國立政治大學 應用數學系

碩士 學位論文

具時間延遲之霍普菲爾神經網路的多重穩定性

Multistability in Hopfield-type neural networks with delays

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中文摘要

這篇論文研究具多重穩定性之時間延遲型霍普菲爾神經網路。我們以兩個神經元所組成的神經網路來表現我們的想法。運用方程式的幾何結構，我們可推導出各種使網路具有不同數量固定點的條件，我們可以進一步建立網路系統的全局收斂性。

關鍵字：神經網路、多重穩定性、時間延遲、收斂性

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Abstract

We study multistable Hopfield-type neural networks with delays in this paper. We illustrate the idea of our approach by considering the networks which consist of two neurons. From the geometric configuration of equations, we derive various criteria which lead to disparate numbers of equilibria. We can further establish the convergence of dynamics for the networks.

Keywords: neural network, multistability, delays, convergence.

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Chapter 1

Introduction

In recent decades, neural networks have attracted considerable research interest. The famous neural network models include Hopfield model [6] and Cohen-Grossberg model [4]. On the other hand, due to the transmission of signal across the units, time delays have been incorporated into neural network models, cf [1], [2], [9]. Multistability is a notion to describe the coexistence of multiple stable equilibria or cycles, and is essential in some applications of neural networks including pattern recognition [6] and associative memory storage [5].

In this paper, we consider the delayed Hopfield-type neural network:

$$\frac{dx_i(t)}{dt} = -\mu_i x_i(t) + \sum_{j=1}^{n}[\alpha_{ij} g_j(x_j(t)) + \beta_{ij} g_j(x_j(t-\tau_{ij}))] + I_i, \tag{1.0.1}$$

where $i = 1, \cdots, n$, $\mu_i > 0$, α_{ij}, β_{ij} are connection weights from neuron j to neuron i, $g_j(\cdot)$ are activation functions, $\tau_{ij} \geq 0$ are time lags which are bounded above by τ_M, and I_i stand for independent bias current sources.

A typical class of activation functions is

$$\text{class } \mathcal{A} : \begin{cases} g_i \in \mathcal{C}^2(\mathbb{R}),\ g_i'(\xi) > 0, \text{ for all } \xi \in \mathbb{R}, \\ (\xi - \sigma_i) g_i''(\xi) < 0, \text{ for all } \xi \in \mathbb{R},\ \xi \neq \sigma_i, \\ \lim_{\xi \to +\infty} g_i(\xi) = v_i,\ \lim_{\xi \to -\infty} g_i(\xi) = u_i, \end{cases}$$

where v_i, u_i, and σ_i are constants with $u_i < v_i$, i= 1, 2, ..., n. Let $\rho_i := max\{|u_i|, |v_i|\}$.

The typical examples of activation functions in class $\mathcal{A}$ include $g_i(\xi) = tanh(\xi)$ and $g_i(\xi) = 1/[1 + exp(-c\xi)]$ for some $c > 0$, cf Figure 1.1.

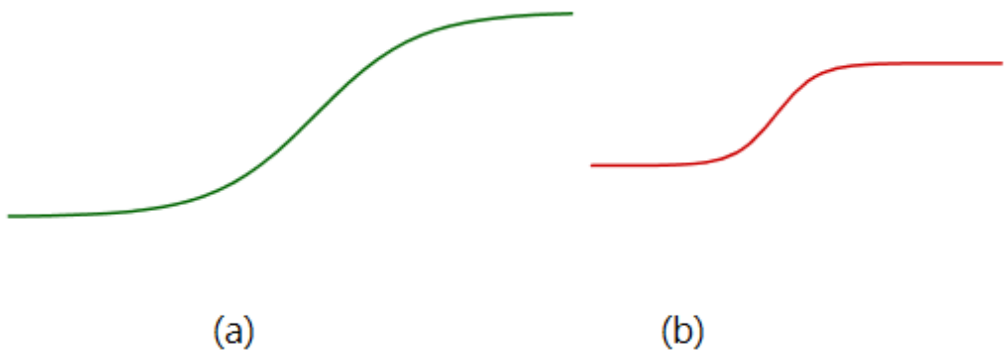

Figure 1.1: (a) $g_i(\xi) = tanh(\xi)$, (b) $g_i(\xi) = 1/[1 + exp(-c\xi)]$.

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Chapter 2

Literature review and study motivation

In this section, we introduce some existing results in [3], and some problems which are not solved in [3]. We shall address these unsloved problems in Section 3.

2.1 General cases

Set $\mathcal{N} = \{1, 2, \cdots, n\}$. The stationary equations for (1.0.1) are

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0, \tag{2.1.1}$$

for $i \in \mathcal{N}$. Set $\mathbf{F} = (F_1, \cdots, F_n)$ and $\mathbf{x} = (x_1, \cdots, x_n)$. For each $i \in \mathcal{N}$,

$$f_i(\xi) := -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + J_i, \tag{2.1.2}$$

where $\xi \in \mathbb{R}$. Then, we define the following vertical shifts of f_i:

$$\hat{f}_i(\xi) := -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + k_i^+, \tag{2.1.3}$$

$$\check{f}_i(\xi) := -\mu_i \xi + (\alpha_{ii} + \beta_{ii}) g_i(\xi) + k_i^-, \tag{2.1.4}$$

for $i \in \mathcal{N}$, where $k_i^+ := \sum_{j=1,j\neq i}^n \rho_j|\alpha_{ij}+\beta_{ij}| + I_i$, $k_i^- := -\sum_{j=1,j\neq i}^n \rho_j|\alpha_{ij}+\beta_{ij}| + I_i$. It follows that

$$\check{f}_i(x_i) \leq F_i(\mathbf{x}) \leq \hat{f}_i(x_i),$$

for all $\mathbf{x} = (x_1, \cdots, x_n)$ and $i \in \mathcal{N}$.

For given parameters μ_i, α_{ii}, β_{ii}, $i \in \mathcal{N}$, we denote

$$\mathcal{M} := \{i \in \mathcal{N} | \max_{\xi\in\mathbb{R}} g_i'(\xi) \leq \frac{\mu_i}{\alpha_{ii}+\beta_{ii}}\},$$
$$\mathcal{B} := \{i \in \mathcal{N} | \inf_{\xi\in\mathbb{R}} g_i'(\xi) < \frac{\mu_i}{\alpha_{ii}+\beta_{ii}} < \max_{\xi\in\mathbb{R}} g_i'(\xi)\}.$$

The notations $\mathcal{M}$ and $\mathcal{B}$ are related to the sense of monostable and (potentially) bistable scenarios, respectively.

Lemma 2.1. *([3]) If $i \in \mathcal{B}$, there exist two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \sigma_i < \tilde{q}_i$ such that $f_i'(\tilde{p}_i) = f_i'(\tilde{q}_i) = 0$, or equivalently, $g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = \mu_i/(\alpha_{ii}+\beta_{ii})$.*

We consider six disjoint subsets of $\mathcal{B}$, cf Figure 2.1:

$$\mathcal{B}_{\mathrm{r}}^{\mathrm{r}} = \{i \in \mathcal{N} | i \in \mathcal{B},\ \check{f}_i(\tilde{p}_i) > 0\},$$
$$\mathcal{B}_{\mathrm{l}}^{\mathrm{l}} = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{q}_i) < 0\},$$
$$\mathcal{B}_3^3 = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) < 0,\ \check{f}_i(\tilde{q}_i) > 0\},$$
$$\mathcal{B}_3^{\mathrm{r}} = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) > 0,\ \check{f}_i(\tilde{p}_i) < 0,\ \check{f}_i(\tilde{q}_i) > 0\},$$
$$\mathcal{B}_{\mathrm{l}}^3 = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) < 0,\ \hat{f}_i(\tilde{q}_i) > 0,\ \check{f}_i(\tilde{q}_i) < 0\},$$
$$\mathcal{B}_{\mathrm{l}}^{\mathrm{r}} = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) > 0,\ \check{f}_i(\tilde{q}_i) < 0\}.$$

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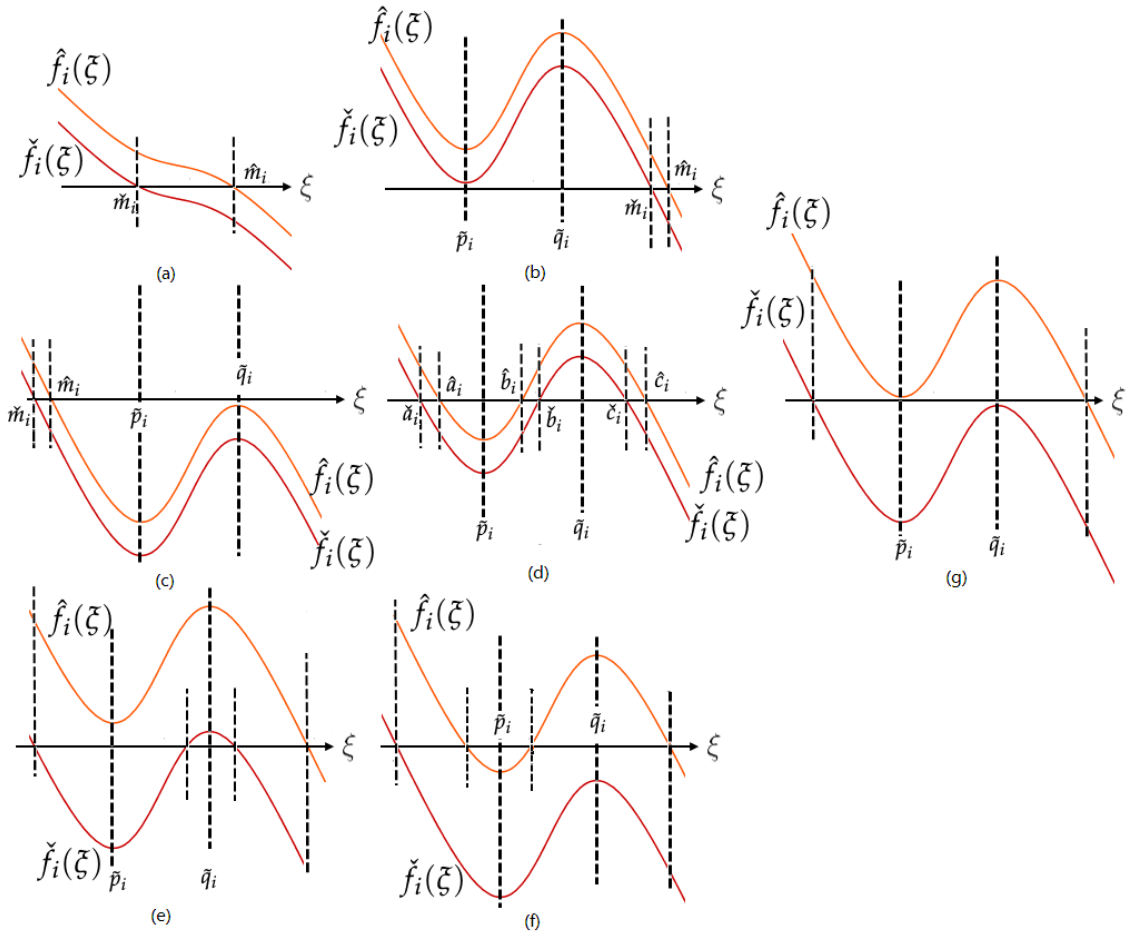

Figure 2.1: Types $\mathcal{M}$, $\mathcal{B}_{\mathrm{r}}^{\mathrm{r}}$, $\mathcal{B}_{1}^{\mathrm{l}}$, $\mathcal{B}_{3}^{3}$, $\mathcal{B}_{\mathrm{r}}^{3}$, $\mathcal{B}_{3}^{\mathrm{l}}$, and $\mathcal{B}_{\mathrm{r}}^{\mathrm{l}}$.

If $\mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{1}^{\mathrm{l}} \cup \mathcal{B}_{3}^{3} = \mathcal{N}$ and $k = card(\mathcal{B}_{3}^{3}) \geq 1$, there exist 3^k disjoint closed regions in $\mathbb{R}^n$, as follows:

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{2.1.5}$$

$$\mathbf{w} = (w_1, \cdots, w_n),$$

$$w_i = \text{“l”}, \text{“m”}, \text{“r”}, \text{ for } i \in \mathcal{B}_3^3,$$

$$w_i = \text{“s”}, \text{ for } i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{1}^{\mathrm{l}},$$

where $\tilde{\Omega}_i^{\mathrm{l}} = [\check{a}_i, \hat{a}_i]$, $\tilde{\Omega}_i^{\mathrm{m}} = [\hat{b}_i, \check{b}_i]$, $\tilde{\Omega}_i^{\mathrm{r}} = [\check{c}_i, \hat{c}_i]$ and $\tilde{\Omega}_i^{\mathrm{s}} = [\check{m}_i, \hat{m}_i]$ are compact intervals. With $\tilde{\Omega}^{\mathbf{w}}$ mentioned above, the author define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$. By applying Brouwer's fixed-point theorem and contraction mapping theorem, they proved in Theorem 2.1 that there

exist exactly 3^k equilibria for (1.0.1). Let $L_i := g_i'(\sigma_i)$, $i \in \mathcal{N}$.

Theorem 2.1. *([3]) If $\mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}} \cup \mathcal{B}_3^3 = \mathcal{N}$ and $k = card(\mathcal{B}_3^3) \geq 1$, then there exist 3^k equilibria in system* (1.0.1)*.*

Theorem 2.2. *([3]) Assume that $\mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}} \cup \mathcal{B}_3^3 = \mathcal{N}$ with $k = card(\mathcal{B}_3^3) \geq 1$. For each $i \in \mathcal{N}$, fix a $\theta_i \in (0, \mu_i)$ and then define*

$$\bar{L}_i := \begin{cases} \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}, \\ L_i, & \text{if } i \in \mathcal{B}_3^3. \end{cases} \tag{2.1.6}$$

If the parameters satisfy

$$\theta_i > \sum_{j=1, j \neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}|, \tag{2.1.7}$$

and

$$g_i'(\xi) \begin{cases} < \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in [\check{m}_i, \hat{m}_i],\ i \in \mathcal{M} \cup \mathcal{B}_{\mathrm{r}}^{\mathrm{r}} \cup \mathcal{B}_{\mathrm{l}}^{\mathrm{l}}, \\ < \frac{\mu_i - \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in (-\infty, \hat{a}_i] \cup [\check{c}_i, \infty),\ i \in \mathcal{B}_3^3, \\ > \frac{\mu_i + \theta_i}{\alpha_{ii} + \beta_{ii}}, & \text{if } \xi \in [\hat{b}_i, \check{b}_i],\ i \in \mathcal{B}_3^3, \end{cases} \tag{2.1.8}$$

for all $i \in \mathcal{N}$, then there exist exactly 3^k equilibria in system (1.0.1)*, and each region $\tilde{\Omega}^{\mathbf{w}}$, defined in* (2.1.5)*, contains exactly one of these 3^k equilibria.*

2.2 Other cases for $n = 2$

This section considers (1.0.1) with $n = 2$. System (1.0.1) with $n = 2$ reads as

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\mu_1 x_1(t) + \alpha_{11} g_1(x_1(t)) + \alpha_{12} g_2(x_2(t)) \\ &\quad + \beta_{11} g_1(x_1(t - \tau_{11})) + \beta_{12} g_2(x_2(t - \tau_{12})) + I_1, \end{aligned} \tag{2.2.1}$$

$$\begin{aligned} \frac{dx_2(t)}{dt} &= -\mu_2 x_2(t) + \alpha_{21} g_1(x_1(t)) + \alpha_{22} g_2(x_2(t)) \\ &\quad + \beta_{21} g_1(x_1(t - \tau_{21})) + \beta_{22} g_2(x_2(t - \tau_{22})) + I_2. \end{aligned} \tag{2.2.2}$$

For this two-neurons system, there are four basic types: $(\mathcal{M},\mathcal{M})$, $(\mathcal{M},\mathcal{B})$, $(\mathcal{B},\mathcal{M})$, and $(\mathcal{B},\mathcal{B})$, which correspond to $1,2 \in \mathcal{M}$, $1 \in \mathcal{M}$ and $2 \in \mathcal{B}$, $1 \in \mathcal{B}$ and $2 \in \mathcal{M}$, and $1,2 \in \mathcal{B}$, respectively. We further denote the following subtypes of these four types:

Notation 2.1. *([3]) Denote subtype* $\binom{\mathrm{m}}{\mathrm{m}}\binom{\mathrm{m}}{\mathrm{m}}$ *if* $1,2 \in \mathcal{M}$, $\binom{\mathrm{m}}{\mathrm{m}}\binom{\star}{\bullet}$ *if* $1 \in \mathcal{M}$ *and* $2 \in \mathcal{B}^{\star}_{\bullet}$, $\binom{\star}{\bullet}\binom{\mathrm{m}}{\mathrm{m}}$ *if* $1 \in \mathcal{B}^{\star}_{\bullet}$ *and* $2 \in \mathcal{M}$, *and* $\binom{\star}{\bullet}\binom{\star'}{\bullet'}$ *if* $1 \in \mathcal{B}^{\star}_{\bullet}$ *and* $2 \in \mathcal{B}^{\star'}_{\bullet'}$.

Type	Subtype	Cases
$(\mathcal{M},\mathcal{M})$	T_1	$\binom{\mathrm{m}}{\mathrm{m}}\binom{\mathrm{m}}{\mathrm{m}}$
$(\mathcal{M},\mathcal{B})$	T_2	$\binom{\mathrm{m}}{\mathrm{m}}\binom{\mathrm{r}}{\mathrm{r}}, \binom{\mathrm{m}}{\mathrm{m}}\binom{\mathrm{l}}{\mathrm{l}}$
	T_3	$\binom{\mathrm{m}}{\mathrm{m}}\binom{3}{3}$
	T_4	$\binom{\mathrm{m}}{\mathrm{m}}\binom{\mathrm{r}}{3}, \binom{\mathrm{m}}{\mathrm{m}}\binom{3}{\mathrm{l}}$
	T_5	$\binom{\mathrm{m}}{\mathrm{m}}\binom{\mathrm{r}}{\mathrm{l}}$
$(\mathcal{B},\mathcal{M})$	T_6	$\binom{\mathrm{r}}{\mathrm{r}}\binom{\mathrm{m}}{\mathrm{m}}, \binom{\mathrm{l}}{\mathrm{l}}\binom{\mathrm{m}}{\mathrm{m}}$
	T_7	$\binom{3}{3}\binom{\mathrm{m}}{\mathrm{m}}$
	T_8	$\binom{\mathrm{r}}{3}\binom{\mathrm{m}}{\mathrm{m}}, \binom{3}{\mathrm{l}}\binom{\mathrm{m}}{\mathrm{m}}$
	T_9	$\binom{\mathrm{r}}{\mathrm{l}}\binom{\mathrm{m}}{\mathrm{m}}$
$(\mathcal{B},\mathcal{B})$	T_{10}	$\binom{\mathrm{r}}{\mathrm{r}}\binom{\mathrm{r}}{\mathrm{r}}, \binom{\mathrm{l}}{\mathrm{l}}\binom{\mathrm{l}}{\mathrm{l}}, \binom{\mathrm{r}}{\mathrm{r}}\binom{\mathrm{l}}{\mathrm{l}}, \binom{\mathrm{l}}{\mathrm{l}}\binom{\mathrm{r}}{\mathrm{r}}$
	T_{11}	$\binom{\mathrm{r}}{\mathrm{r}}\binom{3}{3}, \binom{\mathrm{l}}{\mathrm{l}}\binom{3}{3}, \binom{3}{3}\binom{\mathrm{r}}{\mathrm{r}}, \binom{3}{3}\binom{\mathrm{l}}{\mathrm{l}}$
	T_{12}	$\binom{\mathrm{r}}{\mathrm{r}}\binom{\mathrm{r}}{3}, \binom{\mathrm{r}}{\mathrm{r}}\binom{3}{\mathrm{l}}, \binom{\mathrm{l}}{\mathrm{l}}\binom{\mathrm{r}}{3}, \binom{\mathrm{l}}{\mathrm{l}}\binom{3}{\mathrm{l}}$
		$\binom{\mathrm{r}}{3}\binom{\mathrm{r}}{\mathrm{r}}, \binom{\mathrm{r}}{3}\binom{\mathrm{l}}{\mathrm{l}}, \binom{3}{\mathrm{l}}\binom{\mathrm{r}}{\mathrm{r}}, \binom{3}{\mathrm{l}}\binom{\mathrm{l}}{\mathrm{l}}$
	T_{13}	$\binom{\mathrm{r}}{\mathrm{r}}\binom{\mathrm{r}}{\mathrm{l}}, \binom{\mathrm{l}}{\mathrm{l}}\binom{\mathrm{r}}{\mathrm{l}}, \binom{\mathrm{r}}{\mathrm{l}}\binom{\mathrm{r}}{\mathrm{r}}, \binom{\mathrm{r}}{\mathrm{l}}\binom{\mathrm{l}}{\mathrm{l}}$
	T_{14}	$\binom{3}{3}\binom{3}{3}$
	T_{15}	$\binom{3}{3}\binom{\mathrm{r}}{3}, \binom{3}{3}\binom{3}{\mathrm{l}}, \binom{\mathrm{r}}{3}\binom{3}{3}, \binom{3}{\mathrm{l}}\binom{3}{3}$
	T_{16}	$\binom{3}{3}\binom{\mathrm{r}}{\mathrm{l}}, \binom{\mathrm{r}}{\mathrm{l}}\binom{3}{3}$
	T_{17}	$\binom{\mathrm{r}}{3}\binom{\mathrm{r}}{3}, \binom{3}{\mathrm{l}}\binom{3}{\mathrm{l}}$
	T_{18}	$\binom{\mathrm{r}}{3}\binom{3}{\mathrm{l}}, \binom{3}{\mathrm{l}}\binom{\mathrm{r}}{3}$
	T_{19}	$\binom{\mathrm{r}}{3}\binom{\mathrm{r}}{\mathrm{l}}, \binom{3}{\mathrm{l}}\binom{\mathrm{r}}{\mathrm{l}}, \binom{\mathrm{r}}{\mathrm{l}}\binom{\mathrm{r}}{3}, \binom{\mathrm{r}}{\mathrm{l}}\binom{3}{\mathrm{l}}$
	T_{20}	$\binom{\mathrm{r}}{\mathrm{l}}\binom{\mathrm{r}}{\mathrm{l}}$

Table 2.1: Subtypes in $(\mathcal{M},\mathcal{M})$, $(\mathcal{M},\mathcal{B})$, $(\mathcal{B},\mathcal{M})$, and $(\mathcal{B},\mathcal{B})$.

The authors list these twenty subtypes T_k, $k = 1,\cdots,20$, in Table 2.1. The cases of T_1, T_2, T_3, T_6, T_7, T_{10}, T_{11}, and T_{14} have been discussed in [3]. We need the following notations to further describe the geometric configurations of the upper and lower functions $\check{f}_i$ and $\hat{f}_i$.

Notation 2.2. *([3])*

1. *For an interval* $\mathcal{I} = [\varrho, \varsigma]$, *we say that* $\mathcal{I} > 0$ (< 0) *if* $\varrho > 0$ $(\varsigma < 0)$. *Denote* $c\mathcal{I} := [c\varrho, c\varsigma]$, *if* $c \geq 0$, $c\mathcal{I} := [c\varsigma, c\varrho]$, *if* $c < 0$; $\beta + \mathcal{I} := [\beta + \varrho, \beta + \varsigma]$, *for a real number* β; $g_i(\mathcal{I}) := [g_i(\varrho), g_i(\varsigma)]$ *for increasing function* g_i.

2. *For intervals* $\mathcal{I}_k = [\varrho_k, \varsigma_k]$, $k = 1, 2$, $\mathcal{I}_1 + \mathcal{I}_2 := [\varrho_1 + \varrho_2, \varsigma_1 + \varsigma_2]$, *is an interval.*

3. *For a given* $\eta \in \mathbb{R}$, *define functions* $f_i^{\eta}(\xi) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + (\alpha_{ij} + \beta_{ij})g_j(\eta) + I_i$, *where* $i, j \in \{1, 2\}$ *and* $j \neq i$.

4. *For each* ξ, *define intervals* $K_i(\xi; \mathcal{I}) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + (\alpha_{ij} + \beta_{ij})g_j(\mathcal{I}) + I_i = \{f_i^{\eta}(\xi) : \eta \in \mathcal{I}\}$, *where* $i, j \in \{1, 2\}$, $j \neq i$. *Regarding* ξ *as a variable, denote by* $K_i(\xi; [\varrho, \varsigma])$ *an interval-valued function of* ξ.

Note that the graph of $K_i(\xi; [\varrho, \varsigma])$ is a strip that lies between the graphs of functions $f_i^{\varrho}(\xi)$ and $f_i^{\varsigma}(\xi)$.

Theorem 2.3. *([3]) Consider the case* $\binom{3}{3}\binom{r}{3}$. *Then there exist parameters so that system* (2.2.1)-(2.2.2) *have three, five, seven, or nine equilibria. The criterion for each of these existence is listed in Table 2.2.*

Some notations in Table 2.2 are introduced follow. If $i = 1 \in \mathcal{B}_3^3$, $\check{f}_1$ and $\hat{f}_1$ have exactly three zeros, respectively, say $\check{f}_1(\check{a}_1) = \check{f}_1(\check{b}_1) = \check{f}_1(\check{c}_1) = 0$ and $\hat{f}_1(\hat{a}_1) = \hat{f}_1(\hat{b}_1) = \hat{f}_1(\hat{c}_1) = 0$ with $\check{a}_1 < \hat{a}_1 < \hat{b}_1 < \check{b}_1 < \check{c}_1 < \hat{c}_1$. Denote $A_1 := [\check{a}_1, \hat{a}_1]$, $B_1 := [\hat{b}_1, \check{b}_1]$, and $C_1 := [\check{c}_1, \hat{c}_1]$.

Conditions	♯ Equi.
$\alpha_{21} + \beta_{21} > 0$	
$K_2(\tilde{p}_2; A_1) > 0$	3
$K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$	5
$K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$	7
$K_2(\tilde{p}_2; C_1) < 0$	9
$\alpha_{21} + \beta_{21} < 0$	
$K_2(\tilde{p}_2; C_1) > 0$	3
$K_2(\tilde{p}_2; C_1) < 0 < K_2(\tilde{p}_2; B_1)$	5
$K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; A_1)$	7
$K_2(\tilde{p}_2; A_1) < 0$	9

Table 2.2: Conditions for various numbers of equilibria for the case $\binom{3}{3}\binom{r}{3}$.

In [3], the authors established the existence of equilibria of (2.2.1)-(2.2.2) for case $\binom{3}{3}\binom{r}{3}$. In the next chapter, we further confirm the exact number of equilibria of the system (2.2.1)-(2.2.2) for case $\binom{3}{3}\binom{r}{3}$ and $\binom{3}{l}\binom{r}{3}$.

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Chapter 3

Main results

In this section, we shall study the exact number of equilibria of system (2.2.1)-(2.2.2) for cases $\binom{3}{3}\binom{\mathrm{r}}{3}$ and $\binom{3}{l}\binom{\mathrm{r}}{3}$ in Section 3.1 and 3.2, respectively. We shall establish the global convergence of dynamics of system (2.2.1)-(2.2.2) for case $\binom{3}{3}\binom{\mathrm{r}}{3}$ under condition $K_2(\tilde{p}_2; A_1) > 0$.

3.1 Exact number of equilibria for case $\binom{3}{3}\binom{\mathrm{r}}{3}$

In this case, since $i = 1 \in \mathcal{B}_3^3$, each of $\check{f}_1$ and $\hat{f}_1$ has three zeros, $\check{f}_1(\check{a}_1) = \check{f}_1(\check{b}_1) = \check{f}_1(\check{c}_1) = 0$ and $\hat{f}_1(\hat{a}_1) = \hat{f}_1(\hat{b}_1) = \hat{f}_1(\hat{c}_1) = 0$, with $\check{a}_1 < \hat{a}_1 < \hat{b}_1 < \check{b}_1 < \check{c}_1 < \hat{c}_1$. Denote intervals $A_1 := [\check{a}_1, \hat{a}_1]$, $B_1 := [\hat{b}_1, \check{b}_1]$, and $C_1 := [\check{c}_1, \hat{c}_1]$.

3.1.1 $K_2(\tilde{p}_2; A_1) > 0$

Recall Table 2.2. We shall show that there exist exactly 3 equilibria of system (2.2.1)-(2.2.2) under $K_2(\tilde{p}_2; A_1) > 0$ and some additional conditions. Under the condition $K_2(\tilde{p}_2; A_1) > 0$, each of functions $f_2^{\check{a}_1}(\xi)$ and $f_2^{\hat{a}_1}(\xi)$ has exactly one zero, say $f_2^{\check{a}_1}(\check{m}_2{}^{(a_1)}) = 0$ and $f_2^{\hat{a}_1}(\hat{m}_2{}^{(a_1)}) = 0$, with $\check{m}_2{}^{(a_1)} < \hat{m}_2{}^{(a_1)}$, cf Figure 3.1. Denote intervals $S_2^{A_1} = [\check{m}_2{}^{(a_1)}, \hat{m}_2^{(a_1)}]$, cf Figure 3.2.

