(1) Let A and B be two nxn matrices over field F. Assume that the characteristic polynomials of A and B split into distinct linear factors in F, respectively. Show that if AB = BA, then A and B can be diagonalized simultaneously. (10%)

(2) Let V be the space of nxn matrices over F. Let A be a fixed nxn matrix over F. Let T_1, T_2 and T_3 be the linear operators on V defined by

$$T_1(B) = AB, \quad T_2(B) = B', \quad T_3(B) = \frac{1}{2}(B+B')$$

where $B \in V$ and B' denotes the transpose matrix of B.

(a) True or false? If A is diagonalizable over F, then T_1 is diagonalizable.

(b) Show that both T_2 and T_3 are diagonalizable.

(c) Find a basis for V such that T_2 and T_3 are diagonalied simultaneously.

(30%)

(3) Let $\lambda_1(A)$ denote the smallest eigenvalue of the nxn real matrix A.

(a) Show that $\lambda_1(A+B) \geq \lambda_1(A) + \lambda_1(B)$ for any nxn real symmetric matrices A and B.

(b) Let $A = (a_{ij})$ be an nxn real symmetric matrix. Show that $\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}$ is between the smallest and the largest eigenvalues of A.

(c) Find the minimum value of $R(x) = \dfrac{x_1^2 - x_1x_2 + x_2^2}{2x_1^2 + x_2^2}$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$.

(Hint: Consider the Rayleigh quotients.) (24%)

(4) Let A be an mxn matrix and b be an mx1 vector over the real field.

(a) Show that the linear system Ax=b is solvable if and only if b belongs to the column space of A. (4%)

(b) Show that $A'Ax = A'b$ is always solvable for any A and b. (10%)

(c) Let V be the space of all continuous function defined on [0,1]. Let $\langle f, g \rangle = \int_0^1 fg$ be an inner product on V. Which linear function is closest to $f(x) = x^3$ in the least squares sense over the interval [0,1]? (10%)

(5) Let T be a linear operator on a finite-dimensional vector space V. Then there exist subspaces W_1 and W_2 of V such that

(a) $V = W_1 \oplus W_2$.

(b) W_1 and W_2 are both T-invariant.

(c) The restriction $T|_{W_1}$ is nilponent and $T|_{W_2}$ is invertible.

(Hint: Consider the nullspaces and ranges of $T, T^2, T^3, \ldots$) (12%)

考試科目	分析概論	系所組別	應用數學系	考試時間	6月27日 星期二 上下午第2節

(20%) 1. Are the following statements true? If yes, prove them; if no, give counter examples.

(a) The intersection of any collection of open sets is open in R^p (p-dimensional real Cartesian space.)

(b) G is connected if any pair of points x, y in $G \subseteq R^p$ can be joined by a polygonal curve lying entirely in G.

(15%) 2. If $x_n = \frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}$, $\forall n \in N$, then does the sequence $\langle x_n \rangle = \{x_1, x_2, \cdots, x_n, \cdots\}$ converge?

(10%) 3. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$

(a) Give the set A of all points where f is not continuous.

(b) Show that f is not continuous on A.

(15%) 4. Let $\langle f_n \rangle = \{f_1, f_2, \cdots, f_n, \cdots\}$ be a sequence of continuous functions with domain D in R^p and range in R^q and let this sequence converge uniformly on D to a function f. Prove or disprove that f is continuous on D.

(20%) 5. Prove or disprove the following statement:

If $\langle f_n \rangle$ is a sequence of measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then $\int_E f \le \underline{\lim} \int_E f_n$.

(20%) 6. State and prove the Hölder Inequality.