot, \cdot \rangle : P_3(\mathbb{R}) \times P_3(\mathbb{R}) \to \mathbb{R}$ by

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$

for all $f(x), g(x) \in P_3(\mathbb{R})$.

(a) What are the conditions need to be satisfied for $\langle \cdot, \cdot \rangle$ to be an inner product on $P_3(\mathbb{R})$?

(b) Suppose that $\langle \cdot, \cdot \rangle$ is indeed a inner product on $P_3(\mathbb{R})$. Apply the Gram-Schmid process to $\{1, x, x^2, x^3\}$ to find an orthogonal basis for $P_3(\mathbb{R})$.

5. (20%) Let V be a finite-dimensional vector space, and let T be a linear operator on V. Suppose that rank(T) = rank(T^2).

(a) Show that $R(T) = R(T^2)$ and $N(T) = N(T^2)$. ($R(T)$ is the range of T and $N(T)$ is the null space of T.)

(b) Show that $V = R(T) \oplus N(T)$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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