Because $K_2(\tilde{p}_2; A_1) > 0$ implies $K_2(\tilde{p}_2; B_1) > 0$ and $K_2(\tilde{p}_2; C_1) > 0$, each of $f_2^{\hat{b}_1}$, $f_2^{\check{b}_1}$, $f_2^{\check{c}_1}$, and $f_2^{\hat{c}_1}$ has exactly one zero, $f_2^{\hat{b}_1}(\hat{m}_2{}^{(b_1)}) = 0$, $f_2^{\check{b}_1}(\check{m}_2^{(b_1)}) = 0$, $f_2^{\check{c}_1}(\check{m}_2{}^{(c_1)}) = 0$, and $f_2^{\hat{c}_1}(\hat{m}_2^{(c_1)}) = 0$. We can define $S_2^{B_1} = [\hat{m}_2{}^{(b_1)}, \check{m}_2^{(b_1)}]$ and $S_2^{C_1} = [\check{m}_2{}^{(c_1)}, \hat{m}_2^{(c_1)}]$, cf Figure 3.2.

Denote intervals $S_2 := [\check{m}_2^{(a_1)}, \hat{m}_2^{(c_1)}]$. Notably, $S_2^{A_1} \cup S_2^{B_1} \cup S_2^{C_1} \subseteq S_2$, cf Figure 3.1.

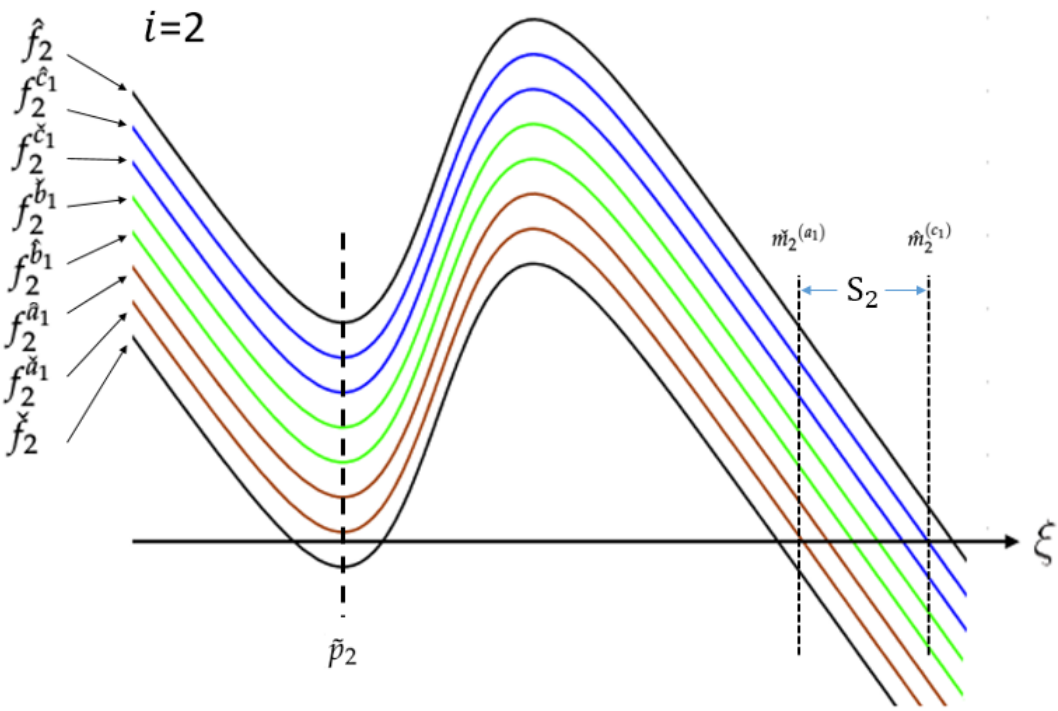

Figure 3.1: The configurations of functions $\hat{f}_2(\xi)$, $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$, $\check{f}_2(\xi)$ under $K_2(\tilde{p}_2; A_1) > 0$.

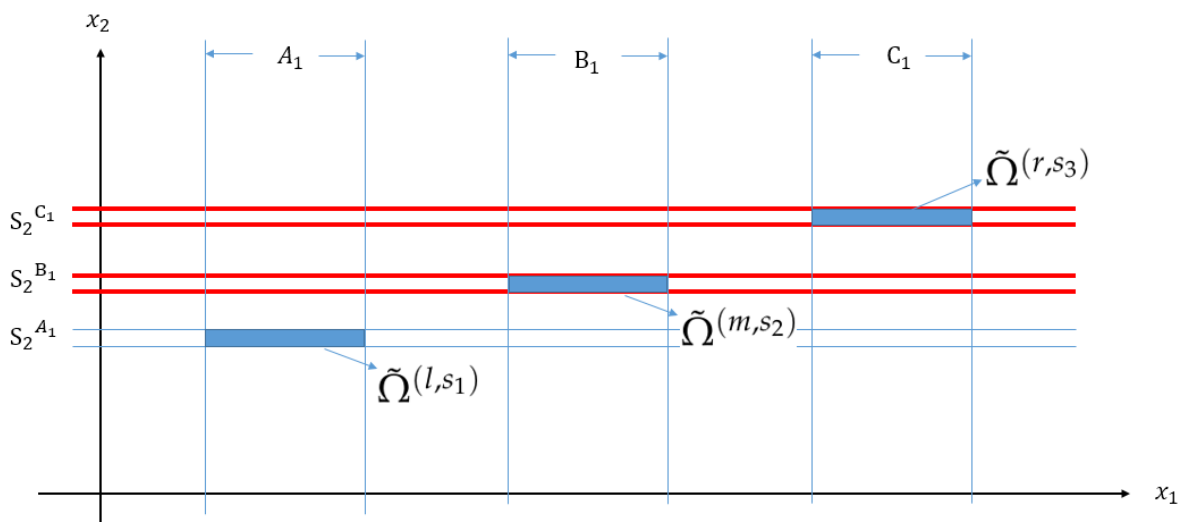

Figure 3.2: $K_2(\tilde{p}_2; A_1) > 0$.

Theorem 3.1. *Consider the case* $\binom{3}{3}\binom{r}{3}$, $\alpha_{21} + \beta_{21} > 0$, *and* $K_2(\tilde{p}_2; A_1) > 0$. *For each*

$i \in \{1,2\}$, *fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := \begin{cases} L_1, & \text{if } i = 1, \\ \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}} & \text{if } i = 2. \end{cases} \tag{3.1.1}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1,2\} \quad \text{and} \quad i \neq j \tag{3.1.2}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in [\hat{b}_1, \check{b}_1], \end{cases} \tag{3.1.3}$$

$$g_2'(\xi) < \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, \quad \text{if } \xi \in [\check{m}_2{}^{(a_1)}, \hat{m}_2^{(c_1)}], \tag{3.1.4}$$

then there exist exactly 3 equilibria in system (2.2.1)-(2.2.2).

Proof. Consider the following three regions,

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.1.5}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m”}, \text{“r”},$$

$$w_2 = \begin{cases} \text{“s}_1\text{”}, & \text{if } w_1 = \text{“l”} \\ \text{“s}_2\text{”}, & \text{if } w_1 = \text{“m”}, \\ \text{“s}_3\text{”}, & \text{if } w_1 = \text{“r”}, \end{cases} \tag{3.1.6}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1 = [\check{a}_1, \hat{a}_1]$, $\tilde{\Omega}_1^{\mathrm{m}} = B_1 = [\hat{b}_1, \check{b}_1]$, $\tilde{\Omega}_1^{\mathrm{r}} = C_1 = [\check{c}_1, \hat{c}_1]$, $\tilde{\Omega}_2^{\mathrm{s}_1} = S_2^{A_1} = [\check{m}_2{}^{(a_1)}, \hat{m}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{s}_2} = S_2^{B_1} = [\hat{m}_2{}^{(b_1)}, \check{m}_2^{(b_1)}]$, and $\tilde{\Omega}_2^{\mathrm{s}_3} = S_2^{C_1} = [\check{m}_2{}^{(c_1)}, \hat{m}_2^{(c_1)}]$ are compact intervals, cf Figure 3.2. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be any one of the 3 regions defined in (3.1.5).

For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.1.7}$$

for $i, j \in \{1, 2\}$ and $j \neq i$. If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,s_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,s_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,s_3)}$, function h_2 is bounded by $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{C_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in C_1 = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.1.7). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist three equilibria for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii}) g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into three cases.

(i)If w_1 =“m” and w_2 =“s_2”, then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1, \check{b}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{m}_2{}^{(b_1)}, \check{m}_2^{(b_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.1.3).

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12}) g_2'(\eta_2^*)(x_2 - y_2)| / |(\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2 |\alpha_{12} + \beta_{12}|]/\theta_1\} \|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2 |\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.1.2).

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.4), which yields

$$|(\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1 |\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1 |\alpha_{21} + \beta_{21}|]/\theta_2\} \|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1 |\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.1.2).

(ii)If w_1 =“l” and w_2 =“s_1”, then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2{}^{(a_1)}, \hat{m}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.1.3), which yields

$$|(\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) > \theta_1,$$

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.4), which yields

$$|(\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.2).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.2).

(iii)If w_1 ="r" and w_2 ="s_3", then $x_1^*, y_1^*, \xi_1^* \in [\check{c}_1, \hat{c}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.3), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1,$$

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2-\theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.1.4), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.2).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.2).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i \bar{x}_i + \alpha_{ii} g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2} \alpha_{ij} g_j(\bar{x}_j) + \sum_{j=1}^{2} \beta_{ij} g_j(\bar{x}_j) + I_i = 0. \tag{3.1.8}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly 3 equilibria. □

3.1.2 $K_2(\tilde{p}_2; C_1) < 0$

Recall Table 2.2. we shall show that there exist exactly 9 equilibria under $K_2(\tilde{p}_2; C_1) < 0$ and some additional conditions.

If $K_2(\tilde{p}_2; C_1) < 0$ and $\alpha_{21} + \beta_{21} > 0$, each of $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$ has exactly three zeros, say $f_2^{\check{c}_1}(\check{a}_2^{(c_1)}) = f_2^{\check{c}_1}(\check{b}_2^{(c_1)}) = f_2^{\check{c}_1}(\check{c}_2^{(c_1)}) = 0$ and $f_2^{\hat{c}_1}(\hat{a}_2^{(c_1)}) = f_2^{\hat{c}_1}(\hat{b}_2^{(c_1)}) = f_2^{\hat{c}_1}(\hat{c}_2^{(c_1)}) = 0$, cf Figures 3.3 and 3.4.

Note that $K_2(\tilde{p}_2; C_1) < 0$ implies $K_2(\tilde{p}_2; A_1) < 0$ and $K_2(\tilde{p}_2; B_1) < 0$. Each of $f_2^{\check{a}_1}$, $f_2^{\hat{a}_1}$, $f_2^{\hat{b}_1}$, and $f_2^{\check{b}_1}$ has exactly three zeros, say $f_2^{\check{a}_1}(\check{a}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{b}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{c}_2^{(a_1)}) = 0$, $f_2^{\hat{a}_1}(\hat{a}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{b}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{c}_2^{(a_1)}) = 0$, $f_2^{\hat{b}_1}(\hat{a}_2^{(b_1)}) = f_2^{\hat{b}_1}(\hat{b}_2^{(b_1)}) = f_2^{\hat{b}_1}(\hat{c}_2^{(b_1)}) = 0$, and $f_2^{\check{b}_1}(\check{a}_2^{(b_1)}) = f_2^{\check{b}_1}(\check{b}_2^{(b_1)}) = f_2^{\check{b}_1}(\check{c}_2^{(b_1)}) = 0$, cf Figures 3.3 and 3.4.

We can define $A_2^{A_1} := [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$, $B_2^{A_1} := [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, $C_2^{A_1} := [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$, $A_2^{B_1} := [\hat{a}_2^{(b_1)}, \check{a}_2^{(b_1)}]$, $B_2^{B_1} := [\check{b}_2^{(b_1)}, \hat{b}_2^{(b_1)}]$, $C_2^{B_1} := [\hat{c}_2^{(b_1)}, \check{c}_2^{(b_1)}]$, $A_2^{C_1} := [\check{a}_2^{(a_1)}, \hat{a}_2^{(c_1)}]$, $B_2^{C_1} := [\hat{b}_2^{(c_1)}, \check{b}_2^{(c_1)}]$, $C_2^{C_1} := [\check{c}_2^{(c_1)}, \hat{c}_2^{(c_1)}]$, cf. Figure 3.5. Denote intervals $A_2 := [\check{a}_2^{(a_1)}, \hat{a}_2^{(c_1)}]$, $B_2 := [\hat{b}_2^{(c_1)}, \check{b}_2^{(a_1)}]$, and $C_2 := [\check{c}_2^{(a_1)}, \hat{c}_2^{(c_1)}]$, cf. Figure 3.3.

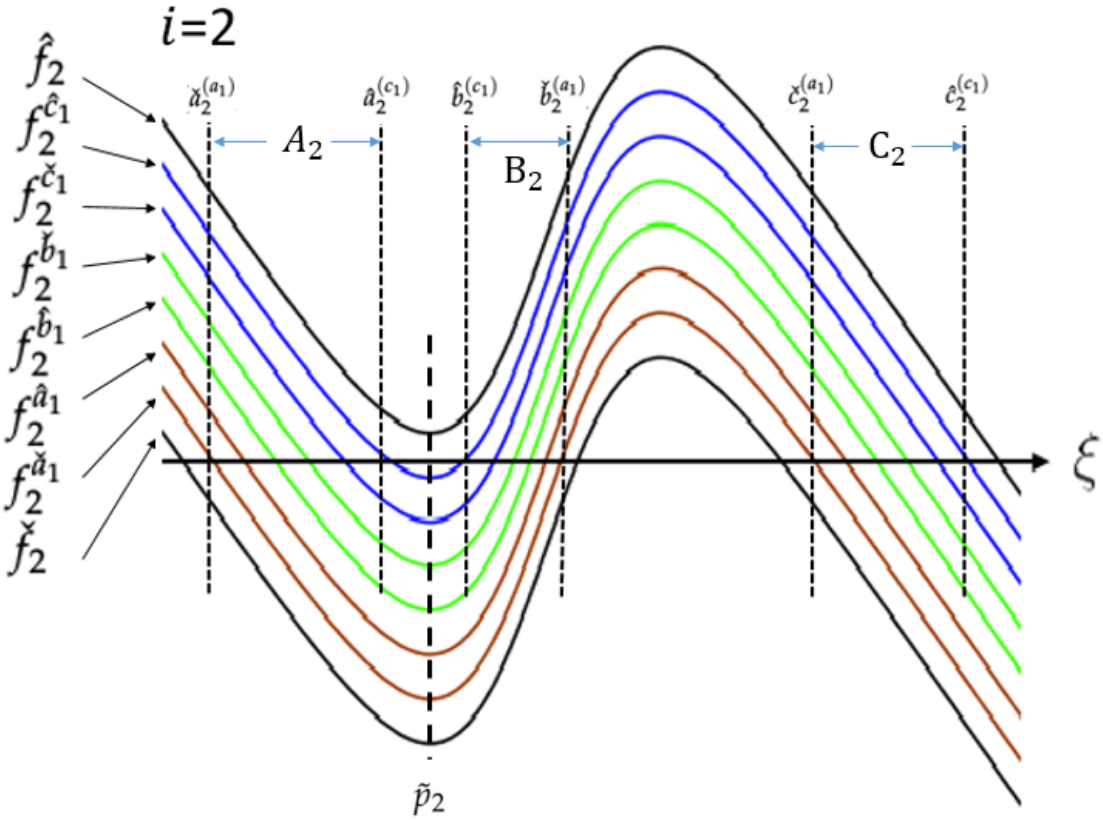

Figure 3.3: The configurations of functions $\hat{f}_2(\xi)$, $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$, $\check{f}_2(\xi)$ under $K_2(\tilde{p}_2; C_1) < 0$.

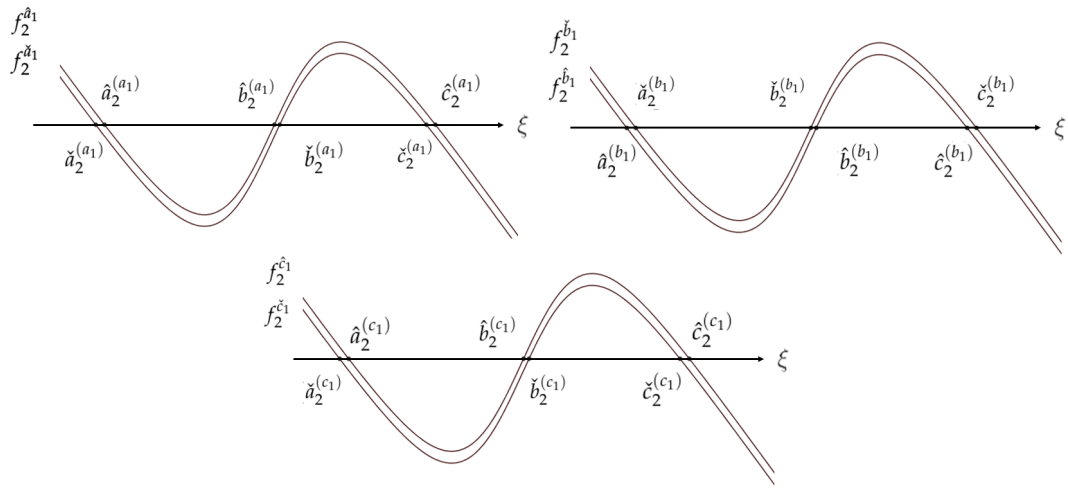

Figure 3.4: The configurations of functions $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$.

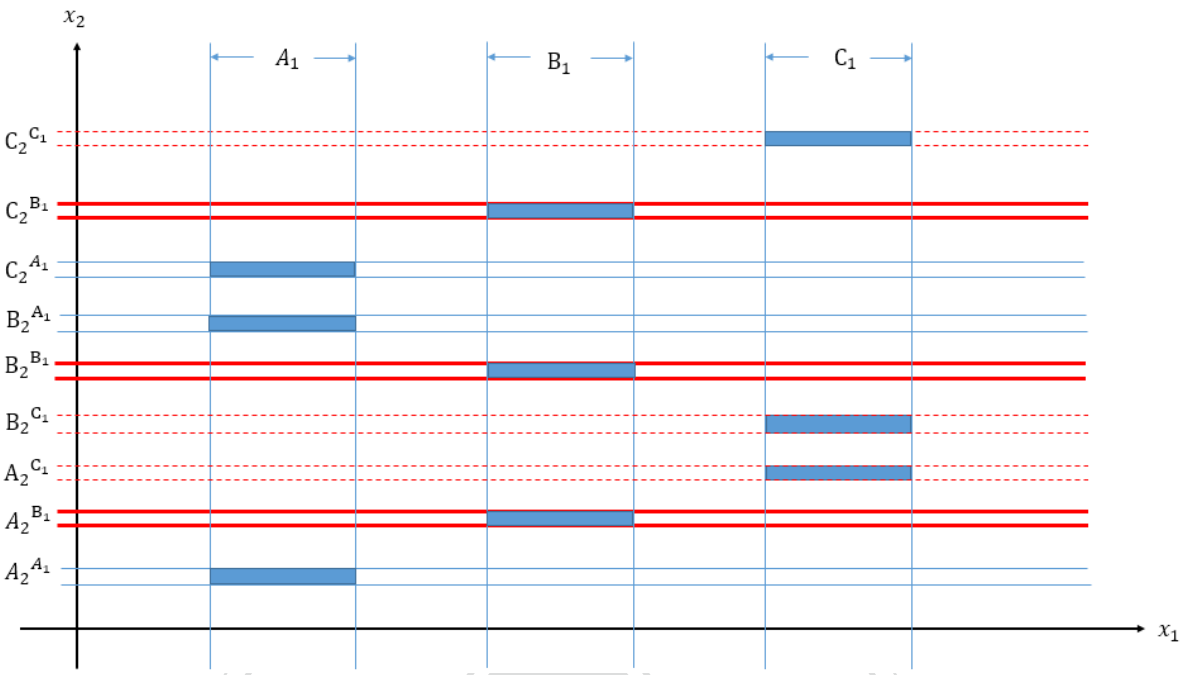

Figure 3.5: $K_2(\tilde{p}_2; C_1) < 0$.

Theorem 3.2. *Consider the case* $\binom{3}{3}\binom{r}{3}$, $\alpha_{21} + \beta_{21} > 0$, *and* $K_2(\tilde{p}_2; C_1) < 0$. *For each* $i \in \{1, 2\}$, *fix a* $\theta_i \in (0, \mu_i)$, *and then define*

$$\bar{L}_i := L_i, i = 1, 2. \tag{3.1.9}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1, 2\} \quad \textit{and} \quad i \neq j, \tag{3.1.10}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \textit{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \textit{if } \xi \in [\hat{b}_1, \check{b}_1], \end{cases} \tag{3.1.11}$$

$$g_2'(\xi) \begin{cases} < \frac{\mu_2-\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in (-\infty, \hat{a}_2^{(c_1)}] \cup [\check{c}_2^{(a_1)}, \infty), \\ > \frac{\mu_2+\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in [\hat{b}_2^{(c_1)}, \check{b}_2^{(a_1)}], \end{cases} \tag{3.1.12}$$

then there exist exactly 9 *equilibria in system* (2.2.1)-(2.2.2).

Proof. Consider the following nine regions

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.1.13}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m”}, \text{“r”},$$

$$w_2 = \begin{cases} \text{“l}_1\text{”}, \text{“m}_1\text{”}, \text{“r}_1\text{”}, & \text{if } w_1 = \text{“l”} \\ \text{“l}_2\text{”}, \text{“m}_2\text{”}, \text{“r}_2\text{”}, & \text{if } w_1 = \text{“m”}, \\ \text{“l}_3\text{”}, \text{“m}_3\text{”}, \text{“r}_3\text{”}, & \text{if } w_1 = \text{“r”}, \end{cases} \tag{3.1.14}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1$, $\tilde{\Omega}_1^{\mathrm{m}} = B_1$, $\tilde{\Omega}_1^{\mathrm{r}} = C_1$, $\tilde{\Omega}_2^{\mathrm{l}_1} = A_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{m}_1} = B_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{r}_1} = C_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{l}_2} = A_2^{B_1}$, $\tilde{\Omega}_2^{\mathrm{m}_2} = B_2^{B_1}$, $\tilde{\Omega}_2^{\mathrm{r}_2} = C_2^{B_1}$, $\tilde{\Omega}_2^{\mathrm{l}_3} = A_2^{C_1}$, $\tilde{\Omega}_2^{\mathrm{m}_3} = B_2^{C_1}$, and $\tilde{\Omega}_2^{\mathrm{r}_3} = C_2^{C_1}$ are compact intervals, cf. Figure 3.5. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be any one of the 9 regions defined in (3.1.13). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.1.15}$$

for $i, j \in \{1, 2\}$ and $j \neq i$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,l_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,m_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,r_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There

exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,l_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,m_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,r_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,l_3)}$, function h_2 is bounded by $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{C_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in C_1 = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,m_3)}$, function h_2 is bounded by $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{C_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in C_1 = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,r_3)}$, function h_2 is bounded by $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{C_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in C_1 = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.1.15). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist nine equilibria for system (2.2.1)-(2.2.2), and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij})g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into nine cases.

(i)If $w_1 =$"m" and $w_2 =$"m_2", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\check{b}_1, \hat{b}_1]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\hat{b}_2^{(b_1)}, \check{b}_2^{(b_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.1.11). On the other hand, $g_2'(\xi_2^*) > (\mu_2 + \theta_2)/(\alpha_{22} + \beta_{22})$ by (3.1.12).

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.1.10).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21} + \beta_{21})g_1'(\eta_1^*)(x_1 - y_1)|/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.1.10).

(ii)If $w_1 =$"m" and $w_2 =$"l_2", then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1, \check{b}_1]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\hat{a}_2^{(b_1)}, \check{a}_2^{(b_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.1.11). On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.12), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.10).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.10).

(iii)The situation for $w_1 =$"m" and $w_2 =$"r_2" is similar to (ii).

(iv)If $w_1 =$"l" and $w_2 =$"m_1", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.11), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1,$$

On the other hand, $g_2'(\xi_2^*) > (\mu_2+\theta_2)/(\alpha_{22}+\beta_{22})$ by (3.1.12).

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.10).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21}+\beta_{21})g_1'(\eta_1^*)(x_1-y_1)|/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.10).

(v)If $w_1 =$"l" and $w_2 =$"l_1", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.11), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

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On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.12), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.1.10).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.1.10).

(vi)The situation for $w_1 =$"l" and $w_2 =$ "r_1" is similar to (v).

(vii) The situation for $w_1 =$"r" and $w_2 =$ "m_3" is similar to case (iv).

(viii)The situation for $w_1 =$"r" and $w_2 =$ "l_3" is similar to case (v).

(ix)The situation for $w_1 =$"r" and $w_2 =$ "r_3" is similar to case (v).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\overline{x}_i + \alpha_{ii}g_i(\overline{x}_i) + \sum_{j=1, j\neq i}^{2} \alpha_{ij}g_j(\overline{x}_j) + \sum_{j=1}^{2} \beta_{ij}g_j(\overline{x}_j) + I_i = 0. \tag{3.1.16}$$

Thus, $\overline{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\overline{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly 9 equilibria. □

3.1.3 $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$

Recall Table 2.2. We shall show that there exist exactly 5 equilibria under $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$ and some additional conditions. Since $K_2(\tilde{p}_2; A_1) < 0$ and $\alpha_{21} + \beta_{21} > 0$, each of functions $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$ has three zeros, say $f_2^{\check{a}_1}(\check{a}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{b}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{c}_2^{(a_1)}) = 0$ and $f_2^{\hat{a}_1}(\hat{a}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{b}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{c}_2^{(a_1)}) = 0$, with $\check{a}_2^{(a_1)} < \hat{a}_2^{(a_1)} < \hat{b}_2^{(a_1)} < \check{b}_2^{(a_1)} < \check{c}_2^{(a_1)} < \hat{c}_2^{(a_1)}$, cf Figures 3.6 and 3.7. Since $K_2(\tilde{p}_2; B_1) > 0$ implies $K_2(\tilde{p}_2; C_1) > 0$, each of functions $f_2^{\hat{b}_1}$, $f_2^{\check{b}_1}$, $f_2^{\check{c}_1}$, and $f_2^{\hat{c}_1}$ has one zero, say $f_2^{\hat{b}_1}(\hat{m}_2^{(b_1)}) = 0$, $f_2^{\check{b}_1}(\check{m}_2^{(b_1)}) = 0$, $f_2^{\check{c}_1}(\check{m}_2^{(c_1)}) = 0$, and $f_2^{\hat{c}_1}(\hat{m}_2^{(c_1)}) = 0$, with $\hat{m}_2^{(b_1)} < \check{m}_2^{(b_1)} < \check{m}_2^{(c_1)} < \hat{m}_2^{(c_1)}$, cf Figures 3.6 and 3.7.

Denote intervals $A_2^{A_1} := [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$, $B_2^{A_1} := [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, $C_2^{A_1} := [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$, $S_2^{B_1} := [\hat{m}_2^{(b_1)}, \check{m}_2^{(b_1)}]$, and $S_2^{C_1} := [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$, cf. Figure 3.6.

There are five regions $A_1 \times A_2^{A_1}$, $A_1 \times B_2^{A_1}$, $A_1 \times C_2^{A_1}$, $B_1 \times S_2^{B_1}$, $C_1 \times S_2^{C_1}$, cf. Figure 3.8.

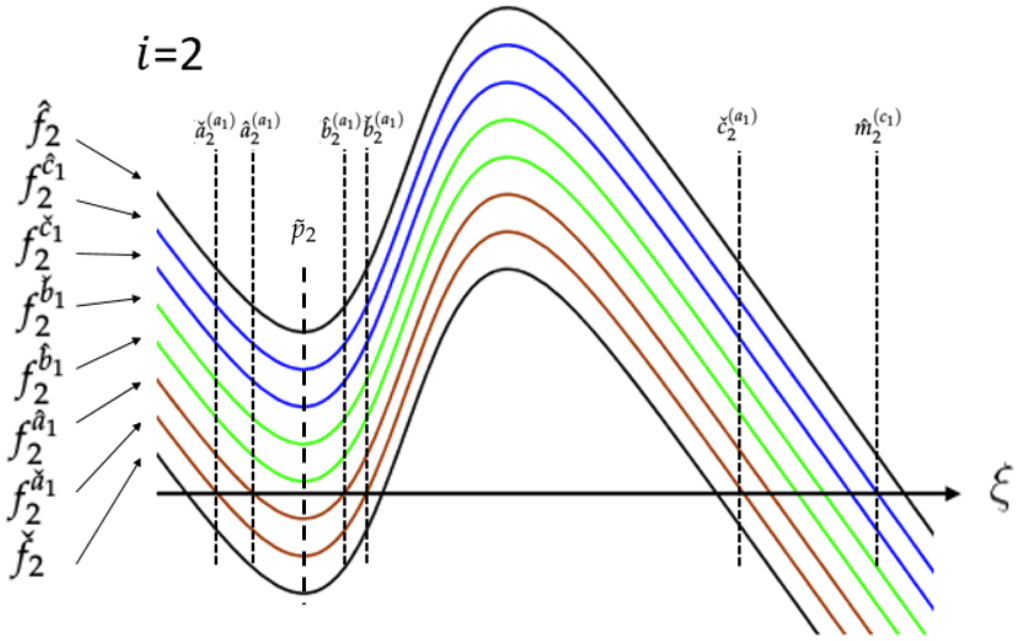

Figure 3.6: The configurations of functions $\hat{f}_2(\xi)$, $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$, $\check{f}_2(\xi)$ under condition $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$.

