(911)

Notation In this examination $\mathbf{x} \in \mathbb{R}^n$ means $\mathbf{x}$ is a real, *column* vector with n components, and $\mathbf{x}^\mathsf{T}$ is the transpose of $\mathbf{x}$.

1. In this problem, A represents a real, symmetric $n \times n$ matrix which is positive definite.

(a) (5%) State a spectral theorem that specifies the properties of eigenvalues and eigenvectors of A.

(b) (5%) Show that $\sqrt{A}$, the square root of A, exists and is positive definite.

(c) (20%) Now assume $A = I + 3\mathbf{u}\mathbf{u}^\mathsf{T}$, where I is the identity matrix and $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u}^\mathsf{T}\mathbf{u} = 1$.

[1] Show that A is positive definite.

[2] Find the eigenvalues and eigenvectors of A.

[3] Compute $\det(A)$ and write out the minimal polynomial of A.

[4] Determine $\sqrt{A}$ and find a polynomial p such that $\sqrt{A} = p(A)$.

(d) (10%) Find the symmetric matrix B so that

$$\mathbf{x}^\mathsf{T} B\mathbf{x} = (x_1 - x_2 + 2x_3)^2,$$

where $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^\mathsf{T} \in \mathbb{R}^3$. Is B positive definite? What is the rank of B? Determine the maximum value of $\mathbf{x}^\mathsf{T}B\mathbf{x}$ under the condition $\mathbf{x}^\mathsf{T}\mathbf{x} = 1$.

2. Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix};$$

and let $\mathcal{R}(A)$ denote the column space of A.

(a) (8%) Find an $\widehat{\mathbf{x}} \in \mathbb{R}^2$ such that $||A\widehat{\mathbf{x}} - \mathbf{b}||_2 \leq ||A\mathbf{x} - \mathbf{b}||_2$ for all $\mathbf{x} \in \mathbb{R}^2$. Is $\widehat{\mathbf{x}}$ unique? Why?

(b) (6%) Find *orthonormal* (column) vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ such that $\mathbf{q}_1$ and $\mathbf{q}_2$ span $\mathcal{R}(A)$.

(c) (10%) Compute the orthogonal projection matrix $P : \mathbb{R}^3 \to \mathcal{R}(A)$. Is P an orthogonal matrix? What are the eigenvalues of P? Use P to determine the vector in $\mathcal{R}(A)$ which is closest to $\mathbf{b}$.

(d) (6%) Let $Q = \mathbf{q}_3\mathbf{q}_3^\mathsf{T}$. Compute $P^3 + Q^3$ and P^3Q^3.

3. Consider a function space $\mathcal{X}$ on which a real inner product $\langle \bullet, \bullet \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is defined.

(a) (5%) State the basic properties that an inner product possesses.

(b) (10%) Given an $f \in \mathcal{X}$ and a set of *linearly independent* functions $\{\phi_k\}_{k=1}^{n}$ in $\mathcal{X}$, show how to determine the coefficients $\{c_k\}_{k=1}^{n}$ in $s_n \equiv \sum_{k=1}^{n} c_k \phi_k$ so that $\|f - s_n\|$ is a minimum, where $\| \bullet \|$ is the norm derived from $\langle \bullet, \bullet \rangle$. Also prove that $G = [g_{jk}]_{j,k=1}^{n}$, a real symmetric $n \times n$ matrix with entries $g_{jk} = \langle \phi_j, \phi_k \rangle$, is positive definite.

(c) (10%) Let $\langle g, h \rangle = \int_0^1 g(x)h(x)\,dx$ for $g, h \in \mathcal{X} \equiv C[0,1]$, the space of functions continuous on $[0,1]$, and let $f(x) = x^3$, $\phi_k(x) = x^{k-1}$, $k = 1, 2, 3$, for $x \in [0,1]$. Compute the coefficients c_1, c_2, and c_3 as described in part (b). Explain why the $n \times n$ matrix H, with entries h_{jk} generated by the formula

$$h_{jk} = \frac{1}{j+k-1} = \int_0^1 x^{j-1} x^{k-1}\,dx, \qquad 1 \le j, k \le n,$$

is positive definite.

(d) (5%) Find a formula for the coefficients $\{c_k\}_{k=1}^{n}$ in part (b) if the given set $\{\phi_k\}_{k=1}^{n}$ is *orthonormal*, and in this case express the orthogonal projection $P : \mathcal{X} \to S$ in terms of $\langle \bullet, \bullet \rangle$ and $\{\phi_k\}_{k=1}^{n}$, where S is the subspace spanned by $\{\phi_k\}_{k=1}^{n}$.

I. Let f be a real value function defined in an open set $\Omega \subset \mathbb{R}^2$. Prove or disprove the following statements:

1.(10%) If the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ both exist, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

2.(10%) If $\nabla f(x,y) := \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = 0$ for all $(x,y) \in \Omega$, then there exists a constant c such that $f(x,y) = c \; \forall (x,y) \in \Omega$.

II. Prove or disprove the following statements:

3.(15%) For any $n \times n$ matrix A with real entries define e^A by $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. Then we have $e^{A+B} = e^A e^B$.

4.(15%) For $u \in L^p(0,1)$ define $\|u\|_p = \sqrt[p]{\int_0^1 |u(x)|^p\, dx}$; suppose that $\frac{1}{p} + \frac{1}{q} = 1$, $f_n \in L^p(0,1)$, $g_n \in L^q(0,1)$ $\forall n \in N$ and that $\lim_{n\to\infty} \|f_n - f\|_p = 0 = \lim_{n\to\infty} \|g_n - g\|_q$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, g_n(x)\, dx = \int_0^1 f(x)\, g(x)\, dx.$$

III. Suppose μ is a positive finite measure on a $\sigma-$algebra $\mathcal{M}$. If $A \in \mathcal{M}$ and $B \in \mathcal{M}$, define

$$\rho(A,B) = \mu(A \triangle B), \; A \triangle B := (A-B) \cup (B-A).$$

5.(15%) Provethat $(\mathcal{M}, \rho)$ is a complete metric space.

6. (10%) The set $\{A \in \mathcal{M} : \rho(A, \phi) \in [0,1]\}$ is closed in $(\mathcal{M} : \rho)$. Here ϕ denotes the empty set.

IV. Let $S_n(a)$ denote the $n-ball$ of radius a given by

$$S_n(a) = \left\{(x_1, x_2, \cdots, x_n) : \sum_{k=1}^{n} x_k^2 \le a^2\right\},$$

and let

$$V_n(a) = \int_{S_n(a)} \cdots \int dx_1 \cdots dx_n,$$

the volume of $S_n(a)$.

7.(10%) Prove that $V_n(1) = \frac{2\pi}{n} V_{n-2}(1)$ for $n \ge 3$.

8.(15%) Prove that $V_n(a) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)} a^n$, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for $x > 0$.