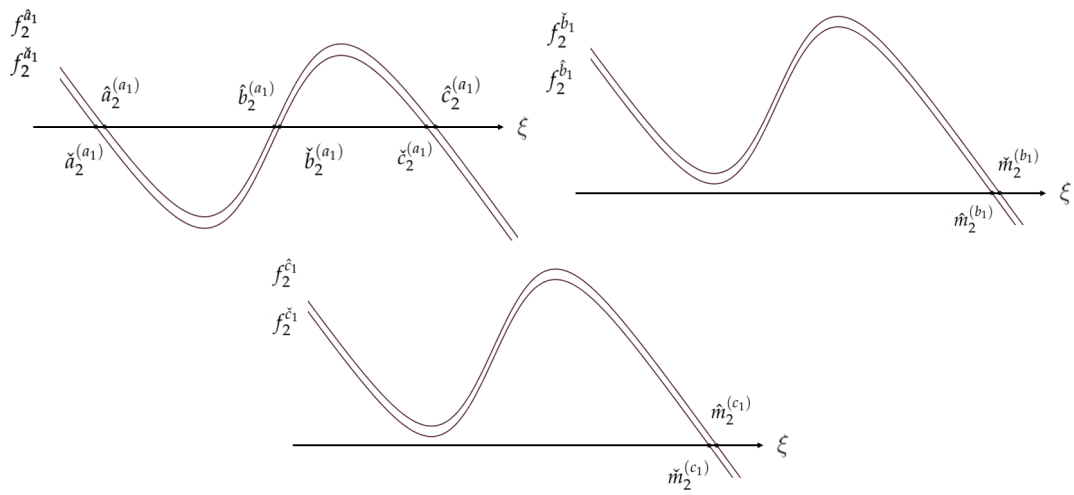

Figure 3.7: The configurations of functions $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$.

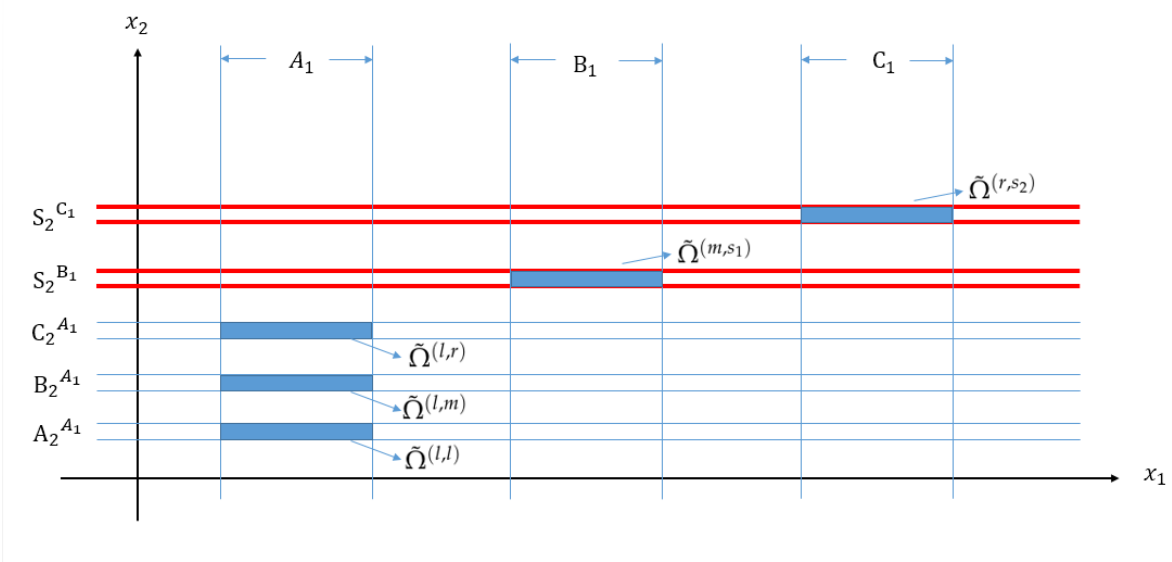

Figure 3.8: $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$.

Theorem 3.3. *Consider the case* $\binom{3}{3}\binom{r}{3}$, $\alpha_{21} + \beta_{21} > 0$ *and* $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$. *For each* $i \in \{1, 2\}$, *fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := L_i, \quad \text{if } i = 1, 2. \tag{3.1.17}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j|\alpha_{ij} + \beta_{ij}|, i, j \in \{1, 2\} \quad \textit{and} \quad i \neq j \tag{3.1.18}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \textit{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \textit{if } \xi \in [\hat{b}_1, \check{b}_1], \end{cases} \tag{3.1.19}$$

$$g_2'(\xi) \begin{cases} < \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, & \textit{if } \xi \in [\hat{m}_2^{(b_1)}, \check{m}_2^{(b_1)}] \cup [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}], \\ < \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, & \textit{if } \xi \in (-\infty, \hat{a}_2^{(a_1)}] \cup [\check{c}_2^{(a_1)}, \infty), \\ > \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}, & \textit{if } \xi \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}], \end{cases} \tag{3.1.20}$$

then there exist exactly 5 *equilibria in system* (2.2.1)-(2.2.2).

Proof. Consider the following five regions

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.1.21}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m”}, \text{“r”},$$

$$w_2 = \begin{cases} \text{“l”}, \text{“m”}, \text{“r”}, & \text{if } w_1 = \text{“l”}, \\ \text{“s}_1\text{”}, & \text{if } w_1 = \text{“m”}, \\ \text{“s}_2\text{”}, & \text{if } w_1 = \text{“r”}, \end{cases} \tag{3.1.22}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1$, $\tilde{\Omega}_1^{\mathrm{m}} = B_1$, $\tilde{\Omega}_1^{\mathrm{r}} = C_1$, $\tilde{\Omega}_2^{\mathrm{l}} = A_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{m}} = B_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{r}} = C_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{s}_1} = S_2^{B_1}$, and $\tilde{\Omega}_2^{\mathrm{s}_2} = S_2^{C_1}$ are compact intervals, cf Figure 3.8. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be any one of the five regions defined in (3.1.21). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.1.23}$$

for $i, j \in \{1, 2\}$ and $j \neq i$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,l)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,m)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,r)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,s_1)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,s_2)}$, function h_2 is bounded by $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{C_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in C_1 = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.1.23). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist five equilibria for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij})g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into five cases.

(i)If w_1 ="m" and w_2 ="s_1", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\hat{b}_1, \check{b}_1]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\hat{m}_2^{(b_1)}, \check{m}_2^{(b_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.1.19). On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.20), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.1.18).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21} + \beta_{21})g_1'(\eta_1^*)(x_1 - y_1)|/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.1.18).

(ii)If w_1 ="l" and w_2 ="m", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.1.19), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $g_2'(\xi_2^*) > (\mu_2 + \theta_2)/(\alpha_{22} + \beta_{22})$ by (3.1.20).

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Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.18).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21}+\beta_{21})g_1'(\eta_1^*)(x_1-y_1)|/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.18).

(iii)If $w_1 =$"l" and $w_2 =$"l", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.19), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2-\theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.1.20), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.18).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.18).

(iv)If $w_1 =$"r" and $w_2 =$ "s_2" , then $x_1^*, y_1^*, \xi_1^* \in [\check{c}_1, \hat{c}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$. Thus, $0 \le g_1^{'}(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.19), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \le g_2^{'}(\xi_2^*) < (\mu_2-\theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.1.20), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^*-y_1^*| &\le \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.18).

$$\begin{aligned} |x_2^*-y_2^*| &\le \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.18).

(v)The situation for $w_1 =$"l" and $w_2 =$ "r" is similar to (iv).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\bar{x}_i + \alpha_{ii}g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2}\alpha_{ij}g_j(\bar{x}_j) + \sum_{j=1}^{2}\beta_{ij}g_j(\bar{x}_j) + I_i = 0. \tag{3.1.24}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly 5 equilibria. □

3.1.4 $K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$

Recall Table 2.2. We shall show that there exist exactly 7 equilibria under $K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$ and some additional conditions.

Since $K_2(\tilde{p}_2; B_1) < 0$, each of functions $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$ has three zeros, say $f_2^{\hat{b}_1}(\hat{a}_2^{(b_1)}) = f_2^{\hat{b}_1}(\hat{b}_2^{(b_1)}) = f_2^{\hat{b}_1}(\hat{c}_2^{(b_1)}) = 0$ and $f_2^{\check{b}_1}(\check{a}_2^{(b_1)}) = f_2^{\check{b}_1}(\check{b}_2^{(b_1)}) = f_2^{\check{b}_1}(\check{c}_2^{(b_1)}) = 0$. Note that $K_2(\tilde{p}_2; B_1) < 0$ implies $K_2(\tilde{p}_2; A_1) < 0$, each of functions $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$ has three zeros, say $f_2^{\check{a}_1}(\check{a}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{b}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{c}_2^{(a_1)}) = 0$, and $f_2^{\hat{a}_1}(\hat{a}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{b}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{c}_2^{(a_1)}) = 0$, with $\check{a}_2^{(a_1)} < \hat{a}_2^{(a_1)} < \hat{a}_2^{(b_1)} < \check{a}_2^{(b_1)} < \check{b}_2^{(b_1)} < \hat{b}_2^{(b_1)} < \hat{b}_2^{(a_1)} < \check{b}_2^{(a_1)} < \check{c}_2^{(a_1)} < \hat{c}_2^{(a_1)} < \hat{c}_2^{(b_1)} < \check{c}_2^{(b_1)}$, cf. Figures 3.9 and 3.10.

Since $K_2(\tilde{p}_2; C_1) > 0$, each of $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$ has exactly one zero, say $f_2^{\check{c}_1}(\check{m}_2^{(c_1)}) = 0$ and $f_2^{\hat{c}_1}(\hat{m}_2^{\hat{c}_1}) = 0$, with $\check{m}_2^{(c_1)} < \hat{m}_2^{(c_1)}$. Denote the interval $S_2^{C_1} := [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$, cf. Figures 3.9 and 3.10.

Denote intervals $A_2^{A_1} := [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$, $B_2^{A_1} := [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, $C_2^{A_1} := [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$, $A_2^{B_1} := [\hat{a}_2^{(b_1)}, \check{a}_2^{(b_1)}]$, $B_2^{B_1} := [\check{b}_2^{(b_1)}, \hat{b}_2^{(b_1)}]$, $C_2^{B_1} := [\hat{c}_2^{(b_1)}, \check{c}_2^{(b_1)}]$, and $S_2^{C_1} := [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$.

There are seven regions $A_1 \times A_2^{A_1}$, $A_1 \times B_2^{A_1}$, $A_1 \times C_2^{A_1}$, $B_1 \times A_2^{B_1}$, $B_1 \times B_2^{B_1}$, $B_1 \times C_2^{B_1}$, $C_1 \times S_2^{C_1}$, cf Figure 3.11.

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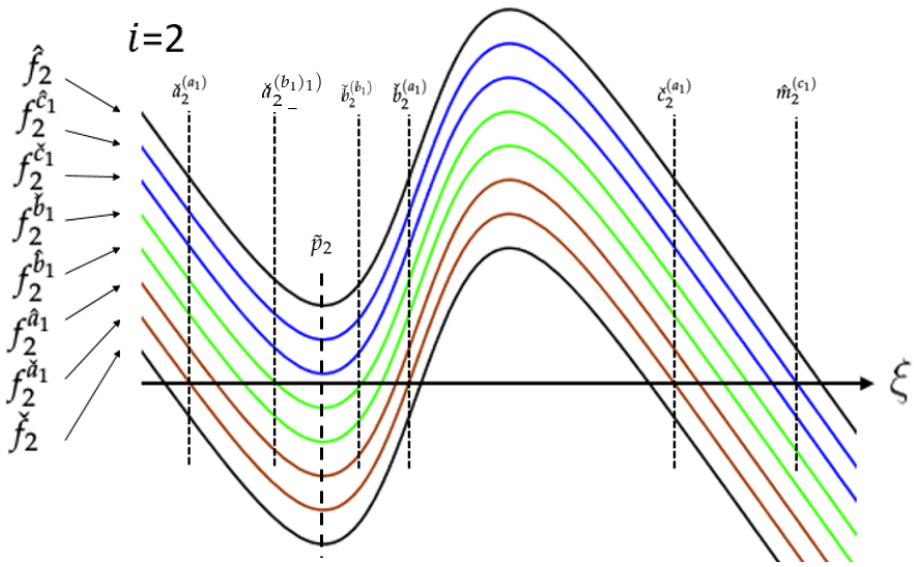

Figure 3.9: The configurations of functions $\hat{f}_2(\xi)$, $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$, $\check{f}_2(\xi)$ under condition $K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$.

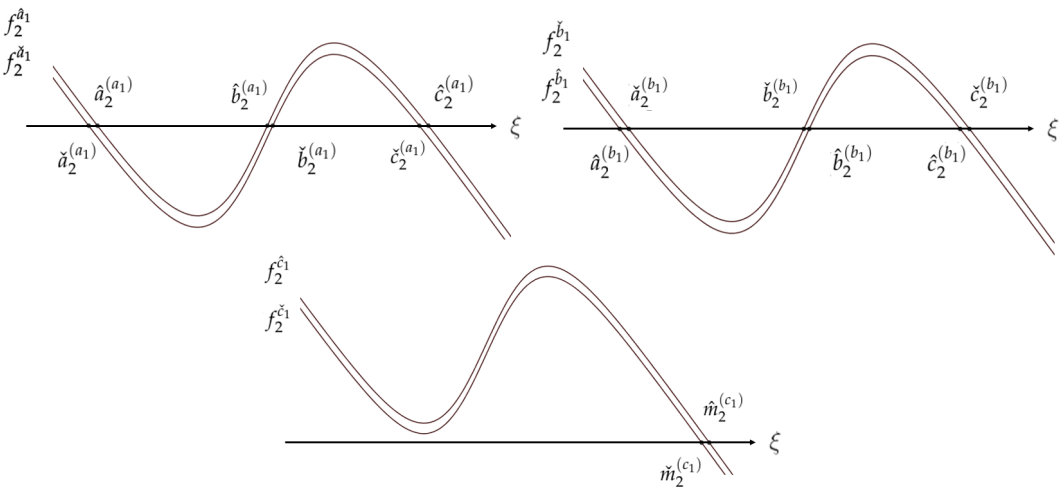

Figure 3.10: The configurations of functions $f_2^{\hat{c}_1}(\xi)$, $f_2^{\check{c}_1}(\xi)$, $f_2^{\check{b}_1}(\xi)$, $f_2^{\hat{b}_1}(\xi)$, $f_2^{\hat{a}_1}(\xi)$, $f_2^{\check{a}_1}(\xi)$.

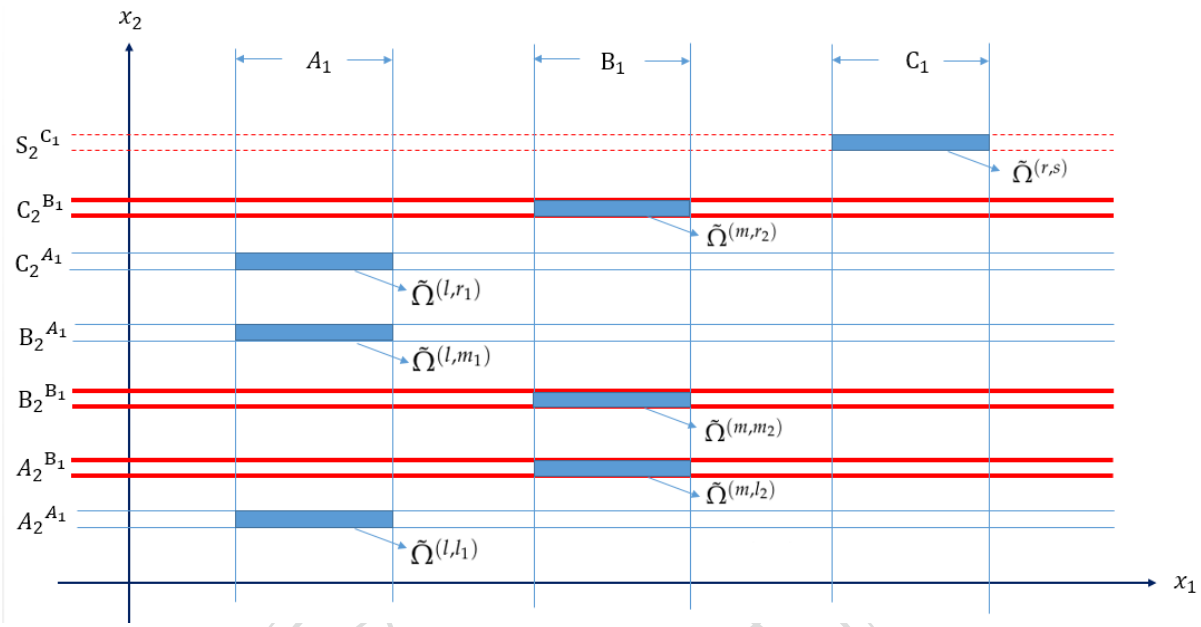

Figure 3.11: $K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$.

Theorem 3.4. *Consider the case* $\binom{3}{3}\binom{r}{3}$, $\alpha_{21} + \beta_{21} > 0$ *and* $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$. *For each* $i \in \{1, 2\}$, *fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := L_i, \quad \text{if } i = 1, 2. \tag{3.1.25}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1, 2\} \quad \text{and} \quad i \neq j \tag{3.1.26}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in [\hat{b}_1, \check{b}_1], \end{cases} \tag{3.1.27}$$

$$g_2'(\xi)\begin{cases} < \frac{\mu_2-\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}] \\ < \frac{\mu_2-\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in (-\infty, \hat{a}_2^{(b_1)}] \cup [\check{c}_2^{(a_1)}, \infty), \\ > \frac{\mu_2+\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in [\check{b}_2^{(b_1)}, \check{b}_2^{(a_1)}], \end{cases} \tag{3.1.28}$$

then there exist exactly 7 *equilibria in system* (2.2.1)-(2.2.2).

Proof. Consider the following seven regions

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.1.29}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m”}, \text{“r”},$$

$$w_2 = \begin{cases} \text{“l}_1\text{”}, \text{“m}_1\text{”}, \text{“r}_1\text{”}, & \text{if } w_1 = \text{“l”}, \\ \text{“l}_2\text{”}, \text{“m}_2\text{”}, \text{“r}_2\text{”}, & \text{if } w_1 = \text{“m”}, \\ \text{“s”}, & \text{if } w_1 = \text{“r”}, \end{cases} \tag{3.1.30}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1$, $\tilde{\Omega}_1^{\mathrm{m}} = B_1$, $\tilde{\Omega}_1^{\mathrm{r}} = C_1$, $\tilde{\Omega}_2^{\mathrm{l}_1} = A_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{m}_1} = B_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{r}_1} = C_2^{A_1}$, $\tilde{\Omega}_2^{\mathrm{l}_2} = A_2^{B_1}$, $\tilde{\Omega}_2^{\mathrm{m}_2} = B_2^{B_1}$, $\tilde{\Omega}_2^{\mathrm{r}_2} = C_2^{B_1}$, and $\tilde{\Omega}_2^{\mathrm{s}} = S_2^{C_1}$ are compact intervals, cf. Figure 3.11. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be any one of the 7 regions defined in (3.1.29). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.1.31}$$

for $i, j \in \{1, 2\}$ and $j \neq i$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,l_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,m_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,r_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has

exactly one zero $\tilde{x}_2$ in $C_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,l_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,m_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,r_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\check{b}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{B_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in B_1 = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,s)}$, function h_2 is bounded by $f_2^{\check{c}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{C_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in C_1 = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.1.31). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist seven equilibria for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii}) g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into seven cases.

(i)If $w_1 =$"m" and $w_2 =$"m_2", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\hat{b}_1, \check{b}_1]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\hat{b}_2^{(b_1)}, \check{b}_2^{(b_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.1.27). On the other hand, $g_2'(\xi_2^*) > (\mu_2 + \theta_2)/(\alpha_{22} + \beta_{22})$ by (3.1.28).

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.1.26).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21} + \beta_{21})g_1'(\eta_1^*)(x_1 - y_1)|/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.1.26).

(ii)If $w_1 =$"m" and $w_2 =$"l_2", then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1, \check{b}_1]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\hat{a}_2^{(b_1)}, \check{a}_2^{(b_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.1.27). On the other hamd, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.28), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.26).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.26).

(iii)The situation for $w_1 =$"m" and $w_2 =$"r_2" is similar to (ii).

(iv)If $w_1 =$"l" and $w_2 =$"m_1", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.27), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $g_2'(\xi_2^*) > (\mu_2+\theta_2)/(\alpha_{22}+\beta_{22})$ by (3.1.28).

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.26).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21}+\beta_{21})g_1'(\eta_1^*)(x_1-y_1)|/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.26).

(v)If $w_1 =$"l" and $w_2 =$"l_1", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.27), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

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On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.28), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.1.26).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.1.26).

(vi)The situation for $w_1 =$"l" and $w_2 =$ "r_1" is similar to (v).

(vii)If $w_1 =$"r" and $w_2 =$ "s", then $x_1^*, y_1^*, \xi_1^* \in [\check{c}_1, \hat{c}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$. Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.1.27), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.28), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.1.26).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21}+\beta_{21})g_1'(\eta_1^*)(x_1-y_1)|/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.1.26).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\bar{x}_i + \alpha_{ii}g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2} \alpha_{ij}g_j(\bar{x}_j) + \sum_{j=1}^{2} \beta_{ij}g_j(\bar{x}_j) + I_i = 0. \tag{3.1.32}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly 7 equilibria. □

3.2 Exact number of equilibria for case $\binom{3}{1}\binom{r}{3}$

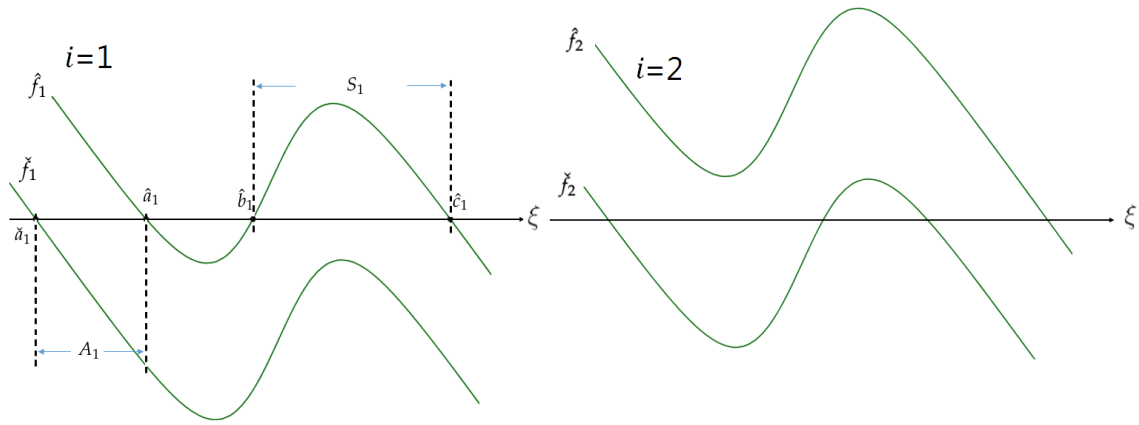

Figure 3.12: $\binom{3}{1}\binom{r}{3}$.

Recall Table 2.1. We shall establish the existnece of equilibria of system (2.2.1)-(2.2.1) for case $\binom{3}{1}\binom{r}{3}$ of subtype T_{18}. In this case, $\binom{3}{1}\binom{r}{3}$ means that $1 \in \mathcal{B}_l^3$ *and* $2 \in \mathcal{B}_3^r$. So, $\hat{f}_1(\xi)$ has

exactly three zeros, say $\hat{f}_1(\hat{a}_1) = \hat{f}_1(\hat{b}_1) = \hat{f}_1(\hat{c}_1) = 0$, and $\check{f}_1(\xi)$ has exactly one zero, say $\check{f}_1(\check{a}_1) = 0$, with $\check{a}_1 < \hat{a}_1 < \hat{b}_1 < \hat{c}_1$, cf. Figure 3.12. Denot intervals $A_1 := [\check{a}_1, \hat{a}_1]$, and $S_1 := [\hat{b}_1, \hat{c}_1]$. We merely consider the case $\alpha_{21} + \beta_{21} > 0$ and $\alpha_{12} + \beta_{12} > 0$. The other three cases can be treated similarly.

Table 3.1 lists the preparatory conditions for the existence of equilibria for case $\binom{3}{1}\binom{\mathrm{r}}{3}$. For example, in Theorem 3.6, we shall show that the system has one equilibrium point under condition $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) < 0$; moreover, it has exactly one equilibrum point under some additional conditions. In the following sections, we shall discuss the existence of equilibria of system (2.2.1)-(2.2.1) based on the conditions in Table 3.1. We note that some notations in Table 3.1 will be introduced in the following subsections. There are three classifications, $K_2(\tilde{p}_2; A_1) > 0$, $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$, and $K_2(\tilde{p}_2; S_1) < 0$. we shall discuss case $K_2(\tilde{p}_2; A_1) > 0$ in Section 3.2.1, which is divided into two parts to further discuss $K_1(\tilde{q}_1; S_2^{S_1}) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) < 0$. In Section 3.2.2, we consider the case $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$. We discuss the case $K_2(\tilde{p}_2; S_1) < 0$ and $K_1(\tilde{q}_1; A_2^{S_1}) > 0$ in Section 3.2.3.

Conditions	♯ Equi.
$\alpha_{12} + \beta_{12} > 0$ and $\alpha_{21} + \beta_{21} > 0$	
$K_2(\tilde{p}_2; A_1) > 0$	
$K_1(\tilde{q}_1; S_2^{S_1}) < 0$	1
$K_1(\tilde{q}_1; S_2^{S_1}) > 0$	3
$K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$	
$K_1(\tilde{q}_1; S_2^{S_1}) < 0$	3
$K_1(\tilde{q}_1; S_2^{S_1}) > 0$	5
$K_2(\tilde{p}_2; S_1) < 0$	
$K_1(\tilde{q}_1; C_2^{S_1}) < 0$	3
$K_1(\tilde{q}_1; B_2^{S_1}) < 0 < K_1(\tilde{q}_1; C_2^{S_1})$	5
$K_1(\tilde{q}_1; A_2^{S_1}) < 0 < K_1(\tilde{q}_1; B_2^{S_1})$	7
$K_1(\tilde{q}_1; A_2^{S_1}) > 0$	9

Table 3.1: Conditons for various numbers of equilibrium points for the case $\binom{3}{l}\binom{\mathrm{r}}{3}$.

3.2.1 $K_2(\tilde{p}_2; A_1) > 0$

If $K_2(\tilde{p}_2; A_1) > 0$, each of functions $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$ has one zero, say $f_2^{\check{a}_1}(\check{m}_2^{(a_1)}) = 0$ and $f_2^{\hat{a}_1}(\hat{m}_2^{(a_1)}) = 0$. For every ξ, $f_2^{\check{a}_1}(\xi) < f_2^{\hat{a}_1}(\xi)$ since $\check{a}_1 < \hat{a}_1$. It follows that $\check{m}_2^{(a_1)} < \hat{m}_2^{(a_1)}$. Denote the interval $S_2^{A_1} := [\check{m}_2^{(a_1)}, \hat{m}_2^{(a_1)}]$. Because $K_2(\tilde{p}_2; A_1) > 0$ implies $K_2(\tilde{p}_2; S_1) > 0$, functions $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$ has one zero at $\check{m}_2^{(b_1)}$ and $\hat{m}_2^{(c_1)}$, respectively, with $\check{m}_2^{(b_1)} < \hat{m}_2^{(c_1)}$, cf. Figure 3.13. Denote the interval $S_2^{S_1} := [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$. In the following two sections, we confirm the exact number of equilibria depending on the sign of $K_1(\tilde{q}_1; S_2^{S_1})$.

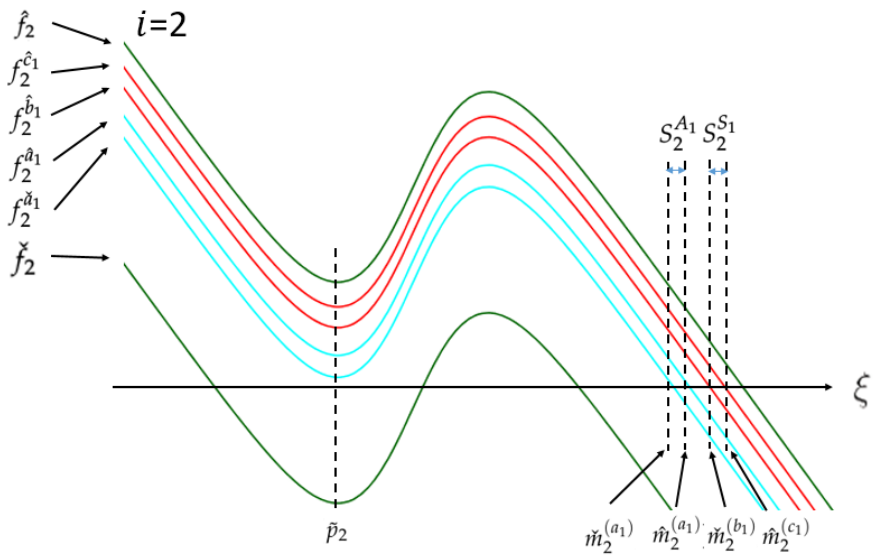

Figure 3.13: $K_2(\tilde{p}_2; A_1) > 0$.

3.2.1.1 $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$

Recall Table 3.1. we shall show that exactly 3 equilibria under $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$ and some additional conditions. If $K_1(\tilde{q}_1; S_2^{S_1}) > 0$ and $\alpha_{12} + \beta_{12} > 0$, each of functions $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$ has three zeros, say $f_1^{\check{m}_2^{(b_1)}}(\check{a}_1^{(b_2)}) = f_1^{\check{m}_2^{(b_1)}}(\check{b}_1^{(b_2)}) = f_1^{\check{m}_2^{(b_1)}}(\check{c}_1^{(b_2)}) = 0$ and $f_1^{\hat{m}_2^{(c_1)}}(\hat{a}_1^{(c_2)}) = f_1^{\hat{m}_2^{(c_1)}}(\hat{b}_1^{(c_2)}) = f_1^{\hat{m}_2^{(c_1)}}(\hat{c}_1^{(c_2)}) = 0$, with $\check{a}_1^{(b_2)} < \hat{a}_1^{(c_2)} < \hat{b}_1^{(c_2)} < \check{b}_1^{(b_2)} < \check{c}_1^{(b_2)} < \hat{c}_1^{(c_2)}$. Denote the interval $A_1^{S_2^{S_1}} := [\check{a}_1^{(b_2)}, \hat{a}_1^{(c_2)}]$, $B_1^{S_2^{S_1}} := [\hat{b}_1^{(c_2)}, \check{b}_1^{(b_2)}]$, and $C_1^{S_2^{S_1}} := [\check{c}_1^{(b_2)}, \hat{c}_1^{(c_2)}]$, cf Figure 3.14. In the following Theorem 3.5, we shall prove that there exists exactly one equilibrium point in $A_1 \times S_2^{A_1}$, $B_1^{S_2^{S_1}} \times S_2^{S_1}$, and $C_1^{S_2^{S_1}} \times S_2^{S_1}$, respectively, cf

Figure 3.15.

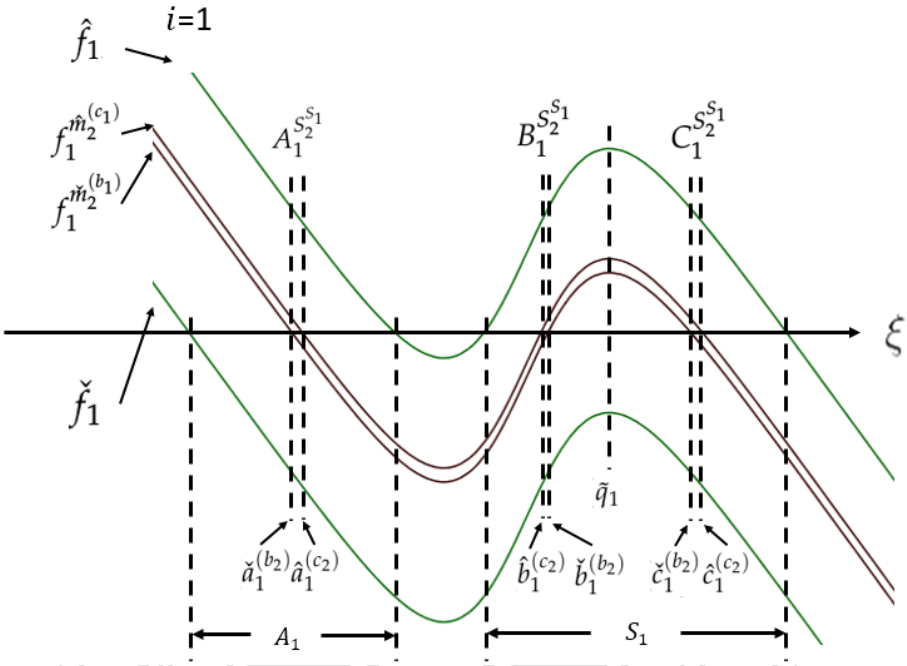

Figure 3.14: $K_1(\tilde{q}_1; S_2^{S_1}) > 0$.

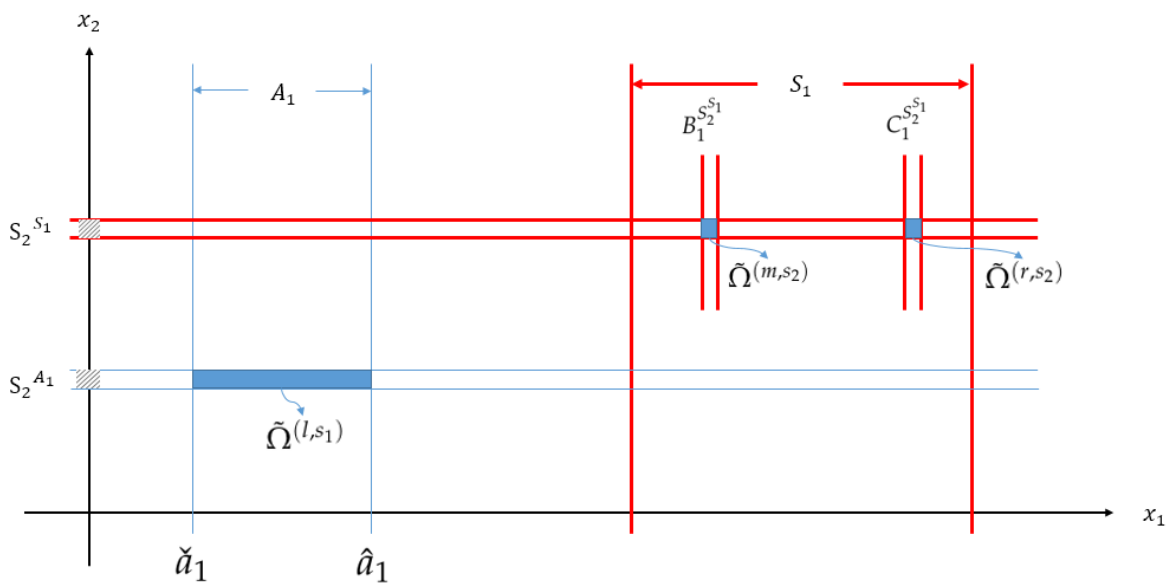

Figure 3.15: $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$.

Theorem 3.5. *Consider the case* $\binom{3}{l}\binom{r}{3}$, $\alpha_{ij} + \beta_{ij} > 0$ *for* $i \neq j$, $i, j \in \{1, 2\}$, *and* $K_2(\tilde{p}_2; A_1) >$

0 *and* $K_1(\tilde{q}_1; S_2^{S_1}) > 0$. *For each* $i \in \{1,2\}$, *fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := \begin{cases} L_1, & \text{if } i = 1, \\ \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}} & \text{if } i = 2. \end{cases} \tag{3.2.1}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1,2\} \quad \text{and} \quad i \neq j \tag{3.2.2}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1^{(b_2)}, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in [\hat{b}_1^{(c_2)}, \check{b}_1^{(b_2)}], \end{cases} \tag{3.2.3}$$

$$g_2'(\xi) < \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, \quad \text{if } \xi \in [\check{m}_2^{(a_1)}, \hat{m}_2^{(c_1)}], \tag{3.2.4}$$

then there exist exactly 3 equilibria in system (2.2.1)-(2.2.2).

Proof. Consider the following three regions

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.2.5}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m”}, \text{“r”},$$

$$w_2 = \begin{cases} \text{“s}_1\text{”}, & \text{if } w_1 = \text{“l”}, \\ \text{“s}_2\text{”}, & \text{if } w_1 = \text{“m” or “r”}, \end{cases} \tag{3.2.6}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1 = [\check{a}_1, \hat{a}_1]$, $\tilde{\Omega}_1^{\mathrm{m}} = B_1^{S_2^{S_1}} = [\hat{b}_1^{(c_2)}, \check{b}_1^{(b_2)}]$, $\tilde{\Omega}_1^{\mathrm{r}} = C_1^{S_2^{S_1}} = [\check{c}_1^{(b_2)}, \hat{c}_1^{(c_2)}]$, $\tilde{\Omega}_2^{\mathrm{s}_1} = S_2^{A_1} = [\check{m}_2^{(a_1)}, \hat{m}_2^{(a_1)}]$, and $\tilde{\Omega}_2^{\mathrm{s}_2} = S_2^{S_1} = [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$ are compact intervals, cf Figure 3.15. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be one of the 3 regions defined in (3.2.5). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$,

we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.2.7}$$

for $i, j \in \{1, 2\}$ and $j \neq i$. If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,s_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. there exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$. If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,s_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$. there exists exactly one solution $\tilde{x}_1 \in B_1^{S_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$. If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,s_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$. there exists exactly one solution $\tilde{x}_1 \in C_1^{S_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.2.7). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist three equilibria for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii}) g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into three cases.

(i)If w_1 =“m” and w_2 =“s_2”, then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1^{(c_2)}, \check{b}_1^{(b_2)}]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.2.3). On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.4), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.2).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.2).

(ii)If w_1 =“l” and w_2 =“s_1”, then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2^{(a_1)}, \hat{m}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.1.3), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.1.4), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.2).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.2).

(iii)If $w_1 =$"r" and $w_2 =$"s_2", then $x_1^*, y_1^*, \xi_1^* \in [\check{c}_1^{(b_2)}, \hat{c}_1^{(c_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.1.3), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2-\theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.1.4), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.2).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.2).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\bar{x}_i + \alpha_{ii}g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2}\alpha_{ij}g_j(\bar{x}_j) + \sum_{j=1}^{2}\beta_{ij}g_j(\bar{x}_j) + I_i = 0. \tag{3.2.8}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly 3 equilibria. □

3.2.1.2 $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) < 0$

Recall Table 3.1. We shall show that exactly there exist one equilibrium point under $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) < 0$ and some additional conditions. If $K_1(\tilde{q}_1; S_2^{S_1}) < 0$ and $\alpha_{12} + \beta_{12} > 0$, each of functions $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$ has excatly one zero, say $f_1^{\check{m}_2^{(b_1)}}(\check{m}_1^{(b_1)}) = 0$ and $f_1^{\hat{m}_2^{(c_1)}}(\hat{m}_1^{(c_2)}) = 0$, with $\check{m}_1^{(b_2)} < \hat{m}_1^{(c_2)}$. Denote the interval $S_1^{S_2^{S_1}} := [\check{m}_1^{(b_2)}, \hat{m}_1^{(c_2)}]$, cf Figure 3.16. In the following Theorem 3.6, we shall prove that there exists exactly one equilibrium point in $A_1 \times S_2^{A_1}$, cf Figure 3.17.

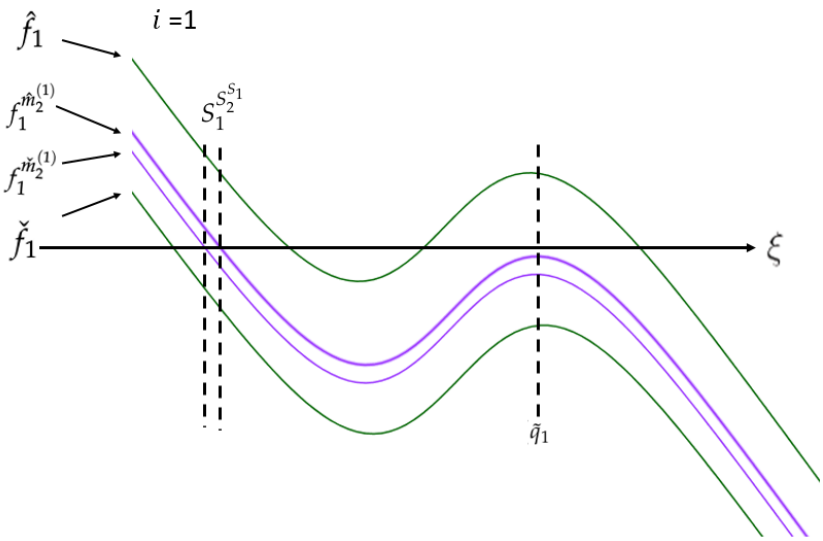

Figure 3.16: $K_1(\tilde{q}_1; S_2^{S_1}) < 0$.

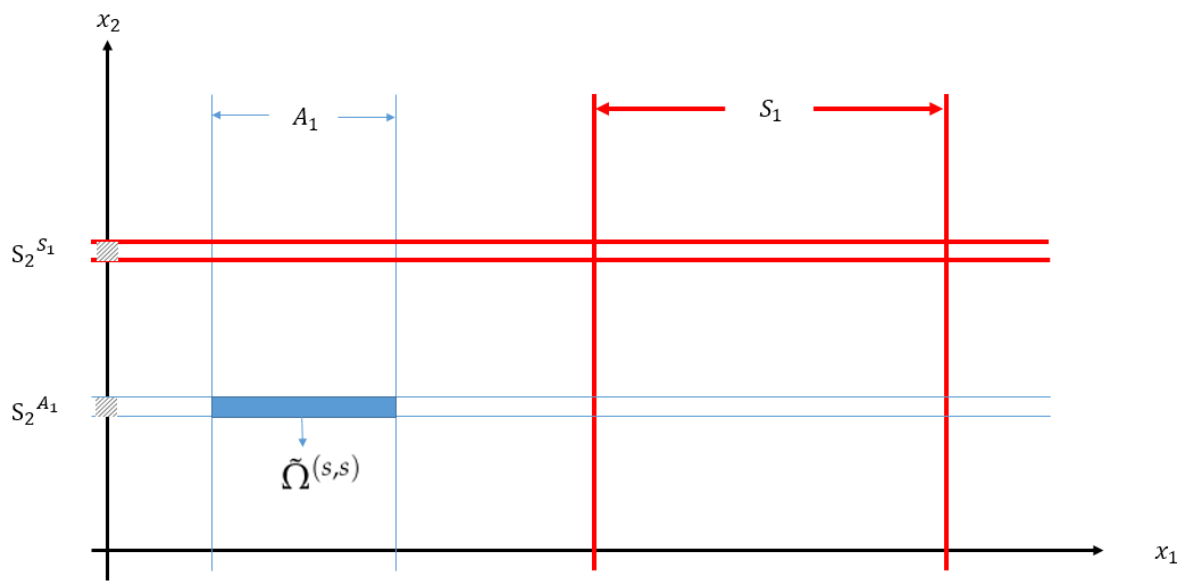

Figure 3.17: $K_2(\tilde{p}_2; A_1) > 0$ and $K_1(\tilde{q}_1; S_2^{S_1}) < 0$.

Theorem 3.6. *Consider the case* $\binom{3}{l}\binom{r}{3}$, $\alpha_{ij} + \beta_{ij} > 0$ *for* $i \neq j$, $i, j \in \{1, 2\}$, *and* $K_2(\tilde{p}_2; S_1) > 0$ *and* $K_1(\tilde{q}_1; S_2^{S_1}) < 0$. *For each* $i \in \{1, 2\}$, *fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := \begin{cases} \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}} & \text{if } i = 1, \\ \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}} & \text{if } i = 2, \end{cases} \tag{3.2.9}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1, 2\} \quad \text{and} \quad i \neq j \tag{3.2.10}$$

and

$$\begin{aligned} g_1'(\xi) &< \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, \text{ if } \xi \in [\check{a}_1, \hat{a}_1], \\ g_2'(\xi) &< \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, \text{ if } \xi \in [\check{m}_2^{(a_1)}, \hat{m}_2^{(a_1)}], \end{aligned} \tag{3.2.11}$$

then there exist exactly one equilibrium point in system (2.2.1)-(2.2.2).

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Proof. Consider the region

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.2.12}$$
$$\mathbf{w} = (w_1, w_2),$$
$$w_i = \text{“s”}, \text{ for } i = 1, 2$$

where $\tilde{\Omega}_1^{\mathrm{s}} = A_1 = [\check{a}_1, \hat{a}_1]$ and $\tilde{\Omega}_2^{\mathrm{s}} = S_2^{A_1} = [\check{m}_2^{(a_1)}, \hat{m}_2^{(a_1)}]$ are compact intervals, cf Figure 3.17. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be the regions defined in (3.2.12). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.2.13}$$

for $i, j \in \{1, 2\}$ and $j \neq i$. If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(s,s)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{s}}$ to $h_1(x_1) = 0$. According, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.2.13). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist one equilibrium point for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium point.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii}) g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j.

w_1 ="s" and w_2 ="s", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{m}_2^{(a_1)}, \hat{m}_2^{(a_1)}]$. Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.11), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.11), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.10).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.10).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\bar{x}_i + \alpha_{ii}g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2} \alpha_{ij}g_j(\bar{x}_j) + \sum_{j=1}^{2} \beta_{ij}g_j(\bar{x}_j) + I_i = 0. \tag{3.2.14}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly one equilibrium point. □

3.2.2 $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$

Under conditions $K_2(\tilde{p}_2; A_1) = K_2(\tilde{p}_2; [\check{a}_1, \hat{a}_1]) < 0$ and $\alpha_{21} + \beta_{21} > 0$, each of $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$ has three zeros, say $f_2^{\check{a}_1}(\check{a}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{b}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{c}_2^{(a_1)}) = 0$ and $f_2^{\hat{a}_1}(\hat{a}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{b}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{c}_2^{(a_1)}) = 0$, with $\check{a}_2^{(a_1)} < \hat{a}_2^{(a_1)} < \hat{b}_2^{(a_1)} < \check{b}_2^{(a_1)} < \check{c}_2^{(a_1)} < \hat{c}_2^{(a_1)}$. Denote intervals $A_2^{A_1} := [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$, $B_2^{A_1} := [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, and $C_2^{A_1} := [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$.

Since $K_2(\tilde{p}_2; S_1) = K_2(\tilde{p}_2; [\hat{b}_1, \hat{c}_1]) > 0$, functions $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$ have unique zeros $\check{m}_2^{(b_1)}$ and $\hat{m}_2^{(c_1)}$, respectively. Denote the interval $S_2^{S_1} := [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$, cf. Figure 3.18. As seen in Table 3.1, there exist different numbers of equilibria depending on $K_1(\tilde{q}_1; S_2^{S_1}) > 0$ or $K_1(q_1; S_2^{S_1}) < 0$.

In the following, let us show that there exist exactly five equilibria under $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$ and some additional conditions. If $K_1(\tilde{q}_1; S_2^{S_1}) > 0$ and $\alpha_{12} + \beta_{12} > 0$, each of $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$ has three zeros, say $f_1^{\check{m}_2^{(b_1)}}(\check{a}_1^{(b_2)}) = f_1^{\check{m}_2^{(b_1)}}(\check{b}_1^{(b_2)}) = f_1^{\check{m}_2^{(b_1)}}(\check{c}_1^{(b_2)}) = 0$ and $f_1^{\hat{m}_2^{(c_1)}}(\hat{a}_1^{(c_2)}) = f_1^{\hat{m}_2^{(c_1)}}(\hat{b}_1^{(c_2)}) = f_1^{\hat{m}_2^{(c_1)}}(\hat{c}_1^{(c_2)}) = 0$, with $\check{a}_1^{(b_2)} < \hat{a}_1^{(c_2)} < \hat{b}_1^{(c_2)} < \check{b}_1^{(b_2)} < \check{c}_1^{(b_2)} < \hat{c}_1^{(c_2)}$. Denote intervals $A_1^{S_2^{S_1}} := [\check{a}_1^{(b_2)}, \hat{a}_1^{(c_2)}]$, $B_1^{S_2^{S_1}} := [\hat{b}_1^{(c_2)}, \check{b}_1^{(b_2)}]$, and $C_1^{S_2^{S_1}} := [\check{c}_1^{(b_2)}, \hat{c}_1^{(c_2)}]$, cf Figure 3.19. In the following Theorem 3.7, we shall prove that there exists exactly one equilibrium point in $A_1 \times A_2^{A_1}$, $A_1 \times B_2^{A_1}$, $A_1 \times C_2^{A_1}$, $B_1^{S_2^{S_1}} \times S_2^{S_1}$, $C_1^{S_2^{S_1}} \times S_2^{S_1}$, respectively, cf Figure 3.20.

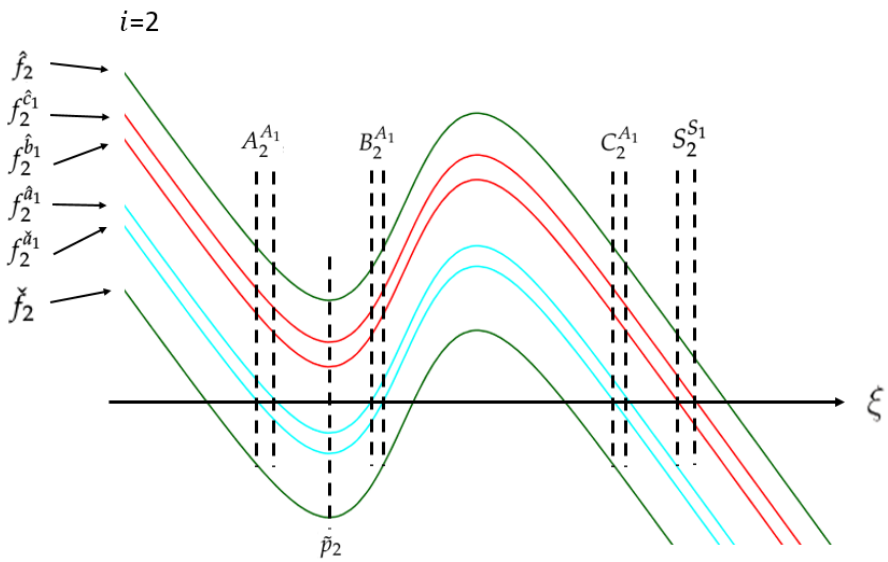

Figure 3.18: $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$.

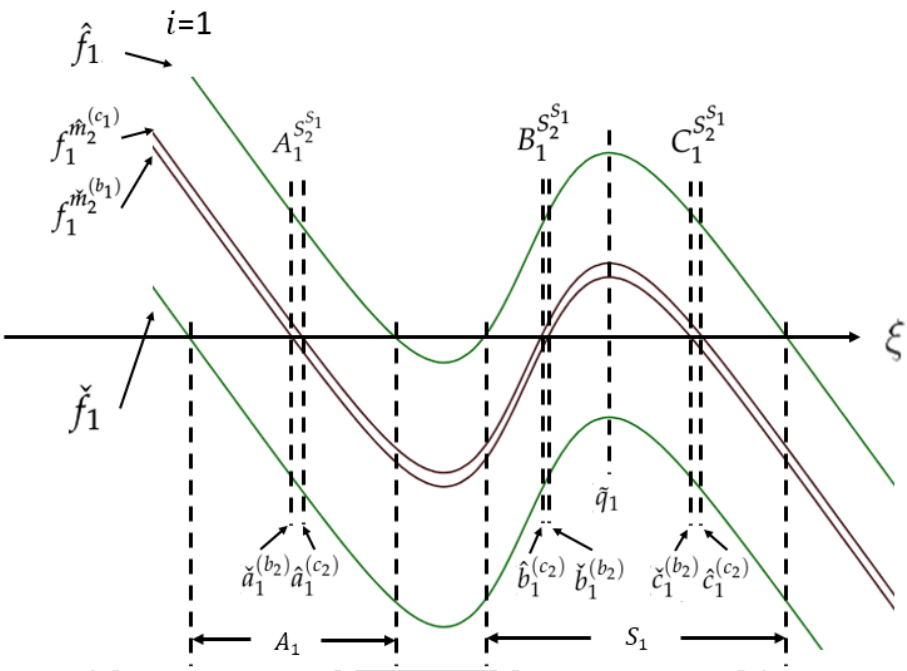

Figure 3.19: $K_1(\tilde{q}_1; S_2^{S_1}) > 0$.

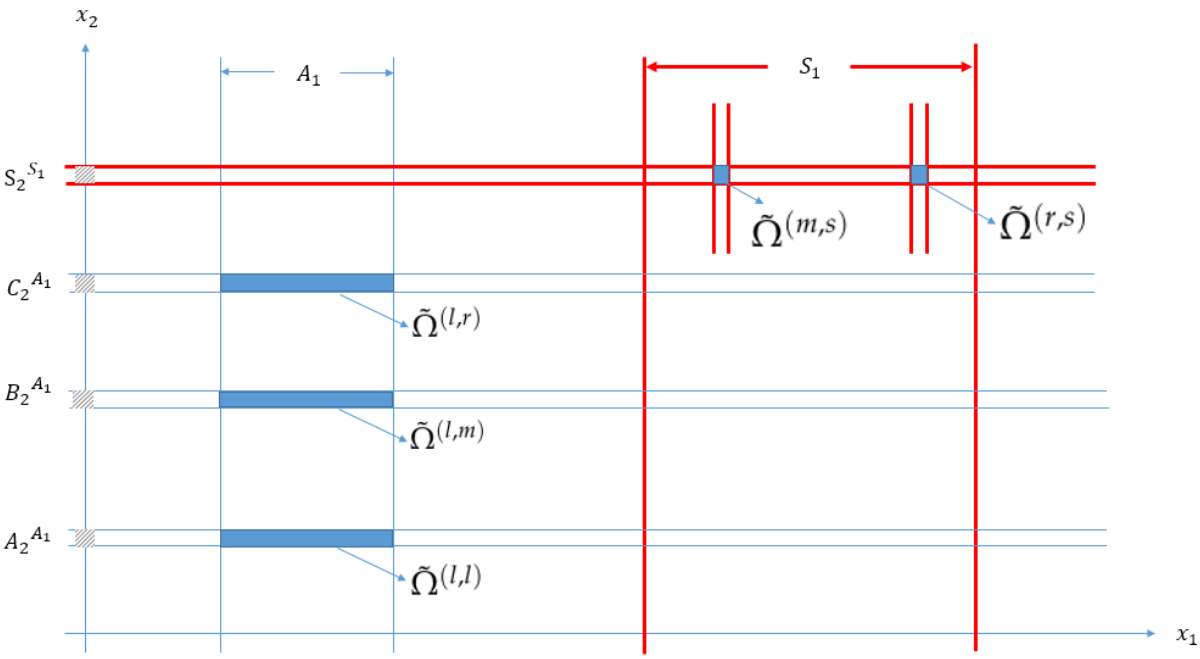

Figure 3.20: $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; S_1)$ and $K_1(\tilde{q}_1; S_2^{S_1}) > 0$.

Theorem 3.7. *Consider the case* $\binom{3}{l}\binom{r}{3}$, $\alpha_{ij} + \beta_{ij} > 0$ *for* $i \neq j$, $i, j \in \{1, 2\}$ *and* $K_2(p_2; S_1) >$

0 *and* $K_1(\tilde{q}_1; S_2^{S_1}) > 0$. *For each i, fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := L_i, i = 1, 2. \tag{3.2.15}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1, 2\} \quad \textit{and} \quad i \neq j \tag{3.2.16}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \textit{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1^{(b_1)}, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \textit{if } \xi \in [\hat{b}_1^{(c_1)}, \check{b}_1^{(b_1)}], \end{cases} \tag{3.2.17}$$

$$g_2'(\xi) \begin{cases} < \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, & \textit{if } \xi \in (-\infty, \hat{a}_2^{(a_1)}] \cup [\check{c}_2^{(a_1)}, \infty), \\ > \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}, & \textit{if } \xi \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}], \end{cases} \tag{3.2.18}$$

then there exist exactly 5 *equilibria in system* (2.2.1)-(2.2.2).

Proof. Consider the following five regions

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.2.19}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m”}, \text{“r”},$$

$$w_2 = \begin{cases} \text{“l”}, \text{“m”}, \text{“r”} & \text{if } w_1 = \text{“l”}, \\ \text{“s”}, & \text{if } w_1 = \text{“m” } \textit{or} \text{ “r”}, \end{cases} \tag{3.2.20}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1 = [\check{a}_1, \hat{a}_1]$, $\tilde{\Omega}_1^{\mathrm{m}} = B_1^{S_2^{S_1}} = [\hat{b}_1^{(c_1)}, \check{b}_1^{(b_1)}]$, $\tilde{\Omega}_1^{\mathrm{r}} = C_1^{S_2^{S_1}} = [\check{c}_1^{(b_1)}, \hat{c}_1^{(c_1)}]$ and $\tilde{\Omega}_2^{\mathrm{l}} = A_2^{A_1} = [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{m}} = B_2^{A_1} = [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{r}} = C_2^{A_1} = [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{s}} = S_2^{S_1} = [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$ are compact intervals, cf Figure 3.20. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be any one

of the 5 regions defined in (3.2.19). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.2.21}$$

for $i, j \in \{1, 2\}$ and $j \neq i$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,l)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,m)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,r)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m,s)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in B_1^{S_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{m}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r,s)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $S_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{m}_2^{(b_1)}}$ and $f_1^{\hat{m}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in C_1^{S_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{r}}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.2.21). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist five equilibria for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for

each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii}) g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into five cases.

(i)If $w_1 =$"m" and $w_2 =$"s" then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1^{(c_1)}, \check{b}_1^{(b_1)}]$ and $x_2^*, y_2^*, \xi_2^*, \eta_2^* \in [\check{m}_2^{(b_1)}, \hat{m}_2^{(c_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.2.17). On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.18), which yields

$$|(\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) > \theta_2.$$

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12}) g_2'(\eta_2^*)(x_2 - y_2)| / |(\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2 |\alpha_{12} + \beta_{12}|]/\theta_1\} \|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2 |\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.16).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1 |\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1 |\alpha_{21} + \beta_{21}|]/\theta_2\} \|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1 |\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.16).

(ii)If $w_1 =$"l" and $w_2 =$"m", then $x_1^*, y_1^*, \xi_1^*, \eta_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$.

Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.17), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $g_2'(\xi_2^*) > (\mu_2 + \theta_2)/(\alpha_{22} + \beta_{22})$ by (3.2.18).

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.16).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21} + \beta_{21})g_1'(\eta_1^*)(x_1 - y_1)|/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.16).

(iii)If $w_1 =$"l" and $w_2 =$"l", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$.

Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.17), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.18), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.16).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.16).

(iv)The situation for w_1 =“l” and w_2 = “r” is similar to case (iii). We can show that $\Phi^{(l,r)}$ is a contraction mapping.

(v)The situation for w_1 =“r” and w_2 = “s” is similar to case (iii). We can show that $\Phi^{(r,s)}$ is a contraction mapping.

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\bar{x}_i + \alpha_{ii}g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2}\alpha_{ij}g_j(\bar{x}_j) + \sum_{j=1}^{2}\beta_{ij}g_j(\bar{x}_j) + I_i = 0. \tag{3.2.22}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (1.0.1) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly five equilibria. □

3.2.3 $K_2(\tilde{p}_2; S_1) < 0$ and $K_1(\tilde{q}_1; A_2^{S_1}) > 0$

Since $\alpha_{21}+\beta_{21} > 0$ and $K_2(\tilde{p}_2; S_1) < 0$, each of $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$ has three zeros, say $f_2^{\hat{b}_1}(\check{a}_2^{(b_1)}) = f_2^{\hat{b}_1}(\check{b}_2^{(b_1)}) = f_2^{\hat{b}_1}(\check{c}_2^{(b_1)}) = 0$ and $f_2^{\hat{c}_1}(\hat{a}_2^{(c_1)}) = f_2^{\hat{c}_1}(\hat{b}_2^{(c_1)}) = f_2^{\hat{c}_1}(\hat{c}_2^{(c_1)}) = 0$, with $\check{a}_2^{(b_1)} < \hat{a}_2^{(c_1)} < \hat{b}_2^{(c_1)} < \check{b}_2^{(b_1)} < \check{c}_2^{(b_1)} < \hat{c}_2^{(c_1)}$. Denote intervals $A_2^{S_1} := [\check{a}_2^{(b_1)}, \hat{a}_2^{(c_1)}]$, $B_2^{S_1} := [\hat{b}_2^{(c_1)}, \check{b}_2^{(b_1)}]$, and $C_2^{S_1} := [\check{c}_2^{(b_1)}, \hat{c}_2^{(c_1)}]$.

Note that $K_2(\tilde{p}_2; S_1) < 0$ implies $K_2(\tilde{p}_2; A_1) < 0$. Each of $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$ has three zeros, say $f_2^{\check{a}_1}(\check{a}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{b}_2^{(a_1)}) = f_2^{\check{a}_1}(\check{c}_2^{(a_1)}) = 0$ and $f_2^{\hat{a}_1}(\hat{a}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{b}_2^{(a_1)}) = f_2^{\hat{a}_1}(\hat{c}_2^{(a_1)}) = 0$, with $\check{a}_2^{(a_1)} < \hat{a}_2^{(a_1)} < \hat{b}_2^{(a_1)} < \check{b}_2^{(a_1)} < \check{c}_2^{(a_1)} < \hat{c}_2^{(a_1)}$. Denote intervals $A_2^{A_1} := [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$,

$B_2^{A_1} := [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, and $C_2^{A_1} := [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$.

As seen in Table 3.1, there exist different numbers of equilibria depending on $K_1(q_1; A_2^{S_1}) > 0$ or $K_1(q_1; A_2^{S_1}) < 0 < K_1(q_1; B_2^{S_1})$ or $K_1(q_1; B_2^{S_1}) < 0 < K_1(q_1; C_2^{S_1})$ or $K_1(q_1; C_2^{S_1}) < 0$.

In the following, let us show that exactly 9 equilibria under $K_2(\tilde{p}_2; S_1) < 0$ and $K_1(\tilde{q}_1; A_2^{S_1}) > 0$ and some additional conditions. If $K_1(\tilde{q}_1; A_2^{S_1}) > 0$, each of $f_1^{\check{a}_2^{(b_1)}}$ and $f_1^{\hat{a}_2^{(c_1)}}$ has three zeros, say $f_1^{\check{a}_2^{(b_1)}}(\check{a}_1^{(a_2)}) = f_1^{\check{a}_2^{(b_1)}}(\check{b}_1^{(a_2)}) = f_1^{\check{a}_2^{(b_1)}}(\check{c}_1^{(a_2)}) = 0$ and $f_1^{\hat{a}_2^{(c_1)}}(\hat{a}_1^{(a_2)}) = f_1^{\hat{a}_2^{(c_1)}}(\hat{b}_1^{(a_2)}) = f_1^{\hat{a}_2^{(c_1)}}(\hat{c}_1^{(a_2)}) = 0$, with $\check{a}_1^{(a_2)} < \hat{a}_1^{(a_2)} < \hat{b}_1^{(a_2)} < \check{b}_1^{(a_2)} < \check{c}_1^{(a_2)} < \hat{c}_1^{(a_2)}$. Denote intervals $A_1^{A_2^{S_1}} := [\check{a}_1^{(a_2)}, \hat{a}_1^{(a_2)}]$, $B_1^{A_2^{S_1}} := [\hat{b}_1^{(a_2)}, \check{b}_1^{(a_2)}]$, and $C_1^{A_2^{S_1}} := [\check{c}_1^{(a_2)}, \hat{c}_1^{(a_2)}]$, cf Figure 3.22.

Note that $\alpha_{12} + \beta_{12} > 0$ and $K_1(\tilde{q}_1; A_2^{S_1}) > 0$ implies $K_1(q_1; B_2^{S_1}) > 0$ and $K_1(q_1; C_2^{S_1}) > 0$, each of $f_1^{\hat{b}_2^{(b_1)}}$, $f_1^{\check{b}_2^{(c_1)}}$, $f_1^{\check{c}_2^{(b_1)}}$, and $f_1^{\hat{c}_2^{(c_1)}}$ has three zeros, say $f_1^{\hat{b}_2^{(b_1)}}(\hat{a}_1^{(b_2)}) = f_1^{\hat{b}_2^{(b_1)}}(\hat{b}_1^{(b_2)}) = f_1^{\hat{b}_2^{(b_1)}}(\hat{c}_1^{(b_2)}) = 0$, $f_1^{\check{b}_2^{(c_1)}}(\check{a}_1^{(b_2)}) = f_1^{\check{b}_2^{(c_1)}}(\check{b}_1^{(b_2)}) = f_1^{\check{b}_2^{(c_1)}}(\check{c}_1^{(b_2)}) = 0$, $f_1^{\check{c}_2^{(b_1)}}(\check{a}_1^{(c_2)}) = f_1^{\check{c}_2^{(b_1)}}(\check{b}_1^{(c_2)}) = f_1^{\check{c}_2^{(b_1)}}(\check{c}_1^{(c_2)}) = 0$, and $f_1^{\hat{c}_2^{(c_1)}}(\hat{a}_1^{(c_2)}) = f_1^{\hat{c}_2^{(c_1)}}(\hat{b}_1^{(c_2)}) = f_1^{\hat{c}_2^{(c_1)}}(\hat{c}_1^{(c_2)}) = 0$, cf Figure 3.22.

Define intervals $A_1^{B_2^{S_1}} := [\hat{a}_1^{(b_2)}, \check{a}_1^{(b_2)}]$, $B_1^{B_2^{S_1}} := [\check{b}_1^{(b_2)}, \hat{b}_1^{(b_2)}]$, and $C_1^{B_2^{S_1}} := [\hat{c}_1^{(b_2)}, \check{c}_1^{(b_2)}]$, $A_1^{C_2^{S_1}} := [\check{a}_1^{(c_2)}, \hat{a}_1^{(c_2)}]$, $B_1^{C_2^{S_1}} := [\hat{b}_1^{(c_2)}, \check{b}_1^{(c_2)}]$, and $C_1^{C_2^{S_1}} := [\check{c}_1^{(c_2)}, \hat{c}_1^{(c_2)}]$, cf Figure 3.23. In the following Theorem 3.8, we shall prove that there exists exactly one equilibrium point in $A_1 \times A_2^{A_1}$, $A_1 \times B_2^{A_1}$, $A_1 \times C_2^{A_1}$, $B_1^{A_2^{S_1}} \times A_2^{S_1}$, $C_1^{A_2^{S_1}} \times A_2^{S_1}$, $B_1^{B_2^{S_1}} \times B_2^{S_1}$, $C_1^{B_2^{S_1}} \times B_2^{S_1}$, $B_1^{C_2^{S_1}} \times C_2^{S_1}$, and $C_1^{C_2^{S_1}} \times C_2^{S_1}$, respectively, cf Figure 3.23.

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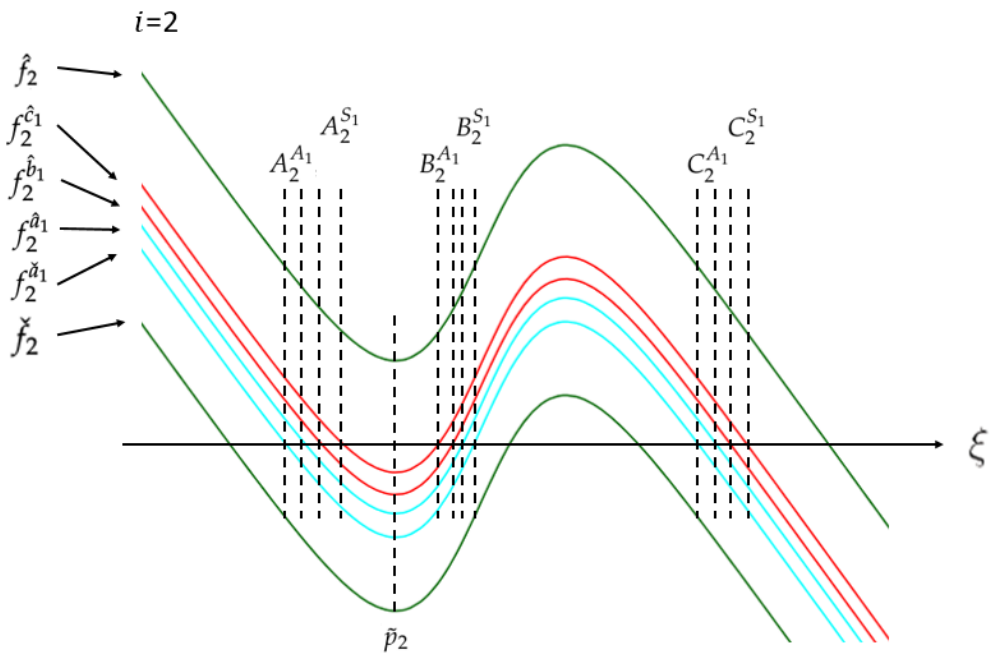

Figure 3.21: $K_2(\tilde{p}_2; S_1) < 0$.

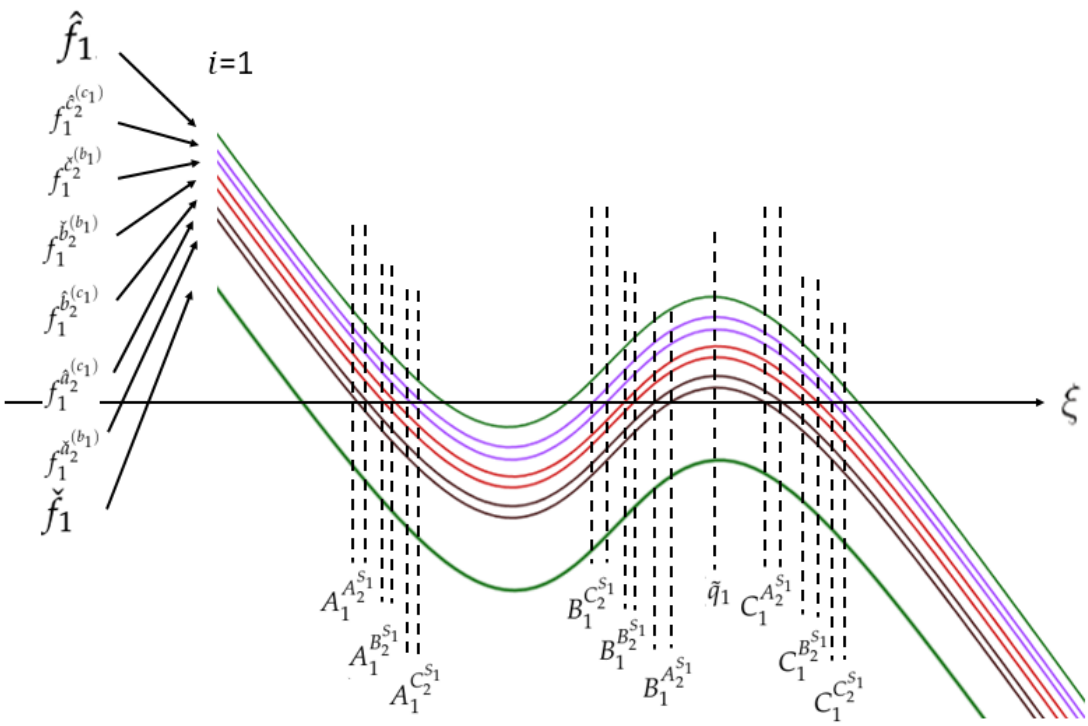

Figure 3.22: $K_1(\tilde{q}_1; S_2^{A_1}) > 0$.

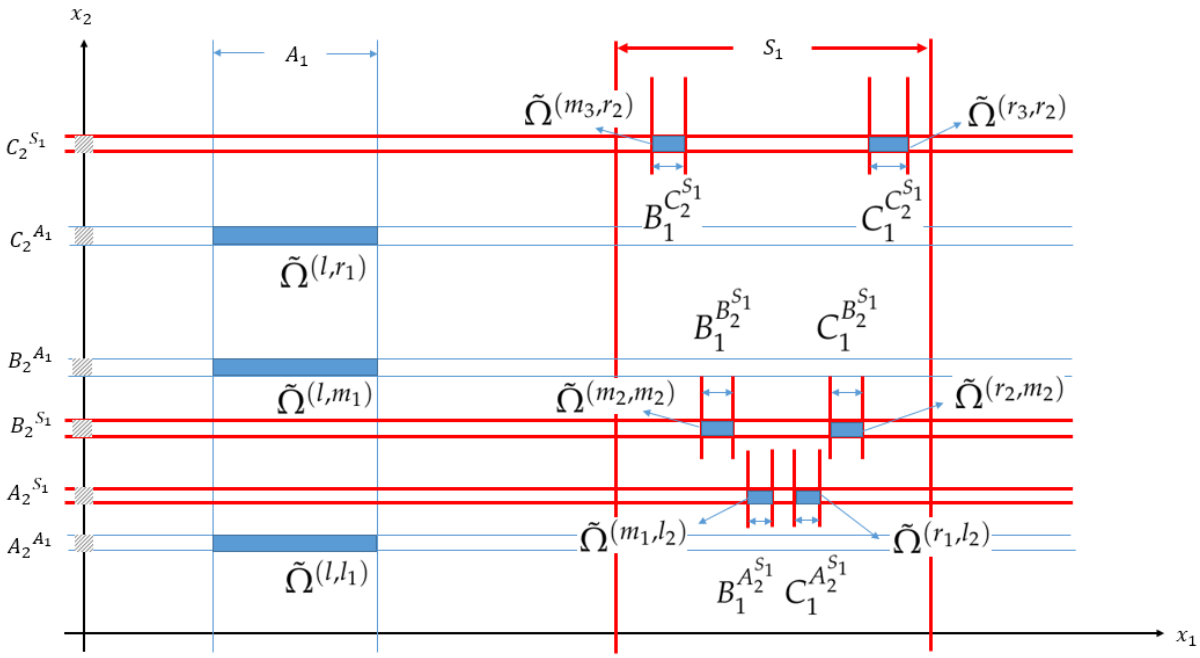

Figure 3.23: $K_2(\tilde{p}_2; S_1) < 0$ and $K_1(\tilde{q}_1; A_2^{S_1}) > 0$.

Theorem 3.8. *Consider the case* $\binom{3}{l}\binom{r}{3}$, $\alpha_{ij} + \beta_{ij} > 0$ *for* $i \neq j$, $i, j \in \{1, 2\}$, *and* $K_2(\tilde{p}_2; S_1) < 0$ *and* $K_1(\tilde{q}_1; A_2^{S_1}) > 0$. *For each* $i \in \{1, 2\}$, *fix a* $\theta_i \in (0, \mu_i)$ *and then define*

$$\bar{L}_i := L_i, i = 1, 2. \tag{3.2.23}$$

If the parameters satisfy

$$\theta_i > \bar{L}_j |\alpha_{ij} + \beta_{ij}|, i, j \in \{1, 2\} \quad \text{and} \quad i \neq j \tag{3.2.24}$$

and

$$g_1'(\xi) \begin{cases} < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1^{(a_2)}, \infty), \\ > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in [\hat{b}_1^{(c_2)}, \check{b}_1^{(a_2)}], \end{cases} \tag{3.2.25}$$

$$g_2'(\xi)\begin{cases} < \frac{\mu_2-\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in (-\infty, \hat{a}_2^{(c_1)}] \cup [\check{c}_2^{(a_1)}, \infty), \\ > \frac{\mu_2+\theta_2}{\alpha_{22}+\beta_{22}}, & \text{if } \xi \in [\hat{b}_2^{(c_1)}, \check{b}_2^{(a_1)}], \end{cases} \tag{3.2.26}$$

then there exist exactly nine equilibria in system (2.2.1)-(2.2.2).

Proof. Consider the following nine regions

$$\tilde{\Omega}^{\mathbf{w}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_i \in \tilde{\Omega}_i^{w_i}\}, \tag{3.2.27}$$

$$\mathbf{w} = (w_1, w_2),$$

$$w_1 = \text{“l”}, \text{“m}_1\text{”}, \text{“m}_2\text{”}, \text{“m}_3\text{”}, \text{“r}_1\text{”}, \text{“r}_2\text{”}, \text{“r}_3\text{”}$$

$$w_2 = \begin{cases} \text{“l}_1\text{”}, \text{“m}_1\text{”}, \text{“r}_1\text{”} & \text{if } w_1 = \text{“l”}, \\ \text{“l}_2\text{”}, & \text{if } w_1 = \text{“m}_1\text{” } or \text{ “r}_1\text{”}, \\ \text{“m}_2\text{”}, & \text{if } w_1 = \text{“m}_2\text{” } or \text{ “r}_2\text{”}, \\ \text{“r}_2\text{”}, & \text{if } w_1 = \text{“m}_3\text{” } or \text{ “r}_3\text{”}, \end{cases} \tag{3.2.28}$$

where $\tilde{\Omega}_1^{\mathrm{l}} = A_1 = [\check{a}_1, \hat{a}_1]$, $\tilde{\Omega}_1^{\mathrm{m}_1} = B_1^{A_2^{S_1}} = [\hat{b}_1^{(a_2)}, \check{b}_1^{(a_2)}]$, $\tilde{\Omega}_1^{\mathrm{m}_2} = B_1^{B_2^{S_1}} = [\check{b}_1^{(b_2)}, \hat{b}_1^{(b_2)}]$, $\tilde{\Omega}_1^{\mathrm{m}_3} = B_1^{C_2^{S_1}} = [\hat{b}_1^{(c_2)}, \check{b}_1^{(c_2)}]$, $\tilde{\Omega}_1^{\mathrm{r}_1} = C_1^{A_2^{S_1}} = [\check{c}_1^{(a_2)}, \hat{c}_1^{(a_2)}]$, $\tilde{\Omega}_1^{\mathrm{r}_2} = C_1^{B_2^{S_1}} = [\hat{c}_1^{(b_2)}, \check{c}_1^{(b_2)}]$, $\tilde{\Omega}_1^{\mathrm{r}_3} = C_1^{C_2^{S_1}} = [\check{c}_1^{(c_2)}, \hat{c}_1^{(c_2)}]$, $\tilde{\Omega}_2^{\mathrm{l}_1} = A_2^{A_1} = [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{m}_1} = B_2^{A_1} = [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{r}_1} = C_2^{A_1} = [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$, $\tilde{\Omega}_2^{\mathrm{l}_2} = A_2^{S_1} = [\check{a}_2^{(b_1)}, \hat{a}_2^{(c_1)}]$, $\tilde{\Omega}_2^{\mathrm{m}_2} = B_2^{S_1} = [\hat{b}_2^{(b_1)}, \check{b}_2^{(c_2)}]$, $\tilde{\Omega}_2^{\mathrm{r}_2} = C_2^{S_1} = [\check{c}_2^{(b_1)}, \hat{c}_2^{(c_1)}]$, are compact intervals, cf Figure 3.23. Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2)$, be any one of the 9 regions defined in (3.2.27). For any given $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + (\alpha_{ij} + \beta_{ij}) g_j(\zeta_j) + I_i = 0, \tag{3.2.29}$$

for $i, j \in \{1, 2\}$ and $j \neq i$. If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,l_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,m_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(l,r_1)}$, function h_2 is bounded by $f_2^{\check{a}_1}$ and $f_2^{\hat{a}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{A_1}$. Function h_1 is a vertical shift of f_1 and lies between $\check{f}_1$ and $\hat{f}_1$. There exists exactly one solution $\tilde{x}_1 \in A_1 = \tilde{\Omega}_1^{\mathrm{l}}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m_1,l_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{a}_2^{(b_1)}}$ and $f_1^{\hat{a}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in B_1^{A_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{m}_1}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r_1,l_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $A_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{a}_2^{(b_1)}}$ and $f_1^{\hat{a}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in C_1^{A_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{r}_1}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m_2,m_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\hat{b}_2^{(b_1)}}$ and $f_1^{\check{b}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in B_1^{B_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{m}_2}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r_2,m_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $B_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\hat{b}_2^{(b_1)}}$ and $f_1^{\check{b}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in C_1^{B_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{r}_2}$ $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(m_3,r_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{c}_2^{(b_1)}}$ and $f_1^{\hat{c}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in B_1^{C_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{m}_3}$ to $h_1(x_1) = 0$.

If $(\zeta_1, \zeta_2) \in \tilde{\Omega}^{(r_3,r_2)}$, function h_2 is bounded by $f_2^{\hat{b}_1}$ and $f_2^{\hat{c}_1}$. The equation $h_2(x_2) = 0$ has exactly one zero $\tilde{x}_2$ in $C_2^{S_1}$. The equation $h_1(x_1) = 0$ is bounded by $f_1^{\check{c}_2^{(b_1)}}$ and $f_1^{\hat{c}_2^{(c_1)}}$. There exists exactly one solution $\tilde{x}_1 \in C_1^{C_2^{S_1}} = \tilde{\Omega}_1^{\mathrm{r}_3}$ to $h_1(x_1) = 0$.

Accordingly, we define a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (3.2.29). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 2.1 in [3]. It follows from the Brouwer's fixed point theorem that there exists a fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2.1.1). Consequently, there exist nine equilibria for system (2.2.1)-(2.2.2) and each region $\tilde{\Omega}^{\mathbf{w}}$ contains one equilibrium point.

To show that $\Phi^{\mathbf{w}}$ is a contraction mapping, assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for

each $i = 1, 2$,

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii}) g_i(y_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(y_j) + I_i = 0, \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii}) g_i(x_i^*) + \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j(x_j) + I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii}) g_i'(\xi_i^*)] - \sum_{j=1, j\neq i}^{2} (\alpha_{ij} + \beta_{ij}) g_j'(\eta_j^*)[x_j - y_j] = 0,$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into nine cases.

(i)If $w_1 =$"l" and $w_2 =$"m_1", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(a_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.25), which yields

$$|(\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $g_2'(\xi_2^*) > (\mu_2 + \theta_2)/(\alpha_{22} + \beta_{22})$ by (3.2.26).
Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11}) g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\} \|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21} + \beta_{21}) g_1'(\eta_1^*)(x_1 - y_1)|/|(\alpha_{22} + \beta_{22}) g_2'(\xi_2^*) - \mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\} \|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(ii)If $w_1 =$"l" and $w_2 =$"l_1", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(a_1)}, \hat{a}_2^{(a_1)}]$.

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Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.25), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.26), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(iii)If $w_1 =$"l" and $w_2 =$"r_1", then $x_1^*, y_1^*, \xi_1^* \in [\check{a}_1, \hat{a}_1]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{c}_2^{(a_1)}, \hat{c}_2^{(a_1)}]$. Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.25), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.25), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

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Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(iv)If $w_1 =$"m_1" and $w_2 =$"l_2", then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1^{(a_2)}, \check{b}_1^{(a_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(b_1)}, \hat{a}_2^{(c_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1+\theta_1)/(\alpha_{11}+\beta_{11})$ by (3.2.25). On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.2.26), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12}+\beta_{12})g_2'(\eta_2^*)(x_2-y_2)|/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(v)If $w_1 =$"r_1" and $w_2 =$"l_2", then $x_1^*, y_1^*, \xi_1^* \in [\check{c}_1^{(a_2)}, \hat{c}_1^{(a_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{a}_2^{(b_1)}, \hat{a}_2^{(c_1)}]$.

Thus, $0 \leq g_1^{'}(\xi_1^*) < (\mu_1 - \theta_1)/(\alpha_{11} + \beta_{11})$, due to (3.2.25), which yields

$$|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| = \mu_1 - (\alpha_{11} + \beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2^{'}(\xi_2^*) < (\mu_2 - \theta_2)/(\alpha_{22} + \beta_{22})$, due to (3.2.26), which yields

$$|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2| = \mu_2 - (\alpha_{22} + \beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/|(\alpha_{22} + \beta_{22})g_2'(\xi_2^*) - \mu_2|\} \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21} + \beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(vi)If $w_1 =$"m_2" and $w_2 =$"m_2", then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1^{(b_2)}, \check{b}_1^{(b_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(b_1)}, \check{b}_2^{(c_2)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1 + \theta_1)/(\alpha_{11} + \beta_{11})$ by (3.2.25) and $g_2'(\xi_2^*) > (\mu_2 + \theta_2)/(\alpha_{22} + \beta_{22})$ by (3.2.26).

Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12} + \beta_{12})g_2'(\eta_2^*)(x_2 - y_2)|/|(\alpha_{11} + \beta_{11})g_1'(\xi_1^*) - \mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12} + \beta_{12}|]/\theta_1\}\|\mathbf{x} - \mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21}+\beta_{21})g_1'(\eta_1^*)(x_1-y_1)|/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(vii)If $w_1 =$"r_2" and $w_2 =$"m_2", then $x_1^*, y_1^*, \xi_1^* \in [\hat{c}_1^{(b_2)}, \check{c}_1^{(b_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\hat{b}_2^{(b_1)}, \check{b}_2^{(c_2)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.2.25), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $g_2'(\xi_2^*) > (\mu_2+\theta_2)/(\alpha_{22}+\beta_{22})$ by (3.2.26).
Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &= |(\alpha_{21}+\beta_{21})g_1'(\eta_1^*)(x_1-y_1)|/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| \\ &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(viii)If $w_1 =$"m_3" and $w_2 =$"r_2", then $x_1^*, y_1^*, \xi_1^* \in [\hat{b}_1^{(c_2)}, \check{b}_1^{(c_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{c}_2^{(b_1)}, \hat{c}_2^{(c_1)}]$. Thus, $g_1'(\xi_1^*) > (\mu_1+\theta_1)/(\alpha_{11}+\beta_{11})$ by (3.2.25). On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2-\theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.2.26), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

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Hence,

$$\begin{aligned} |x_1^* - y_1^*| &= |(\alpha_{12}+\beta_{12})g_2'(\eta_2^*)(x_2-y_2)|/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| \\ &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

(ix)If $w_1 =$"r_3" and $w_2 =$"r_2", then $x_1^*, y_1^*, \xi_1^* \in [\check{c}_1^{(c_2)}, \hat{c}_1^{(c_2)}]$ and $x_2^*, y_2^*, \xi_2^* \in [\check{c}_2^{(b_1)}, \hat{c}_2^{(c_1)}]$. Thus, $0 \leq g_1'(\xi_1^*) < (\mu_1-\theta_1)/(\alpha_{11}+\beta_{11})$, due to (3.2.25), which yields

$$|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1| = \mu_1-(\alpha_{11}+\beta_{11})g_1'(\xi_1^*) > \theta_1.$$

On the other hand, $0 \leq g_2'(\xi_2^*) < (\mu_2-\theta_2)/(\alpha_{22}+\beta_{22})$, due to (3.2.26), which yields

$$|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2| = \mu_2-(\alpha_{22}+\beta_{22})g_2'(\xi_2^*) > \theta_2.$$

Subsequently,

$$\begin{aligned} |x_1^* - y_1^*| &\leq \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/|(\alpha_{11}+\beta_{11})g_1'(\xi_1^*)-\mu_1|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

where $[\bar{L}_2|\alpha_{12}+\beta_{12}|]/\theta_1 < 1$, owing to (3.2.24).

$$\begin{aligned} |x_2^* - y_2^*| &\leq \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/|(\alpha_{22}+\beta_{22})g_2'(\xi_2^*)-\mu_2|\}\cdot\|\mathbf{x}-\mathbf{y}\|_\infty \\ &< \{[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2\}\|\mathbf{x}-\mathbf{y}\|_\infty, \end{aligned}$$

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where $[\bar{L}_1|\alpha_{21}+\beta_{21}|]/\theta_2 < 1$, owing to (3.2.24).

Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, 2$,

$$-\mu_i\bar{x}_i + \alpha_{ii}g_i(\bar{x}_i) + \sum_{j=1, j\neq i}^{2} \alpha_{ij}g_j(\bar{x}_j) + \sum_{j=1}^{2} \beta_{ij}g_j(\bar{x}_j) + I_i = 0. \tag{3.2.30}$$

Thus, $\bar{\mathbf{x}}$ is the unique equilibrium point of (2.2.1)-(2.2.2) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\bar{\mathbf{x}}$ is an equilibrium point of system (2.2.1)-(2.2.2), then its components satisfy (2.1.1), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (2.2.1)-(2.2.2) therefore admits exactly 9 equilibria. □

3.3 Convergence of dynamics for case $\binom{3}{3}\binom{r}{3}$ under conditions $K_2(\tilde{p}_2; A_1) > 0$

In this subsection, we establish the global convergence of system (2.2.1)-(2.2.2) which means that every solution of system (2.2.1)-(2.2.2) converges to one of the equilibria.

In the following, we fix an arbitrary initial condition $\phi \in C([-\tau_M, 0]; \mathbb{R}^2)$. Its evolution $\mathbf{x}(t) = (x_1(t), x_2(t))$ is then a fixed function defined on $[t_0, \infty)$, and the ith component $x_i(t)$, $i = 1, 2$, satisfies

$$\dot{x}_i(t) = -\mu_i x_i(t) + \alpha_{ii}g_i(x_i(t)) + \beta_{ii}g_i(x_i(t-\tau_{ii})) + w_i(t), \tag{3.3.1}$$

for all $t \geq t_0$, where $w_i(t) := \sum_{j=1, j\neq i}^{2}\{\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}))\} + I_i$. We define for each $i = 1, 2$,

$$w_i^{\max}(\infty) := \lim_{T\to\infty} w_i^{\max}(T), w_i^{\min}(\infty) := \lim_{T\to\infty} w_i^{\min}(T),$$

where $w_i^{\max}(T) := \sup\{w_i(t) \mid t \geq T\}$ is nonincreasing in T, and $w_i^{\min}(T) := \inf\{w_i(t) \mid t \geq T\}$ is nondecreasing in T. We shall show that $x_i(t)$ converges to some singleton, for $i = 1, 2$. Hence, system (2.2.1)-(2.2.2) achieves global convergence. For later use, for $i = 1, 2$,

we define the following functions successively.

$$\hat{H}_i(\xi) := -\mu_i\xi + 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + \sum_{j=1,j\neq i}^{2}(|\alpha_{ij}| + |\beta_{ij}|)\rho_j + I_i, \tag{3.3.2}$$

$$\check{H}_i(\xi) := -\mu_i\xi - 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i - \sum_{j=1,j\neq i}^{2}(|\alpha_{ij}| + |\beta_{ij}|)\rho_j + I_i, \tag{3.3.3}$$

$$\hat{h}_i(\xi) := -\mu_i\xi + 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0), \tag{3.3.4}$$

$$\check{h}_i(\xi) := -\mu_i\xi - 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\min}(t_0), \tag{3.3.5}$$

$$\begin{aligned}\hat{F}_i(\xi) &:= -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + \sum_{j=1,j\neq i}^{2}(|\alpha_{ij}| + |\beta_{ij}|)\rho_j + I_i \\ &\quad - \begin{cases} \beta_{ii}L_i\tau_{ii}\check{H}_i(\hat{A}_i^H) & \text{if } \beta_{ii} \geq 0, \\ \beta_{ii}L_i\tau_{ii}\hat{H}_i(\check{A}_i^H) & \text{if } \beta_{ii} < 0, \end{cases}\end{aligned} \tag{3.3.6}$$

$$\begin{aligned}\check{F}_i(\xi) &:= -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) - \sum_{j=1,j\neq i}^{2}(|\alpha_{ij}| + |\beta_{ij}|)\rho_j + I_i \\ &\quad - \begin{cases} \beta_{ii}L_i\tau_{ii}\hat{H}_i(\check{A}_i^H) & \text{if } \beta_{ii} \geq 0, \\ \beta_{ii}L_i\tau_{ii}\check{H}_i(\hat{A}_i^H) & \text{if } \beta_{ii} < 0, \end{cases}\end{aligned} \tag{3.3.7}$$

$$\begin{aligned}\hat{f}_i^{(0)}(\xi, T) &:= -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + w_i^{\max}(T) \\ &\quad - \begin{cases} \beta_{ii}L_i\tau_{ii}\check{h}_i(\hat{A}_i^h) & \text{if } \beta_{ii} \geq 0, \\ \beta_{ii}L_i\tau_{ii}\hat{h}_i(\check{A}_i^h) & \text{if } \beta_{ii} < 0, \end{cases}\end{aligned} \tag{3.3.8}$$

$$\begin{aligned}\check{f}_i^{(0)}(\xi, T) &:= -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + w_i^{\min}(T) \\ &\quad - \begin{cases} \beta_{ii}L_i\tau_{ii}\hat{h}_i(\check{A}_i^h) & \text{if } \beta_{ii} \geq 0, \\ \beta_{ii}L_i\tau_{ii}\check{h}_i(\hat{A}_i^h) & \text{if } \beta_{ii} < 0, \end{cases}\end{aligned} \tag{3.3.9}$$

where $\hat{A}_i^H$, $\check{A}_i^H$,$\hat{A}_i^h$, $\check{A}_i^h$ are the unique zeros to functions $\hat{H}_i$, $\check{H}_i$, $\hat{h}_i$, and $\check{h}_i$, respectively. Recall $\check{f}_i$ and $\hat{f}_i$ in (2.1.3). Notably, $\hat{F}_i(\cdot)$, $\check{F}_i(\cdot)$, $\hat{f}_i^{(0)}(\cdot, T)$, and $\check{f}_i^{(0)}(\cdot, T)$ are vertical shifts of $\hat{f}_i(\cdot)$, and $\check{f}_i(\cdot)$; $\hat{h}_i(\cdot)$ and $\check{h}_i(\cdot)$ are vertical shifts of $\hat{H}_i(\cdot)$ and $\check{H}_i(\cdot)$. It is not difficult to verify

$$\check{H}_i(\xi) \leq \check{h}_i(\xi) \leq \hat{h}_i(\xi) \leq \hat{H}_i(\xi), \tag{3.3.10}$$

$$\check{F}_i(\xi) \leq \check{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, T) \leq \hat{F}_i(\xi), \tag{3.3.11}$$

for all $T \geq t_0$ and $\xi \in \mathbb{R}$. Moreover,

$$\check{F}_i(\xi) \leq \check{f}_i(\xi) \leq \hat{f}_i(\xi) \leq \hat{F}_i(\xi), \tag{3.3.12}$$

for all $\xi \in \mathbb{R}$, since $\hat{H}_i(\check{A}_i^H) \geq 0$ and $\check{H}_i(\hat{A}_i^H) \leq 0$.

Now, let us introduce three conditions which are for monostable scenario:

Condition (M1): $|\beta_{ii}|\tau_{ii} < (|\alpha_{ii}| + |\beta_{ii}|)\rho_i / \{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0)]\}$.

Condition (M2): There exists a $T_0 \geq t_0$ such that $\hat{f}_i^{(0)}(\cdot, T_0)$ and $\check{f}_i^{(0)}(\cdot, T_0)$ have unique zeros, $\hat{m}_i^{(0)}(T_0)$ and $\check{m}_i^{(0)}(T_0)$, respectively.

Condition (M3): $g_i'(\xi) < (\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii})$ for all $\xi \in [\check{m}_i^{(0)}(T_0), \hat{m}_i^{(0)}(T_0)]$ for some $\theta_i \in (0, \mu_i)$.

Next, let us further introduce following three conditions which are for bistable scenario:

Condition (B1): $L_i > \mu_i/(\alpha_{ii} + \beta_{ii}) > 0$, $|\beta_{ii}|\tau_{ii} < (|\alpha_{ii}| + |\beta_{ii}|)\rho_i / \{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0)]\}$.

Notably, condition (B1) implies $L_i > \mu/(\alpha_{ii} + \beta_{ii})$. Hence, there exist exactly two points $\tilde{p}_i$ and $\tilde{q}_i$, with $\tilde{p}_i < \sigma_i < \tilde{q}_i$, satisfying

$$g_i'(\tilde{p}_i) = g'(\tilde{q}_i) = \mu_i/(\alpha_{ii} + \beta_{ii}).$$

Condition (B2): There exists a $T_0 \geq t_0$ such that $\check{f}_i^{(0)}(\tilde{q}_i, T_0) > 0$ and $\hat{f}_i^{(0)}(\tilde{p}_i, T_0) < 0$.

Under condition (B2), there exist exactly three zeros $\hat{a}_i(T_0)$, $\hat{b}_i(T_0)$, and $\hat{c}_i(T_0)$ (resp., $\check{a}_i(T_0)$, $\check{b}_i(T_0)$, and $\check{c}_i(T_0)$) of $\hat{f}_i^{(0)}(\cdot, T_0) = 0$ (resp., $\check{f}_i^{(0)}(\cdot, T_0) = 0$), where $\check{a}_i(T_0) \leq \hat{a}_i(T_0) < \tilde{p}_i < \hat{b}_i(T_0) \leq \check{b}_i(T_0) < \tilde{q}_i < \check{c}_i(T_0) \leq \hat{c}_i(T_0)$. Let $\theta_i \in (0, \mu_i)$ be a fixed number. The third condition is concerned with the slope of function g_i:

Condition (B3):

$$g_i'(\xi) \begin{cases} > (\mu_i + \theta_i)/(\alpha_{ii} + \beta_{ii}) & \text{if } \xi \in [\hat{b}_i(T_0), \check{b}_i(T_0)], \\ < (\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii}) & \text{if } \xi \in (-\infty, \hat{a}_i(T_0)] \cup [\check{c}_i(T_0), \infty). \end{cases}$$

Now, let us introduce three propositions in [3]:

Proposition 3.1. *([3]) Assume that conditions (M1)-(M3) hold for some $i \in \mathcal{N}$. Then $x_i(t)$ satisfying* (3.3.1) *converges to $[\underline{m}_i, \overline{m}_i]$, where*

$$\overline{m}_i - \underline{m}_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i].$$

Proposition 3.2. *([3]) Assume that conditions (B1)-(B3) hold for some $i \in \mathcal{N}$ and some $\theta_i \in (0, \mu_i)$. Then $x_i(t)$ satisfying* (3.3.1) *converges to one of the three disjoint intervals: $[\underline{a}_i, \overline{a}_i]$, $[\underline{b}_i, \overline{b}_i]$, and $[\underline{c}_i, \overline{c}_i]$, where*

$$0 \leq \overline{a}_i - \underline{a}_i, \overline{b}_i - \underline{b}_i, \overline{c}_i - \underline{c}_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i].$$

Proposition 3.3. *([3]) Let $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$ be a fixed solution of* (1.0.1)*. Assume that for every $i \in \mathcal{N}$, there exists a compact interval J_i of length d_i, such that $x_i(t)$ converges to J_i and d_i satisfies*

$$d_i \leq [w_i^{\max}(\infty) - w_i^{\min}(\infty)]/\eta_i,$$

for some $\eta_i > 0$, and there exist a compact interval $\tilde{J}_i$ and a $\tilde{L}_i \geq 0$, such that $J_i \subseteq \tilde{J}_i$ and

$$g_i'(\xi) \leq \tilde{L}_i \text{ for all } \xi \in \tilde{J}_i.$$

Let $\mathbf{M} := [m_{ij}]_{1 \leq i,j \leq n}$ with $m_{ii} := \eta_i$, $m_{ij} := -(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j$ for $i \neq j$. If the Gauss-Seidel iteration for solving the linear system

$$\mathbf{M}\mathbf{v} = \mathbf{0}, \tag{3.3.13}$$

converges to zero, the unique solution of (3.3.13)*, or equivalently, $\tilde{\lambda}_M < 1$, then every d_i degenerates into zero and the solution $\mathbf{x}(t)$ of system* (1.0.1) *converges to a singleton, where*

$$\tilde{\lambda}_M := \max_{1 \leq i \leq n} \{|\lambda_i| : \lambda_i : \text{ eigenvalue of } (D_{\mathbf{M}} - L_{\mathbf{M}})^{-1} U_{\mathbf{M}}\},$$

and $\mathbf{M} = D_{\mathbf{M}} - L_{\mathbf{M}} - U_{\mathbf{M}}$ with $D_{\mathbf{M}}$, $-L_{\mathbf{M}}$ and $-U_{\mathbf{M}}$ the diagonal, strictly lower-triangular

and strictly upper-triangular parts of $\mathbf{M}$ *respectively.*

Recall from Theorem 3.1 that there exist exactly three equilibria of system (2.2.1)-(2.2.2) which belongs to the type $\binom{3}{3}\binom{r}{3}$, and satisfies $\alpha_{21}+\beta_{21} > 0$ and $K_2(\tilde{p}_1; A_1) > 0$. If, in addition

$$\hat{F}_1(\tilde{p}_1) < 0 \quad \textit{and} \quad \check{F}_1(\tilde{q}_1) > 0, \tag{3.3.14}$$

then there exist exactly three zeros $\hat{a}_1^F$, $\hat{b}_1^F$, and $\hat{c}_1^F$ (resp., $\check{a}_1^F$, $\check{b}_1^F$, and $\check{c}_1^F$) for $\hat{F}_1$(resp., $\check{F}_1$), where $\check{a}_1^F < \hat{a}_1^F < \tilde{p}_1 < \hat{b}_1^F < \check{b}_1^F < \tilde{q}_1 < \check{c}_1^F < \hat{c}_1^F$. To further establish the convergent dynamics, we define the following functions which are vertical shifts of $\hat{F}_2$ and $\check{F}_2$:

$$\begin{aligned}
\hat{F}_2^A(\xi) &:= \hat{\Gamma}_2(\xi) + (\alpha_{21}+\beta_{21})g_1(\hat{a}_1^F) + I_2,\\
\check{F}_2^A(\xi) &:= \check{\Gamma}_2(\xi) + (\alpha_{21}+\beta_{21})g_1(\check{a}_1^F) + I_2,\\
\hat{F}_2^B(\xi) &:= \hat{\Gamma}_2(\xi) + (\alpha_{21}+\beta_{21})g_1(\check{b}_1^F) + I_2,\\
\check{F}_2^B(\xi) &:= \check{\Gamma}_2(\xi) + (\alpha_{21}+\beta_{21})g_1(\hat{b}_1^F) + I_2,\\
\hat{F}_2^C(\xi) &:= \hat{\Gamma}_2(\xi) + (\alpha_{21}+\beta_{21})g_1(\hat{c}_1^F) + I_2,\\
\check{F}_2^C(\xi) &:= \check{\Gamma}_2(\xi) + (\alpha_{21}+\beta_{21})g_1(\check{c}_1^F) + I_2,
\end{aligned} \tag{3.3.15}$$

where

$$\hat{\Gamma}_i(\xi) := \begin{cases} -\mu_i\xi + (\alpha_{ii}+\beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\check{H}_i(\hat{A}_i^H) & \text{if } \beta_{ii} \geq 0,\\ -\mu_i\xi + (\alpha_{ii}+\beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\hat{H}_i(\check{A}_i^H) & \text{if } \beta_{ii} < 0, \end{cases}$$

$$\check{\Gamma}_i(\xi) := \begin{cases} -\mu_i\xi + (\alpha_{ii}+\beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\hat{H}_i(\check{A}_i^H) & \text{if } \beta_{ii} \geq 0,\\ -\mu_i\xi + (\alpha_{ii}+\beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\check{H}_i(\hat{A}_i^H) & \text{if } \beta_{ii} < 0. \end{cases}$$

If $\alpha_{21} \geq 0$ and $\beta_{21} \geq 0$, then

$$\check{F}_2^*(\xi) \leq \hat{F}_2^*(\xi), * = A, B, C, \tag{3.3.16}$$

$$\check{F}_2(\xi) \leq \check{F}_2^A(\xi) \leq \check{F}_2^B(\xi) \leq \check{F}_2^C(\xi), \hat{F}_2^A(\xi) \leq \hat{F}_2^B(\xi) \leq \hat{F}_2^C(\xi) \leq \hat{F}_2(\xi) \tag{3.3.17}$$

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for all $\xi \in \mathbb{R}$. Now, let us consider the following condition:

$$\alpha_{21} \geq 0, \beta_{21} \geq 0, \check{F}_2^A(\tilde{p}_2) > 0. \tag{3.3.18}$$

Indeed, (3.3.18) implies $K_2(\tilde{p}_1; A_1) > 0$. Under condition (3.3.18), we can verified from (3.3.16) and (3.3.17) that there exists a unique zero $\check{m}_2^{FA}$ (resp., $\hat{m}_2^{FA}$) to $\check{F}_2^A$ (resp., $\hat{F}_2^A$); there exists a unique zero $\check{m}_2^{FB}$ (resp., $\hat{m}_2^{FB}$) to $\check{F}_2^B$ (resp., $\hat{F}_2^B$); there exists a unique zero $\check{m}_2^{FC}$ (resp., $\hat{m}_2^{FC}$) to $\check{F}_2^C$ (resp., $\hat{F}_2^C$).

Theorem 3.9. *Consider system* (2.2.1)-(2.2.2) *of subtype* $\binom{3}{3}\binom{r}{3}$. *Assume that* $\alpha_{21} + \beta_{21} > 0$, $\alpha_{21}^2 + \beta_{21}^2 \neq 0$,

$$|\beta_{ii}|\tau_{ii} < \tau_{ii}^c := \frac{(|\alpha_{ii}| + |\beta_{ii}|)\rho_i}{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + 2\sum_{j=1, j\neq i}^n (|\alpha_{ij}| + |\beta_{ij}|)\rho_j]}, \tag{3.3.19}$$

(3.3.14) *and* (3.3.18) *hold, and*

$$\begin{cases} g_1'(\xi) < \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in (-\infty, \hat{a}_1^F] \cup [\check{c}_1^F, \infty), \\ g_1'(\xi) > \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}, & \text{if } \xi \in [\hat{b}_1^F, \check{b}_1^F], \\ g_2'(\xi) < \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, & \text{if } \xi \in [\check{m}_2^{FA}, \hat{m}_2^{FA}] \cup [\check{m}_2^{FB}, \hat{m}_2^{FB}] \cup [\check{m}_2^{FC}, \hat{m}_2^{FC}], \end{cases} \tag{3.3.20}$$

for some $\theta_i \in (0, \mu_i), i = 1, 2$, *Then every solution of system* (2.2.1)-(2.2.2) *converges to an equilibrium provided*

$$\prod_{i=1,2} (1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i > \frac{L_1(\mu_2 - \theta_2)}{\alpha_{22} + \beta_{22}}(|\alpha_{12}| + |\beta_{12}|)(|\alpha_{21}| + |\beta_{21}|). \tag{3.3.21}$$

Proof. Recall that $(x_1(t), x_2(t))$ is an arbitrary solution of (2.2.1)-(2.2.2), and $x_i(t)$ satisties (3.3.1) for $i = 1, 2$. Recalling the inequalities in (3.3.11), there exist exactly three zeros $\hat{a}_1^f$, $\hat{b}_1^f$, and $\hat{c}_1^f$ (resp., $\check{a}_1^f$, $\check{b}_1^f$, and $\check{c}_1^f$) of $\hat{f}_1^{(0)}(\cdot, t_0)$ (resp., $\check{f}_1^{(0)}(\cdot, t_0)$) under condition (3.3.14), which yields $[\check{a}_1^f, \hat{a}_1^f] \subset [\check{a}_1^F, \hat{a}_1^F]$, $[\hat{b}_1^f, \check{b}_1^f] \subset [\hat{b}_1^F, \check{b}_1^F]$, and $[\check{c}_1^f, \hat{c}_1^f] \subset [\check{c}_1^F, \hat{c}_1^F]$, due to $\check{F}_1(\xi) < \check{f}_1^{(0)}(\xi, t_0) < \hat{f}_1^{(0)}(\xi, t_0) < \hat{F}_1(\xi)$ for all $\xi \in \mathbb{R}$, under conditon $\alpha_{21}^2 + \beta_{21}^2 \neq 0$. Then, it is not difficult to verify that (3.3.1) with $i = 1$ satisfies (B1)-(B3) under (3.3.19), (3.3.14), and

the first two inequalities in (3.3.20). By Proposition 3.2, $x_1(t)$ converges to one of the three disjoint intervals: $[\underline{a}_1, \overline{a}_1], [\underline{b}_1, \overline{b}_1]$, and $[\underline{c}_1, \overline{c}_1]$ with $[\underline{a}_1, \overline{a}_1] \subset [\check{a}_1^F, \hat{a}_1^F]$, $[\underline{b}_1, \overline{b}_1] \subset [\hat{b}_1^F, \check{b}_1^F]$, $[\underline{c}_1, \overline{c}_1] \subset [\check{c}_1^F, \hat{c}_1^F]$, and

$$0 \leq \overline{a}_1 - \underline{a}_1, \overline{b}_1 - \underline{b}_1, \overline{c}_1 - \underline{c}_1 \leq [w_1^{max}(\infty) - w_1^{min}(\infty)]/[(1 - 2|\beta_{11}|L_1\tau_{11})\theta_1]. \quad (3.3.22)$$

Let us discuss the following three cases:

(i) If $x_1(t)$ converges to $[\underline{a}_1, \overline{a}_1] \subset [\check{a}_1^F, \hat{a}_1^F]$, there exists a T_1 such that $\check{f}_2^{(0)}(\tilde{p}_2, T_1) > \check{F}_2^A(\tilde{p}_2) > 0$, recalling $\alpha_{21} + \beta_{21} > 0$, $\alpha_{21} \geq 0$, and $\beta_{21} \geq 0$. Accordingly, there exists exactly one zero $\hat{m}_2^f$ (resp., $\check{m}_2^f$) of $\hat{f}_2^{(0)}(\cdot, T_1)$ (resp., $\check{f}_2^{(0)}(\cdot, T_1)$), where $[\check{m}_2^f, \hat{m}_2^f] \subset [\check{m}_2^{FA}, \hat{m}_2^{FA}]$. It is not difficult to verify that (3.3.1) with $i = 2$ satisfies conditions (M1)-(M3) under (3.3.19), the last inequality in (3.3.18), and the third inequality in (3.3.20). By Proposition 3.3, $x_2(t)$ converges to an interval $[\underline{m}_2, \overline{m}_2]$ with $[\underline{m}_2, \overline{m}_2] \subset [\check{m}_2^{FA}, \hat{m}_2^{FA}]$ satisfying

$$0 \leq \overline{m}_2 - \underline{m}_2 \leq [w_2^{max}(\infty) - w_2^{min}(\infty)]/[(1 - 2|\beta_{22}|L_2\tau_{22})\theta_2]. \quad (3.3.23)$$

Notice that $[\underline{a}_1, \overline{a}_1] \subset [\check{a}_1^F, \hat{a}_1^F]$, $[\underline{m}_2, \overline{m}_2] \subset [\check{m}_2^{FA}, \hat{m}_2^{FA}]$, and

$$g_i'(\xi) \leq \begin{cases} \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & i = 1, \xi \in [\check{a}_1^F, \hat{a}_1^F], \\ \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, & i = 2, \xi \in [\check{m}_2^{FA}, \hat{m}_2^{FA}]. \end{cases} \quad (3.3.24)$$

By Proposition 3.3, $(x_1(t), x_2(t))$ converges to a singleton if the Gauss-Seidel iteration for solving the linear system:

$$\begin{pmatrix} (1 - 2|\beta_{11}|L_1\tau_{11})\theta_1 & \frac{-(\mu_2 - \theta_2)(|\alpha_{12}| + |\beta_{12}|)}{\alpha_{22} + \beta_{22}} \\ \frac{-(\mu_1 - \theta_1)(|\alpha_{21}| + |\beta_{21}|)}{\alpha_{11} + \beta_{11}} & (1 - 2|\beta_{22}|L_2\tau_{22})\theta_2 \end{pmatrix} \mathbf{v} = \mathbf{0} \quad (3.3.25)$$

converges to zero. We note that if $\mathbf{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, with $a, d > 0$, $b, c < 0$, then $\tilde{\lambda}_M = bc/(ad)$.

Thus, the convergence to zero is assured if

$$\prod_{i=1,2}(1-2|\beta_{ii}|L_i\tau_{ii})\theta_i > L_1(\mu_2-\theta_2)(|\alpha_{12}|+|\beta_{12}|)(|\alpha_{21}|+|\beta_{21}|)/(\alpha_{22}+\beta_{22}),$$

which holds under (3.3.21).

(ii) If $x_1(t)$ converges to $[\underline{b}_1,\overline{b}_1] \subset [\hat{b}_1^F,\check{b}_1^F]$, there exists a T_1 such that $\check{f}_2^{(0)}(\tilde{p}_2,T_1) > \check{F}_2^B(\tilde{p}_2) > 0$, recalling $\alpha_{21}+\beta_{21} > 0$, $\alpha_{21} \geq 0$, and $\beta_{21} \geq 0$. Accordingly, there exists exactly one zero $\hat{m}_2^f$ (resp., $\check{m}_2^f$) of $\hat{f}_2^{(0)}(\cdot,T_1)$ (resp., $\check{f}_2^{(0)}(\cdot,T_1)$), where $[\check{m}_2^f,\hat{m}_2^f] \subset [\check{m}_2^{FB},\hat{m}_2^{FB}]$. It is not difficult to verify that (3.3.1) with $i=2$ satisfies conditions (M1)-(M3) under (3.3.19), the last inequality in (3.3.18), and the third inequality in (3.3.20). By Proposition 3.3, $x_2(t)$ converges to an interval $[\underline{m}_2,\overline{m}_2]$ with $[\underline{m}_2,\overline{m}_2] \subset [\check{m}_2^{FB},\hat{m}_2^{FB}]$ satisfying

$$0 \leq \overline{m}_2 - \underline{m}_2 \leq [w_2^{max}(\infty) - w_2^{min}(\infty)]/[(1-2|\beta_{22}|L_2\tau_{22})\theta_2]. \tag{3.3.26}$$

Notice that $[\underline{b}_1,\overline{b}_1] \subset [\hat{b}_1^F,\check{b}_1^F]$, $[\underline{m}_2,\overline{m}_2] \subset [\check{m}_2^{FB},\hat{m}_2^{FB}]$, and

$$g_i'(\xi) \leq \begin{cases} L_1, & i=1, \xi \in [\hat{b}_1^F,\check{b}_1^F], \\ \frac{\mu_2-\theta_2}{\alpha_{22}+\beta_{22}}, & i=2, \xi \in [\check{m}_2^{FB},\hat{m}_2^{FB}]. \end{cases} \tag{3.3.27}$$

By Proposition 3.3, $(x_1(t),x_2(t))$ converges to a singleton if the Gauss-Seidel iteration for solving the linear system:

$$\begin{pmatrix} (1-2|\beta_{11}|L_1\tau_{11})\theta_1 & \frac{-(\mu_2-\theta_2)(|\alpha_{12}|+|\beta_{12}|)}{\alpha_{22}+\beta_{22}} \\ -L_1(|\alpha_{21}|+|\beta_{21}|) & (1-2|\beta_{22}|L_2\tau_{22})\theta_2 \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{3.3.28}$$

converges to zero. Thus, the convergence to zero is assured if

$$\prod_{i=1,2}(1-2|\beta_{ii}|L_i\tau_{ii})\theta_i > L_1(\mu_2-\theta_2)(|\alpha_{12}|+|\beta_{12}|)(|\alpha_{21}|+|\beta_{21}|)/(\alpha_{22}+\beta_{22}),$$

which holds under (3.3.21).

(iii) If $x_1(t)$ converges to $[\underline{c}_1,\overline{c}_1] \subset [\check{c}_1^F,\hat{c}_1^F]$, there exists a T_1 such that $\check{f}_2^{(0)}(\tilde{p}_2,T_1) >$

$\check{F}_2^C(\tilde{p}_2) > 0$, recalling $\alpha_{21} + \beta_{21} > 0$, $\alpha_{21} \geq 0$, and $\beta_{21} \geq 0$. Accordingly, there exists exactly one zero $\hat{m}_2^f$ (resp., $\check{m}_2^f$) of $\hat{f}_2^{(0)}(\cdot, T_1)$ (resp., $\check{f}_2^{(0)}(\cdot, T_1)$), where $[\check{m}_2^f, \hat{m}_2^f] \subset [\check{m}_2^{FC}, \hat{m}_2^{FC}]$. It is not difficult to verify that (3.3.1) with $i = 2$ satisfies conditions (M1)-(M3) under (3.3.19), the last inequality in (3.3.18), and the third inequality in (3.3.20). By Proposition 3.3, $x_2(t)$ converges to an interval $[\underline{m}_2, \overline{m}_2]$ with $[\underline{m}_2, \overline{m}_2] \subset [\check{m}_2^{FC}, \hat{m}_2^{FC}]$ satisfying

$$0 \leq \overline{m}_2 - \underline{m}_2 \leq [w_2^{max}(\infty) - w_2^{min}(\infty)]/[(1 - 2|\beta_{22}|L_2\tau_{22})\theta_2]. \tag{3.3.29}$$

Notice that $[\underline{c}_1, \overline{c}_1] \subset [\check{c}_1^F, \hat{c}_1^F]$, $[\underline{m}_2, \overline{m}_2] \subset [\check{m}_2^{FC}, \hat{m}_2^{FC}]$, and

$$g_i'(\xi) \leq \begin{cases} \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, & i = 1, \xi \in [\check{c}_1^F, \hat{c}_1^F], \\ \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}, & i = 2, \xi \in [\check{m}_2^{FC}, \hat{m}_2^{FC}]. \end{cases} \tag{3.3.30}$$

By Proposition 3.3, $(x_1(t), x_2(t))$ converges to a singleton if the Gauss-Seidel iteration for solving the linear system:

$$\begin{pmatrix} (1 - 2|\beta_{11}|L_1\tau_{11})\theta_1 & \frac{-(\mu_2 - \theta_2)(|\alpha_{12}| + |\beta_{12}|)}{\alpha_{22} + \beta_{22}} \\ \frac{-(\mu_1 - \theta_1)(|\alpha_{21}| + |\beta_{21}|)}{\alpha_{11} + \beta_{11}} & (1 - 2|\beta_{22}|L_2\tau_{22})\theta_2 \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{3.3.31}$$

converges to zero. Thus, the convergence to zero is assured if

$$\prod_{i=1,2} (1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i > L_1(\mu_2 - \theta_2)(|\alpha_{12}| + |\beta_{12}|)(|\alpha_{21}| + |\beta_{21}|)/(\alpha_{22} + \beta_{22}),$$

which holds under (3.3.21). This completes the proof. □

Chapter 4

Numerical examples

In this chapter, we demonstrate four examples for (1.0.1) with $n = 2$ i.e.,(2.2.1)-(2.2.2) to illustrate the theorems in Sections 3.1-3.2. We choose the activation function as $g_i(\xi) = tanh(\xi)$, and thus $\sigma_i = 0$, $\rho_i = 1$, $L_i = 1$, for $i = 1, 2$.

Example 1. This example illustrates Theorems 3.1 and 3.9. We consider the parameters

$$\begin{aligned}
&\mu_1 = 1, \alpha_{11} = 1.5, \alpha_{12} = 0.07, \beta_{11} = 0.1, \beta_{12} = 0.08,\\
&\mu_2 = 1, \alpha_{21} = 0.1, \alpha_{22} = 1.5, \beta_{21} = 0.02, \beta_{22} = 0.25,\\
&I_1 = -0.05,\ I_2 = 0.48,\\
&\tau_{11} = 0.009,\ \tau_{22} = 0.001\ \tau_{12} = \tau_{21} = 10.
\end{aligned}$$

Then the system (2.2.1)-(2.2.2) becomes:

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -x_1(t) + 1.5g_1(x_1(t)) + 0.07g_2(x_2(t)) && (4.0.1)\\
&\quad +0.1g_1(x_1(t-0.009)) + 0.08g_2(x_2(t-10)) - 0.05,\\
\frac{dx_2(t)}{dt} &= -x_2(t) + 0.1g_1(x_1(t)) + 1.5g_2(x_2(t)) && (4.0.2)\\
&\quad +0.02g_1(x_1(t-10)) + 0.25g_2(x_2(t-0.001)) + 0.48.
\end{aligned}$$

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The stationary equations are

$$\begin{aligned} F_1(\mathbf{x}) &:= -x_1 + 1.5g_1(x_1) + 0.07g_2(x_2) \\ &\quad +0.1g_1(x_1) + 0.08g_2(x_2) - 0.05 = 0, \\ F_2(\mathbf{x}) &:= -x_2 + 0.1g_1(x_1) + 1.5g_2(x_2) \\ &\quad +0.02g_1(x_1) + 0.25g_2(x_2) + 0.48 = 0, \end{aligned}$$

where $\mathbf{x} = (x_1, x_2)$. Herein, $\alpha_{12} + \beta_{12} = 0.15 > 0$, $\alpha_{21} + \beta_{21} = 0.12 > 0$. As introduced in Section 2.1, we have $k_1^+ = 0.1$, $k_1^- = -0.2$, $\tilde{p}_1 = -0.7127085$, $\tilde{q}_1 = 0.7127085$, $k_2^+ = 0.6$, $k_2^- = 0.36$, $\tilde{p}_2 = -0.7833995$, $\tilde{q}_2 = 0.7833995$, and

$$\begin{aligned} \hat{f}_1(\xi) &:= -\xi + 1.6g_i(\xi) + 0.1, \\ \check{f}_1(\xi) &:= -\xi + 1.6g_i(\xi) - 0.2, \\ \hat{f}_2(\xi) &:= -\xi + 1.75g_i(\xi) + 0.6, \\ \check{f}_2(\xi) &:= -\xi + 1.75g_i(\xi) + 0.36. \end{aligned}$$

Since $\hat{f}_1(\tilde{p}_1) = -1.6709 < 0$ and $\check{f}_1(\tilde{q}_1) = 0.06709 > 0$, $i = 1 \in \mathcal{B}_3^3$ by the definition of $\mathcal{B}$ in Section 2.1. Since $\hat{f}_2(\tilde{p}_2) = 0.23776 > 0$, $\check{f}_2(\tilde{p}_2) = -0.002244 < 0$, and $\check{f}_2(\tilde{q}_2) = 0.72224$, $i = 2 \in \mathcal{B}_3^r$ by the definition of $\mathcal{B}$ in Section 2.1. We have $A_1 := [\check{a}_1, \hat{a}_1] = [-1.69581, -1.26304]$, $B_1 := [\hat{b}_1, \check{b}_1] = [-0.17106, 0.37915]$, and $C_1 := [\check{c}_1, \hat{c}_1] = [1.05235, 1.56630]$. By Notation 2.2, we can compute that

$$K_2(\tilde{p}_2; A_1) = [0.005569, 0.22773] > 0,$$

which verifies $K_2(\tilde{p}_2; A_1) > 0$.

Now, let us choose $\theta_1 = 0.32$ and $\theta_2 = 0.8$. For (3.1.2), we have

$$\theta_1 = 0.32 > \bar{L}_2|\alpha_{12} + \beta_{12}| = 1|0.15|, \theta_2 = 0.8 > \bar{L}_1|\alpha_{21} + \beta_{21}| = 1|0.12|. \tag{4.0.3}$$

Thus, we confirm that condition (3.1.2) holds.

As seen from Figure 4.1, for $\xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty)$

$$g_1'(\xi) < g_1'(\check{c}_1) = 0.38735 < 0.425 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, \tag{4.0.4}$$

and for $\xi \in [\hat{b}_1, \check{b}_1]$,

$$g_1'(\xi) > g_1'(\check{b}_1) = 0.86898 > 0.825 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}. \tag{4.0.5}$$

Thus, we confirm that condition (3.1.3) holds.

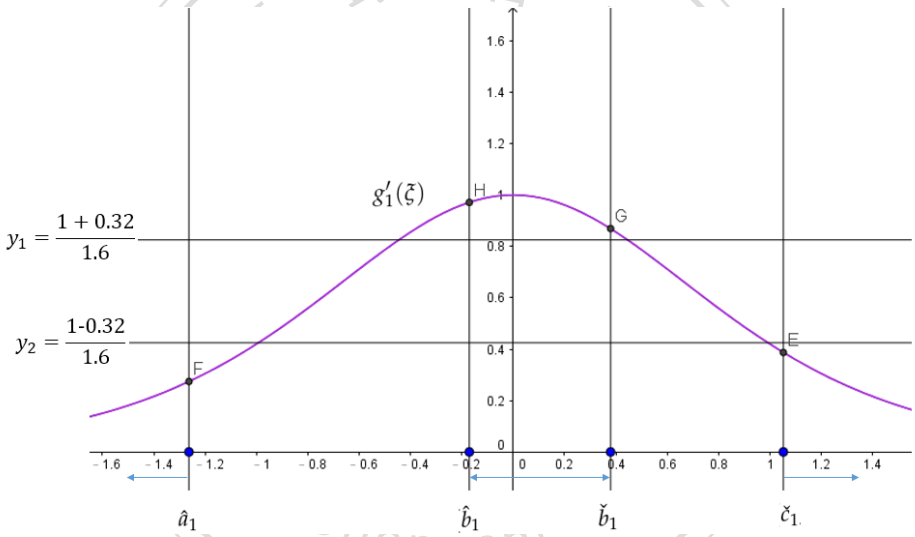

Figure 4.1: g_1' with $\theta_1 = 0.32$.

As seen from Figure 4.2, for $\xi \in [\check{m}_2^{(a_1)}, \hat{m}_2^{(c_1)}]$

$$g_2'(\xi) < g_2'(\check{m}_2^{(a_1)}) = 0.06266 < 0.11429 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}. \tag{4.0.6}$$

Thus, we confirm that condition (3.1.4) holds.

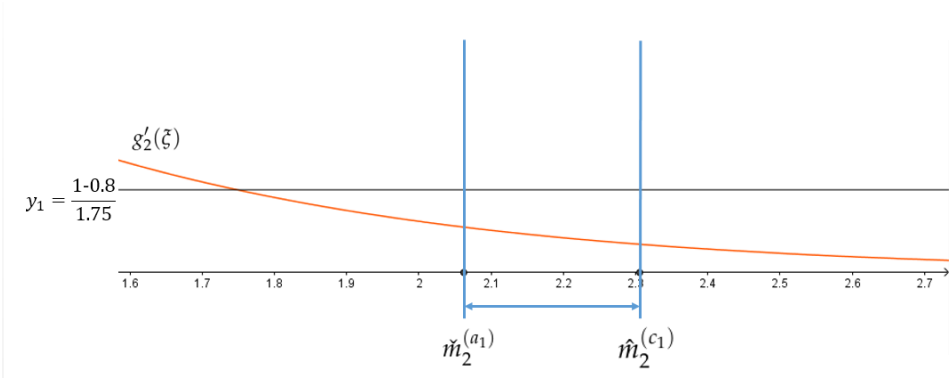

Figure 4.2: g_2' with $\theta_2 = 0.8$.

We have verified all conditions in Theorem 3.1. Next, let us examine the conditons in Theorem 3.9.

For (3.3.19), we have

$$\beta_{11}\tau_{11} = 0.0009 < \tau_{11}^c = 0.23881,$$

$$\beta_{22}\tau_{22} = 0.00025 < \tau_{11}^c = 0.241712.$$

We compute that

$$\hat{F}_1(\tilde{p}_1) = -0.1610574 < 0,\ \check{F}_1(\tilde{q}_1) = 0.061057 > 0.$$

Thus, (3.3.14) holds. We can compute (3.3.15) to obtain

$$\check{F}_2^A(\tilde{p}_2) = 0.003646 > 0,$$

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which justifies (3.3.18). For (3.3.20), we first compute the following quantities:

$$\hat{a}_1^F = -1.25222, \hat{b}_1^F = -0.182006, \hat{c}_1^F = 1.574382,$$
$$\check{a}_1^F = -1.703353, \check{b}_1^F = 0.39491915, \check{c}_1^F = 1.574382,$$
$$\hat{m}_2^{FA} = 2.071642, \check{m}_2^{FA} = 2.059937,$$
$$\hat{m}_2^{FB} = 2.233534, \check{m}_2^{FB} = 2.1607133,$$
$$\hat{m}_2^{FC} = 2.303742, \check{m}_2^{FC} = 2.285509.$$

As seen from Figure 4.3, for $\xi \in (-\infty, \hat{a}_1^F] \cup [\check{c}_1^F, \infty)$,

$$g_1'(\xi) < g_1'(\check{c}_1^F) = 0.38735 < 0.425 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, \tag{4.0.7}$$

and for $\xi \in (\hat{b}_1^F, \check{b}_1^F]$

$$g_1'(\xi) > g_1'(\check{b}_1^F) = 0.85893 > 0.825 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}. \tag{4.0.8}$$

As seen from Figure 4.4, for $\xi \in [\check{m}_2^{FA}, \hat{m}_2^{FA}] \cup [\check{m}_2^{FB}, \hat{m}_2^{FB}] \cup [\check{m}_2^{FC}, \hat{m}_2^{FC}]$

$$g_2'(\xi) < g_2'(\check{m}_2^{FA}) = 0.062925 < 0.11429 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}. \tag{4.0.9}$$

By (4.0.7)-(4.0.9), condition (3.3.20) hlods for $\theta_1 = 0.32$ and $\theta_2 = 0.8$. For (3.3.21),

$$\begin{aligned}\prod_{i=1,2}(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i &= 0.2953194664 > 0.0070875 \\ &= \frac{L_1(\mu_2 - \theta_2)}{\alpha_{22} + \beta_{22}}(|\alpha_{12}| + |\beta_{12}|)(|\alpha_{21}| + |\beta_{21}|)\end{aligned}$$

We have verified that all conditions in Theorem 3.9 hold.

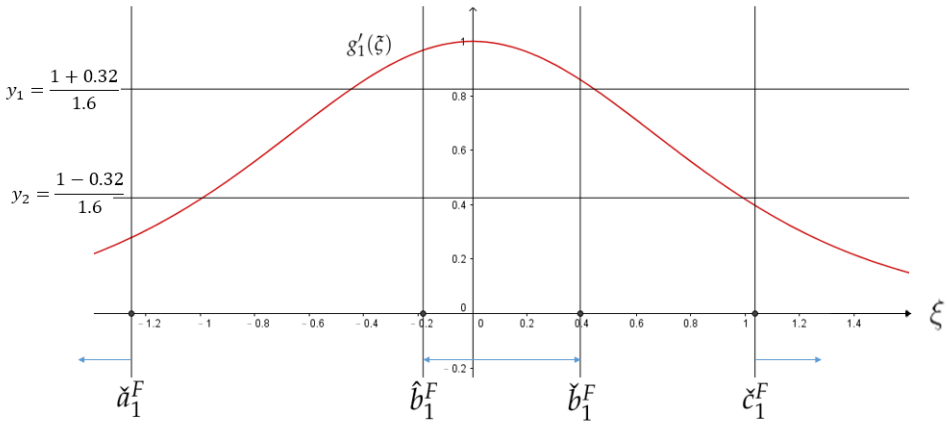

Figure 4.3: g_1' with $\theta_1 = 0.32$ for Theorem 3.9.

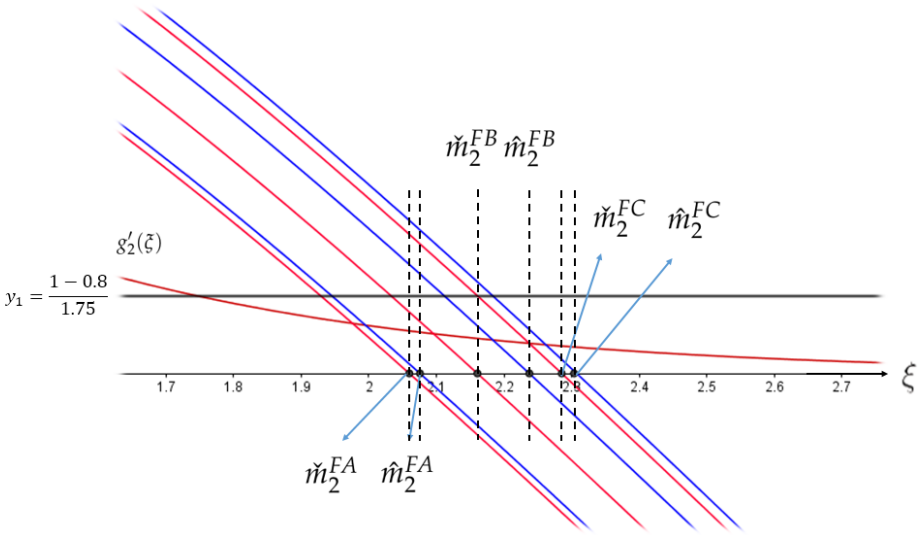

Figure 4.4: g_2' with $\theta_2 = 0.8$ for Theorem 3.9.

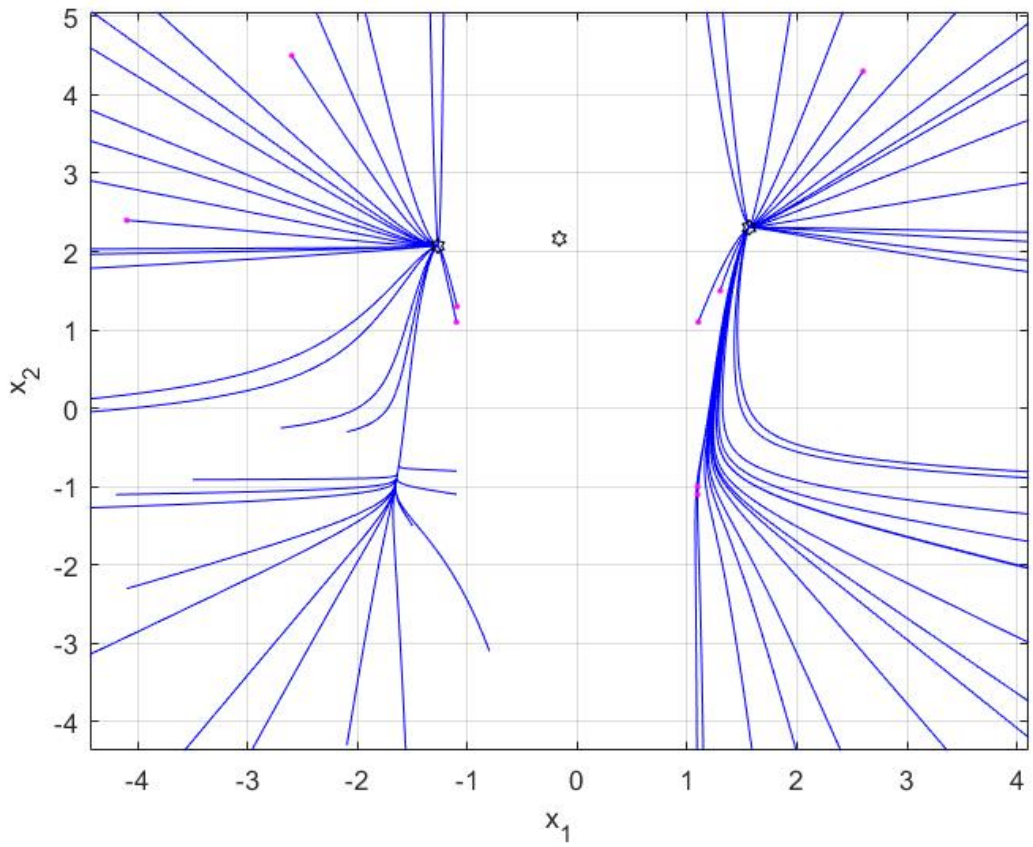

Figure 4.5: Numerical simulation for Example 1.

By Theorem 3.1 and 3.9, the system (4.0.1)-(4.0.2) admits exactly three equilibria and every solution converges to one of equilibria. We note three equilibria can be solved numerically and are listed below: $\mathbf{x_1^*}$:=(-1.27132, 2.07296), $\mathbf{x_2^*}$:=(-0.16405, 2.165001), $\mathbf{x_3^*}$:=(1.562319, 2.30544), cf Figure 4.5. Moreover, Figure 4.5 shows that every solution converges to $\mathbf{x_1^*}$ or $\mathbf{x_3^*}$. This indicates that $\mathbf{x_1^*}$ and $\mathbf{x_3^*}$ are stable, and $\mathbf{x_2^*}$ is unstable.

Example 2. This example illustrates Theorem 3.4. We consider the parameters

$$\mu_1 = 1, \alpha_{11} = 1.5, \alpha_{12} = 0.07, \beta_{11} = 0.1, \beta_{12} = 0.08,$$

$$\mu_2 = 1, \alpha_{21} = 0.1, \alpha_{22} = 1.4, \beta_{21} = 0.1, \beta_{22} = 0.5,$$

$$I_1 = -0.05,\ I_2 = 0.32,$$

$$\tau_{11} = \tau_{22} = 0.05,\ \tau_{12} = \tau_{21} = 10.$$

Then the system (2.2.1)-(2.2.2) becomes:

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 1.5g_1(x_1(t)) + 0.07g_2(x_2(t)) \qquad (4.0.10)\\ &\quad +0.1g_1(x_1(t-0.05)) + 0.08g_2(x_2(t-10)) - 0.05,\\ \frac{dx_2(t)}{dt} &= -x_2(t) + 0.1g_1(x_1(t)) + 1.4g_2(x_2(t)) \qquad (4.0.11)\\ &\quad +0.1g_1(x_1(t-10)) + 0.5g_2(x_2(t-0.05)) + 0.32.\end{aligned}$$

The stationary equations are

$$F_1(\mathbf{x}) := -x_1 + 1.5g_1(x_1) + 0.07g_2(x_2) + 0.1g_1(x_1) + 0.08g_2(x_2) - 0.05 = 0,$$
$$F_2(\mathbf{x}) := -x_2 + 0.1g_1(x_1) + 1.4g_2(x_2) + 0.1g_1(x_1) + 0.5g_2(x_2) + 0.32 = 0.$$

Herein, $\alpha_{12} + \beta_{12} = 0.15 > 0$, $\alpha_{21} + \beta_{21} = 0.2 > 0$. As introduced in Section 2.1, we have $k_1^+ = 0.1$, $k_1^- = -0.2$, $\tilde{p}_1 = -0.7127085$, $\tilde{q}_1 = 0.7127085$, $k_2^+ = 0.52$, $k_2^- = 0.12$, $\tilde{p}_2 = -0.7127085$, $\tilde{q}_2 = 0.7127085$, and the single-variable upper and lower functions are:

$$\begin{aligned}\hat{f}_1(\xi) &:= -\xi + 1.6g_i(\xi) + 0.1,\\ \check{f}_1(\xi) &:= -\xi + 1.6g_i(\xi) - 0.2,\\ \hat{f}_2(\xi) &:= -\xi + 1.9g_i(\xi) + 0.52,\\ \check{f}_2(\xi) &:= -\xi + 1.9g_i(\xi) + 0.12.\end{aligned}$$

Since $\hat{f}_1(\tilde{p}_1) = -1.6709 < 0$ and $\check{f}_1(\tilde{q}_1) = 0.06709 > 0$, $i = 1 \in \mathcal{B}_3^3$ by the definition of $\mathcal{B}$ in Section 2.1. Since $\hat{f}_2(\tilde{p}_2) = 0.05695 > 0$, $\check{f}_2(\tilde{p}_2) = -0.34305 < 0$, and $\check{f}_2(\tilde{q}_2) = 0.58305$, $i = 2 \in \mathcal{B}_3^r$ by the definition of $\mathcal{B}$ in Section 2.1. We have $A_1 := [\check{a}_1, \hat{a}_1] = [-1.69581, -1.26304]$ and $B_1 := [\hat{b}_1, \check{b}_1] = [-0.17106, 0.37915]$, and $C_1 := [\check{c}_1, \hat{c}_1] = [1.05235, 1.56630]$. By Notation 2.2, we can compute that

$$K_2(\tilde{p}_2; B_1) = [-0.17694, -0.07066] < 0,$$
$$K_2(\tilde{p}_2; C_1) = [0.01349, 0.04023] > 0.$$

which verify $K_2(\tilde{p}_2; B_1) < 0 < K_2(\tilde{p}_2; C_1)$.

Now, let us choose $\theta_1 = 0.37$ and $\theta_2 = 0.4$. For (3.1.26), we have

$$\theta_1 = 0.37 > \bar{L}_2|\alpha_{12} + \beta_{12}| = 1|0.15|, \theta_2 = 0.4 > \bar{L}_1|\alpha_{21} + \beta_{21}| = 1|0.2|. \tag{4.0.12}$$

Thus, we confirm that condition (3.1.26) holds.

As seen from Figure 4.6, for $\xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty)$

$$g_1'(\xi) < g_1'(\check{c}_1) = 0.38735 < 0.39375 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}} \tag{4.0.13}$$

and for $\xi \in [\hat{b}_1, \check{b}_1]$,

$$g_1'(\xi) > g_1'(\check{b}_1) = 0.86898 > 0.85625 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}.$$

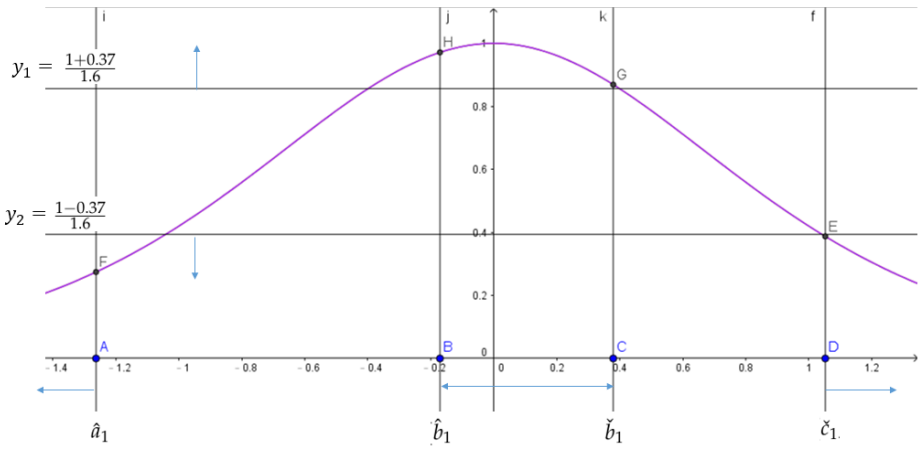

Figure 4.6: g_1' with $\theta_1 = 0.37$.

As seen from Figure 4.7, for $\xi \in [\check{m}_2^{(c_1)}, \hat{m}_2^{(c_1)}]$

$$g_2'(\xi) < g_2'(\check{m}_2^{(c_1)}) = 0.03631 < 0.31579 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}} \tag{4.0.14}$$

and for $\xi \in (-\infty, \hat{a}_2^{(b_1)}] \cup [\check{c}_2^{(a_1)}, \infty)$

$$g_2'(\xi) < g_2'(\hat{a}_2^{(b_1)}) = 0.31563 < 0.31579 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}.$$

and for $\xi \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(b_1)}]$

$$g_2'(\xi) > g_2'(\hat{b}_2^{(b_1)}) = 0.76412 > 0.73684 = \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}.$$

We have verified all conditions in Theorem 3.4. Therefore, the system (4.0.10) and (4.0.11) has exactly seven equilibira.

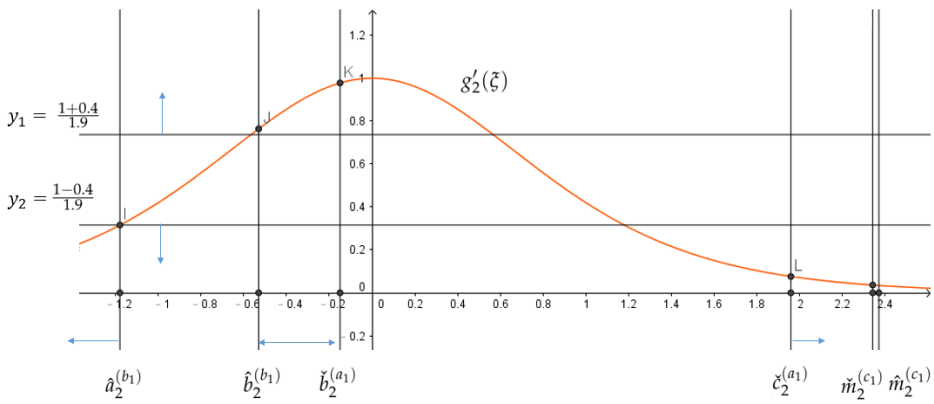

Figure 4.7: g_2' with $\theta_2 = 0.4$.

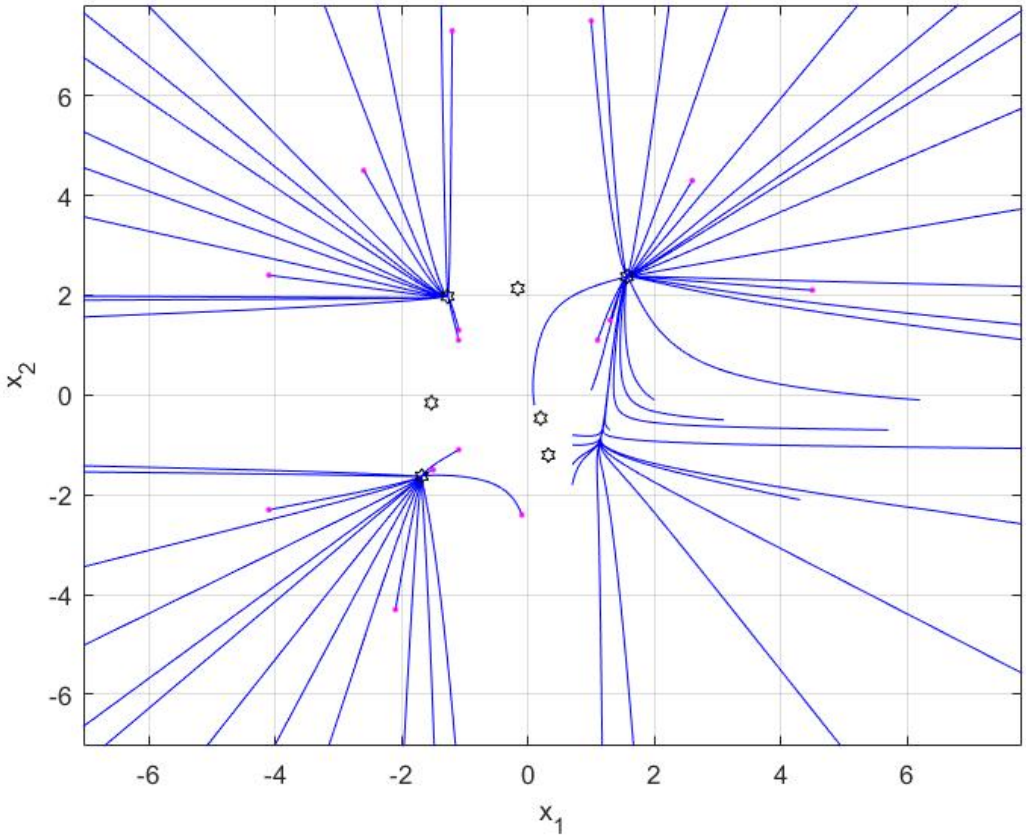

Figure 4.8: Numerical simulation for Example 2.

We note seven equilibria can be solved numerically and are listed below: (0.19844, -0.46369), (-1.27302, 1.97767), (-1.52983, -0.15589), (-1.68173, -1.62469), (-0.16363, 2.13518), (0.32012, -1.20478), (1.5628, 2.37027), cf Figure 4.8(marked by stars). By numerical stimulation, we observe that four are unstable equilibria and three are stable equilibria.

Example 3. This example illustrates Theorem 3.2. We consider the parameters

$$\mu_1 = 1, \alpha_{11} = 1.5, \alpha_{12} = 0.07, \beta_{11} = 0.1, \beta_{12} = 0.08,$$

$$\mu_2 = 1, \alpha_{21} = 0.05, \alpha_{22} = 1.4, \beta_{21} = 0.058, \beta_{22} = 0.63,$$

$$I_1 = -0.05,\ I_2 = 0.448,$$

$$\tau_{11} = \tau_{22} = 0.05,\ \tau_{12} = \tau_{21} = 10.$$

Then the system becomes:

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 1.5g_1(x_1(t)) + 0.07g_2(x_2(t)) \qquad (4.0.15)\\ &\quad +0.1g_1(x_1(t-0.05)) + 0.08g_2(x_2(t-10)) - 0.05,\\ \frac{dx_2(t)}{dt} &= -x_2(t) + 0.05g_1(x_1(t)) + 1.4g_2(x_2(t)) \qquad (4.0.16)\\ &\quad +0.058g_1(x_1(t-10)) + 0.63g_2(x_2(t-0.05)) + 0.448.\end{aligned}$$

The stationary equations are

$$\begin{aligned}F_1(\mathbf{x}) &:= -x_1 + 1.5g_1(x_1) + 0.07g_2(x_2) \qquad (4.0.17)\\ &\quad +0.1g_1(x_1) + 0.08g_2(x_2) - 0.05 = 0,\\ F_2(\mathbf{x}) &:= -x_2 + 0.05g_1(x_1) + 1.4g_2(x_2) \qquad (4.0.18)\\ &\quad +0.058g_1(x_1) + 0.63g_2(x_2) + 0.448 = 0.\end{aligned}$$

Herein, $\alpha_{12} + \beta_{12} = 0.15 > 0$, $\alpha_{21} + \beta_{21} = 0.108 > 0$. As introduced in Section 2.1, we have $k_1^+ = 0.1$, $k_1^- = -0.2$, $\tilde{p}_1 = -0.71271$, $\tilde{q}_1 = 0.71271$, $k_2^+ = 0.556$, $k_2^- = 0.34$, $\tilde{p}_2 = -0.89186$, $\tilde{q}_2 = 0.89186$, and the single-variable upper and lower functions are:

$$\begin{aligned}\hat{f}_1(\xi) &:= -\xi + 1.6g_i(\xi) + 0.1,\\ \check{f}_1(\xi) &:= -\xi + 1.6g_i(\xi) - 0.2,\\ \hat{f}_2(\xi) &:= -\xi + 2.03g_i(\xi) + 0.556,\\ \check{f}_2(\xi) &:= -\xi + 2.03g_i(\xi) + 0.34.\end{aligned}$$

Since $\hat{f}_1(\tilde{p}_1) = -1.6709 < 0$ and $\check{f}_1(\tilde{q}_1) = 0.06709 > 0$, $i = 1 \in \mathcal{B}_3^3$ by the definition of $\mathcal{B}$ in Section 2.1. Since $\hat{f}_2(\tilde{p}_2) = 0.00187 > 0$, $\check{f}_2(\tilde{p}_2) = -0.21413 < 0$, and $\check{f}_2(\tilde{q}_2) = 0.89413 > 0$, $i = 2 \in \mathcal{B}_3^r$ by the definition of $\mathcal{B}$ in Section 2.1. We have $A_1 := [\check{a}_1, \hat{a}_1] = [-1.69581, -1.26304]$ and $B_1 := [\hat{b}_1, \check{b}_1] = [-0.17106, 0.37915]$, and

$C_1 := [\check{c}_1, \hat{c}_1] = [1.05235, 1.56630]$. By Notation 2.2, we can compute that

$$K_2(\tilde{p}_2; C_1) = [-0.03786, -0.01111] < 0.$$

which verify $K_2(\tilde{p}_2; C_1) < 0$.

Now, let us choose $\theta_1 = 0.37$ and $\theta_2 = 0.132$. For (3.1.10), we have

$$\theta_1 = 0.37 > \bar{L}_2|\alpha_{12} + \beta_{12}| = 1|0.15|, \theta_2 = 0.132 > \bar{L}_1|\alpha_{21} + \beta_{21}| = 1|0.108|. \quad (4.0.19)$$

Thus, we confirm that condition (3.1.10) holds.

As seen from Figure 4.9, for $\xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty)$

$$g_1'(\xi) < g_1'(\check{c}_1) = 0.38735 < 0.39375 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}} \quad (4.0.20)$$

and for $\xi \in [\hat{b}_1, \check{b}_1]$,

$$g_1'(\xi) > g_1'(\check{b}_1) = 0.86898 > 0.85625 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}.$$

Thus, we confirm that condition (3.1.11) holds.

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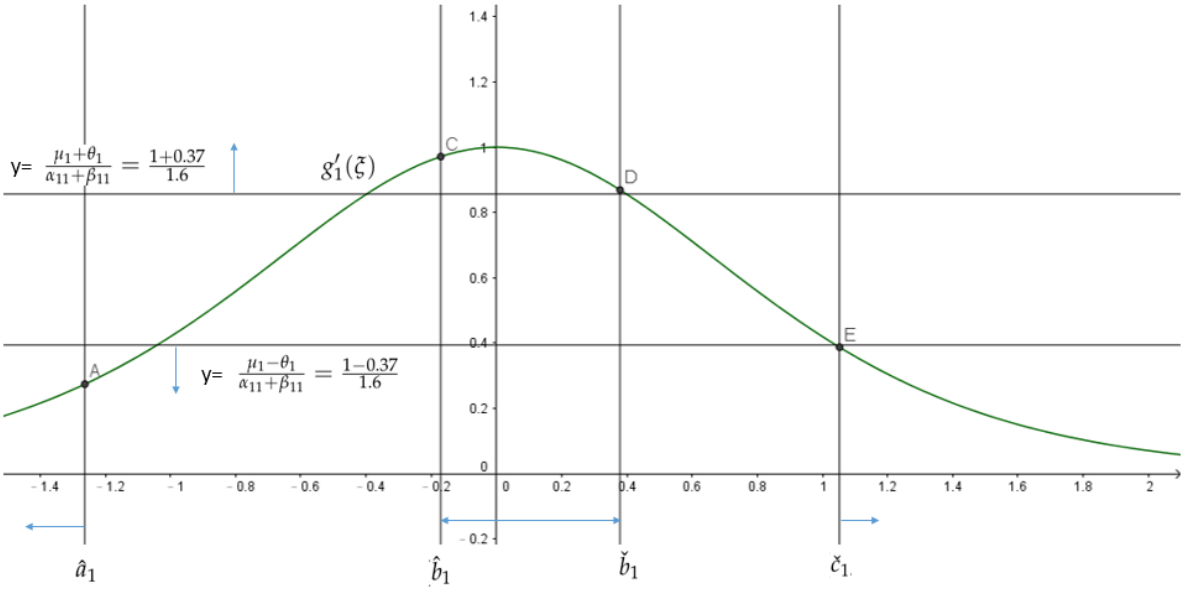

Figure 4.9: g_1' with $\theta_1 = 0.37$.

As seen from Figure 4.10, for $\xi \in (-\infty, \hat{a}_2^{(c_1)}] \cup [\check{c}_2^{(a_1)}, \infty)$

$$g_2'(\xi) < g_2'(\hat{a}_2^{(c_1)}) = 0.42418 < 0.42759 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}.$$

and for $\xi \in [\hat{b}_2^{(a_1)}, \check{b}_2^{(b_1)}]$

$$g_2'(\xi) > g_2'(\hat{b}_2^{(a_1)}) = 0.56446 > 0.55764 = \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}.$$

Thus, we confirm that condition (3.1.12) holds.

We have verified all conditions in Theorem 3.2. Therefore, the system (4.0.15)-(4.0.16) admits exactly nine equilibira.

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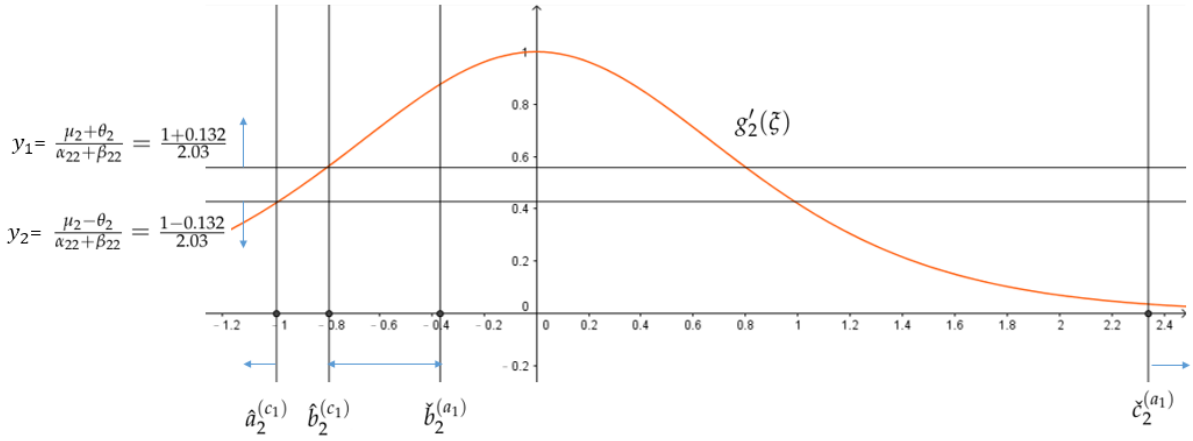

Figure 4.10: g_2' with $\theta_2 = 0.132$.

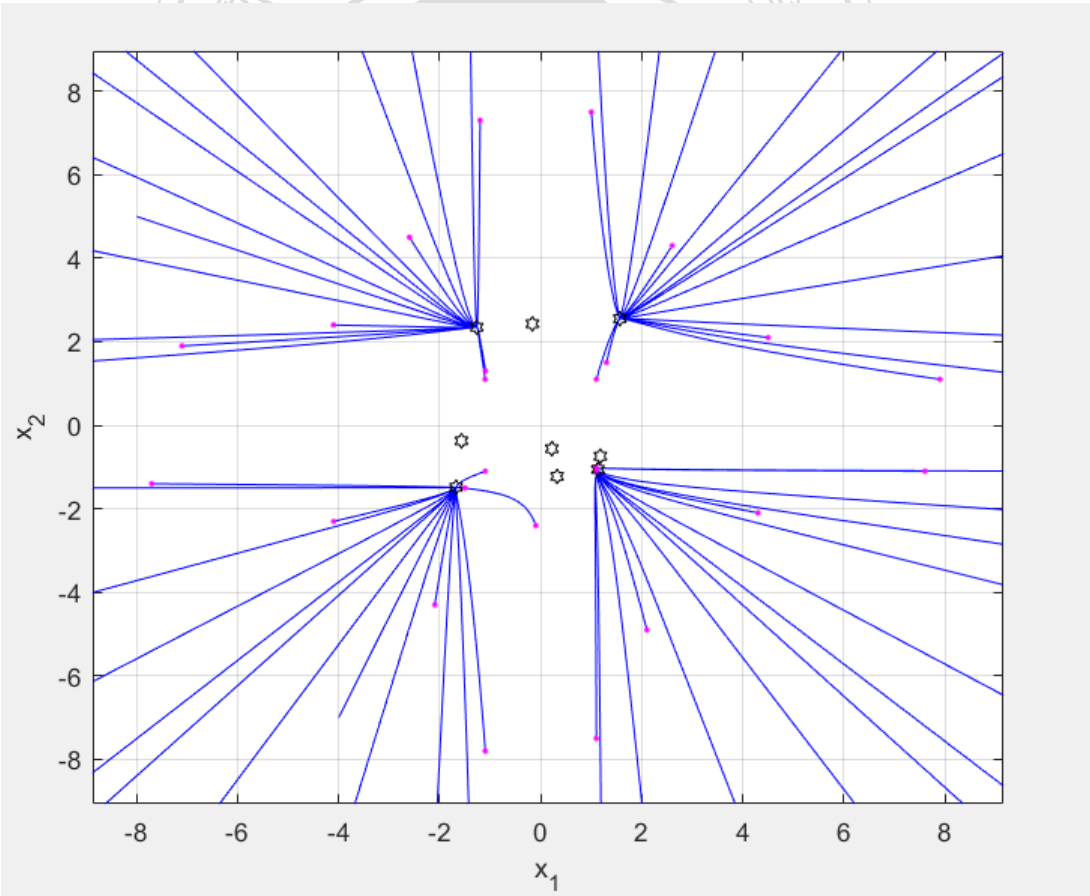

Figure 4.11: Numerical simulation for Example 3.

These nine equilibria can be solved numerically and are listed below: (-1.6774, -1.4843),

(-1.5705, -0.3704), (-1.2679, 2.3492), (-0.1669, 2.4288), (0.2196, -0.5603), (0.3223, -1.227), (1.1299, -1.0569), (1.179, -0.7407), (1.5639, 2.5524), cf Figure 4.11(marked by stars). By numerical stimulation, we observe that five are unstable equilibria and four are stable equilibria.

Example 4. This example illustrates Theorem 3.8 for the case $n = 2$ and $\binom{3}{l}\binom{r}{3}$. We consider the parameters

$$\begin{aligned}
&\mu_1 = 1.7, \alpha_{11} = 1.05, \alpha_{12} = 0.0386, \beta_{11} = 1, \beta_{12} = 0.030095,\\
&\mu_2 = 1.65, \alpha_{21} = 0.02, \alpha_{22} = 1, \beta_{21} = 0.10009, \beta_{22} = 1.1,\\
&I_1 = -0.040185,\ I_2 = 0.04291,\\
&\tau_{11} = \tau_{22} = 0.05,\ \tau_{12} = \tau_{21} = 10.
\end{aligned}$$

Then the system becomes:

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -1.7x_1(t) + 1.05g_1(x_1(t)) + 0.0386g_2(x_2(t)) \qquad (4.0.21)\\
&\quad + g_1(x_1(t-0.05)) + 0.030095g_2(x_2(t-10)) - 0.040185,\\
\frac{dx_2(t)}{dt} &= -1.65x_2(t) + 0.02g_1(x_1(t)) + g_2(x_2(t)) \qquad (4.0.22)\\
&\quad + 0.10009g_1(x_1(t-10)) + 1.1g_2(x_2(t-0.05)) + 0.04291.
\end{aligned}$$

The stationary equations are

$$\begin{aligned}
F_1(\mathbf{x}) : &= -1.7x_1 + 1.05g_1(x_1) + 0.0386g_2(x_2) \qquad (4.0.23)\\
&\quad + g_1(x_1) + 0.030095g_2(x_2) - 0.040185 = 0,\\
F_2(\mathbf{x}) : &= -1.65x_2 + 0.02g_1(x_1) + g_2(x_2) \qquad (4.0.24)\\
&\quad + 0.10009g_1(x_1) + 1.1g_2(x_2) + 0.04291 = 0.
\end{aligned}$$

Herein, $\alpha_{12} + \beta_{12} = 0.068695 > 0$, $\alpha_{21} + \beta_{21} = 0.12009 > 0$. As introduced in Section 2.1, we have $k_1^+ = 0.02851$, $k_1^- = -0.10888$, $\tilde{p}_1 = -0.43946$, $\tilde{q}_1 = 0.43946$, $k_2^+ = 0.163$, $k_2^- = -0.07718$, $\tilde{p}_2 = -0.89186$, $\tilde{q}_2 = 0.89186$, and the single-variable upper and lower

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functions are:

$$\begin{aligned}
\hat{f}_1(\xi) &:= -1.7\xi + 2.05g_i(\xi) + 0.02851, \\
\check{f}_1(\xi) &:= -1.7\xi + 2.05g_i(\xi) - 0.10888, \\
\hat{f}_2(\xi) &:= -1.65\xi + 2.1g_i(\xi) + 0.163, \\
\check{f}_2(\xi) &:= -1.65\xi + 2.1g_i(\xi) - 0.07718.
\end{aligned}$$

Since $\hat{f}_1(\tilde{p}_1) = -0.07146 < 0$, $\hat{f}_1(\tilde{q}_1) = 0.12848 > 0$ and $\check{f}_1(\tilde{q}_1) = -0.00891 < 0$, $i = 1 \in \mathcal{B}_l^3$ by the definition of $\mathcal{B}$ in Section 2.1. Since $\hat{f}_2(\tilde{p}_2) = 0.01755 > 0$, $\check{f}_2(\tilde{p}_2) = -0.22263 < 0$, and $\check{f}_2(\tilde{q}_2) = 0.06827$, $i = 2 \in \mathcal{B}_3^r$ by the definition of $\mathcal{B}$ in Section 2.1. With the upper and lower functions, we have $A_1 := [\check{a}_1, \hat{a}_1] = [-0.96336, -0.74694]$ and $S_1 := [\hat{b}_1, \hat{c}_1] = [-0.08255, 0.85032]$.

First, we show that $K_2(\tilde{p}_2; S_1) < 0$. $f_2^{\hat{c}_1}(\xi) = -1.65\xi + 2.1tanh\xi + 0.12009tanh(\hat{c}_1) + 0.04291$ and $f_2^{\hat{b}_1}(\xi) = -1.65\xi + 2.1tanh\xi + 0.12009tanh(\hat{b}_1) + 0.04291$. By Notation 2.2, we can compute that

$$K_2(\tilde{p}_2; S_1) = [f_2^{\hat{c}_1}(\tilde{p}_2), f_2^{\hat{b}_1}(\tilde{p}_2)] = [-0.11243, -0.01953] < 0.$$

which verify $K_2(\tilde{p}_2; S_1) < 0$.

Since $K_2(\tilde{p}_2; S_1) < 0$, we have $A_2^{S_1} = [\check{a}_2^{(b_1)}, \hat{a}_2^{(c_1)}] = [-0.87742, -0.65858]$, $B_2^{S_1} = [\hat{b}_2^{(c_1)}, \check{b}_2^{(b_1)}] = [-0.33655, -0.074]$, and $C_2^{S_1} = [\check{c}_2^{(b_1)}, \hat{c}_2^{(c_1)}] = [0.97656, 1.09098]$.

Secondly, we show that $K_1(\tilde{q}_1; A_2^{S_1}) > 0$. Similarly, $f_1^{\check{a}_2^{(b_1)}}(\xi) = -1.7\xi + 2.05tanh\xi + 0.068695tanh(\check{a}_2^{(b_1)}) - 0.040185$ and $f_1^{\hat{a}_2^{(c_1)}}(\xi) = -1.7\xi + 2.05tanh\xi + 0.068695tanh(\hat{a}_2^{(c_1)}) - 0.040185$. By Notation 2.2, we can compute that

$$K_1(\tilde{q}_1; A_2^{S_1}) = [f_1^{\check{a}_2^{(b_1)}}(\tilde{q}_1), f_1^{\hat{a}_2^{(c_1)}}(\tilde{q}_1)] = [0.01135, 0.02011] > 0.$$

which verify $K_1(\tilde{q}_1; A_2^{S_1}) > 0$.

Now, let us choose $\theta_1 = 0.37$ and $\theta_2 = 0.132$. For (3.2.24), we have

$$\theta_1 = 0.17 > \bar{L}_2|\alpha_{12} + \beta_{12}| = 1|0.068695|, \theta_2 = 0.2 > \bar{L}_1|\alpha_{21} + \beta_{21}| = 1|0.12009|. \tag{4.0.25}$$

Thus, we confirm that condition (3.2.24) holds.

As seen from Figure 4.13, for $\xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1^{(a_2)}, \infty)$

$$g_1'(\xi) < g_1'(\check{c}_1^{(a_2)}) = 0.73893 < 0.75122 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}, \tag{4.0.26}$$

and for $\xi \in [\hat{b}_1^{(c_2)}, \check{b}_1^{(a_2)}]$,

$$g_1'(\xi) > g_1'(\check{b}_1^{(a_2)}) = 0.91063 > 0.90732 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}.$$

Thus, we confirm that condition (3.2.25) holds.

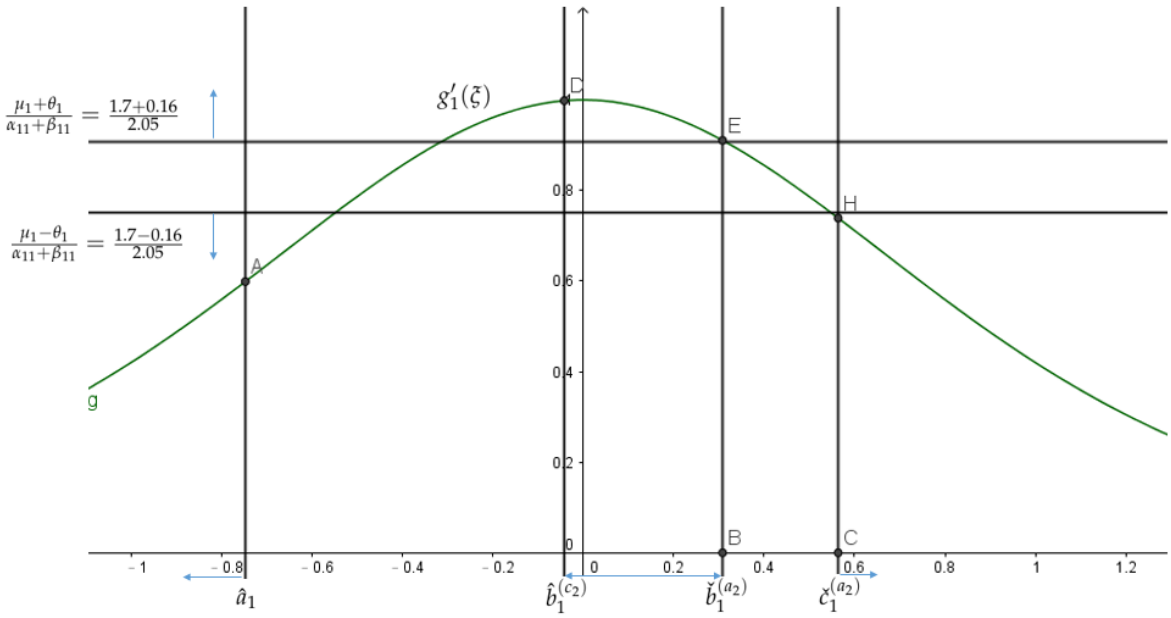

Figure 4.12: g_1' with $\theta_1 = 0.16$.

For (3.2.26), for $\xi \in (-\infty, \hat{a}_2^{(c_1)}] \cup [\check{c}_2^{(a_1)}, \infty)$

$$g_2'(\xi) < g_2'(\hat{a}_2^{(c_1)}) = 0.66659 < 0.69048 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}.$$

and for $\xi \in [\hat{b}_2^{(c_1)}, \check{b}_2^{(a_1)}]$

$$g_2'(\xi) > g_2'(\hat{b}_2^{(c_1)}) = 0.89477 > 0.88095 = \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}.$$

Thus, we confirm that condition (3.2.26) holds. We have verified all conditions in Theorem 3.2. Therefore, the system (4.0.21)-(4.0.22) has exactly nine equilibira.

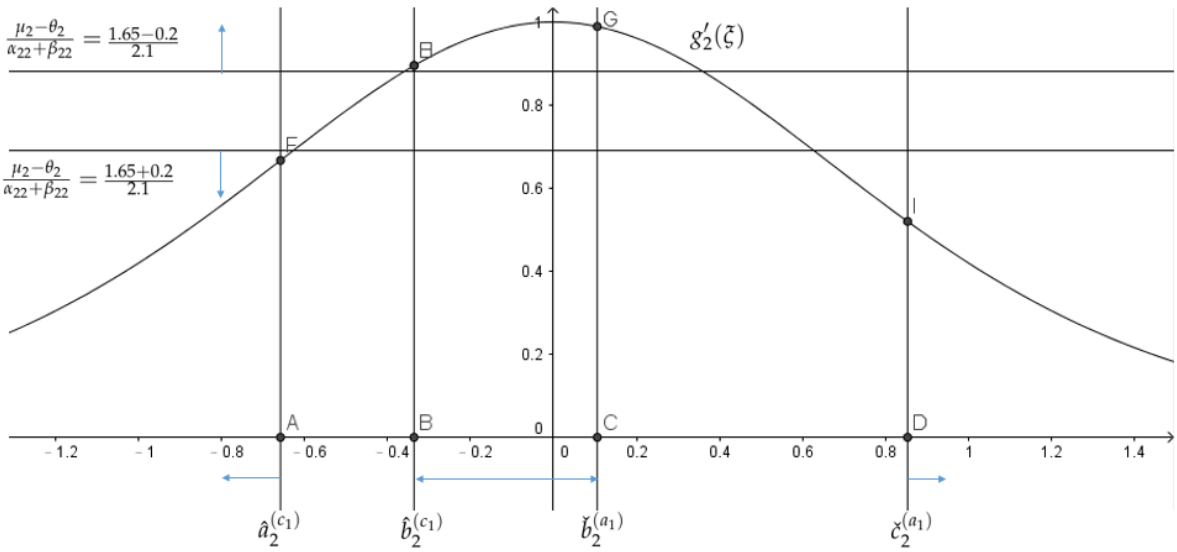

Figure 4.13: g_2' with $\theta_2 = 0.2$.

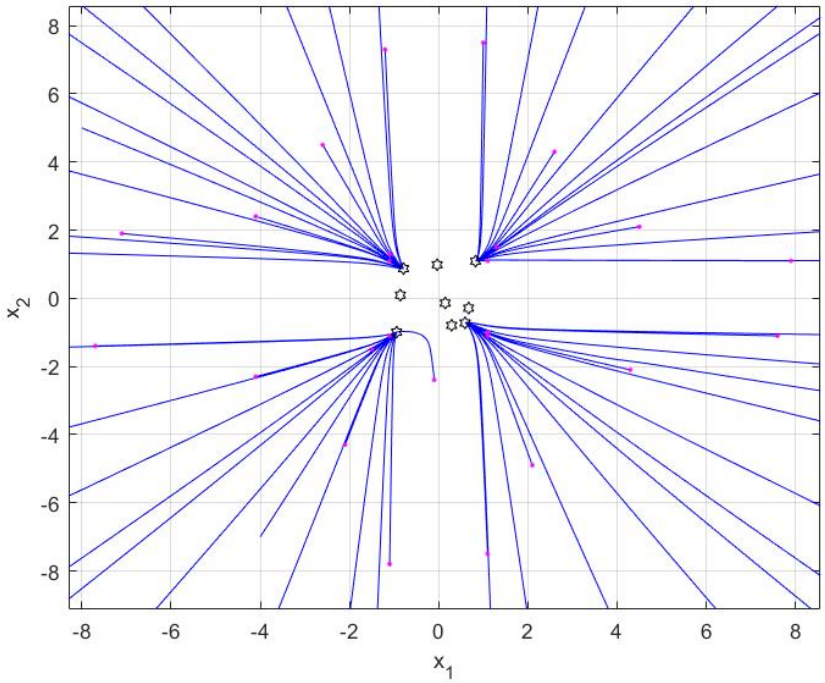

Figure 4.14: Numerical simulation for Example 4.

These nine equilibria can be solved numerically and are listed below: (-0.94199, -0.99324), (-0.7875, 0.87239), (0.82752, 1.08933), (0.59279, -0.72206), (-0.85883, 0.09148), (-0.03344, 0.98448), (0.67223, -0.28772), (0.14816, -0.13873), (0.29121, -0.79481), cf Figure 4.14(marked by stars). By numerical stimulation, we observe that five are unstable equilibria and four are stable equilibria..

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