

The random cluster model and new summation and integration identities

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Abstract

We explicitly evaluate the free energy of the random cluster model at its critical point for $0 < q < 4$ using an exact result due to Baxter, Temperley and Ashley. It is found that the resulting expression assumes a form which depends on whether $\pi/2 \cos^{-1}(\sqrt{q}/2)$ is a rational number, and if it is a rational number whether the denominator is an odd integer. Our consideration leads to new summation identities and, for $q = 2$, a closed-form evaluation of the integral

$$\begin{aligned} & \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln [A + B + C - A \cos \theta - B \cos \phi - C \cos(\theta + \phi)] \\ & = -\ln(2S) + (2/\pi)[\text{Ti}_2(AS) + \text{Ti}_2(BS) + \text{Ti}_2(CS)], \end{aligned}$$

where $A, B, C \geq 0$ and $S = 1/\sqrt{AB + BC + CA}$.

Key words: Random cluster model, critical point, summation and integration identities.

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1 Introduction

The consideration of exact solutions of lattice models in statistical mechanics is a fertile field in both physics and mathematics. In physics it is well-known that (see, for example, [1]) exact solutions of lattice models lead to the finding of different classes of critical phenomena. In mathematics exact solutions in statistical mechanics often lead to the discovery of new mathematical identities. A prominent example is the solution of the hard hexagon model by Baxter [2] leads to the discovery of new Rogers-Ramanujan summation identities in combinatorics [3]. As another example consider the integral

$$I(A, B, C) = \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln \left[A + B + C - A \cos \theta - B \cos \phi - C \cos(\theta + \phi) \right] \quad (1)$$

which arises often in lattice statistical studies. For $A = B = 2, C = 0$ it is well-known [4, 5] that the integral gives the per-site entropy of dimers on a square lattice and the integral can be directly evaluated [4, 5] to yield

$$I(2, 2, 0) = \frac{4}{\pi} \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots \right). \quad (2)$$

For $A = B = C = 2$ the integral gives the per-site entropy of spanning trees on the triangular lattice, and the exact solution of the latter problem [6, 7] yields the closed form evaluation

$$I(2, 2, 2) = \frac{6}{\pi} \text{Ti}_2 \left(\frac{1}{\sqrt{3}} \right) + \frac{1}{2} \ln 3, \quad (3)$$

where Ti_2 is the inverse tangent integral function [8] defined by

$$\begin{aligned} \text{Ti}_2(a) &= \int_0^a \frac{\tan^{-1} t}{t} dt \\ &= a - \frac{a^3}{3^2} + \frac{a^5}{5^2} - \frac{a^7}{7^2} + \dots \end{aligned} \quad (4)$$

Closed-form evaluation of $I(A, B, C)$ for general A, B, C is not known.

In this paper we report a closed-form evaluation of $I(A, B, C)$, by using a result due to Baxter, Temperley and Ashley [9] on the random cluster model [10, 11]. We also show that the explicit expression of its critical free

energy depends crucially on the value of q . This implies that the free energy of the random cluster model, if solved, would also share this property. For convenience of references, we first state the integration result:

For arbitrary $A, B, C \geq 0$ and $S = 1/\sqrt{AB + BC + CA}$, we have

$$I(A, B, C) = -\ln(2S) + \frac{2}{\pi} \left[\text{Ti}_2(AS) + \text{Ti}_2(BS) + \text{Ti}_2(CS) \right]. \quad (5)$$

2 The random cluster model

Consider a q -state Potts model on the triangular lattice of N_s sites with anisotropic (reduced) interactions K_1, K_2, K_3 along the three principal lattice directions. Its partition function is

$$Z_{N_s}^P(K_1, K_2, K_3) = \sum_{\sigma_i=1}^q \prod_E \exp[K_\alpha \delta_{\text{Kr}}(\sigma_i, \sigma_j)] \quad (6)$$

where the summation is over all q^{N_s} spin states σ_i , $i = 1, 2, \dots, q$, and the product is over nearest-neighbor set E connecting sites i and j , $\alpha = 1, 2, 3$, and δ_{Kr} is the Kronecker delta function. We are interested in the ferromagnetic regime $K_\alpha \geq 0$.

The partition function can be rewritten [9] in a graph expansion as

$$Z_{N_s}^{RC}(K_1, K_2, K_3) = \sum_G q^{c(G)} (e^{K_1} - 1)^{\ell_1} (e^{K_2} - 1)^{\ell_2} (e^{K_3} - 1)^{\ell_3} \quad (7)$$

where the summation is over all subgraphs G of the triangular lattice, $c(G)$ is the number of connected clusters in G including isolated points, and ℓ_α is the number of lines in G in the direction $\alpha = 1, 2, 3$. In this form the partition function describes that of a random cluster model [10, 11] for which q can be continuous. One is interested in the per-site “free energy” of the random cluster model

$$f^{RC} = \lim_{N_s \rightarrow \infty} N_s^{-1} \ln Z_{N_s}^{RC}(K_1, K_2, K_3). \quad (8)$$

The triangular random cluster model is known [9, 12] to be critical at

$$\sqrt{q} x_1 x_2 x_3 + x_1 x_2 + x_2 x_3 + x_3 x_1 = 1 \quad (9)$$

where

$$x_\alpha = (e^{K_\alpha} - 1)/\sqrt{q} \geq 0. \quad (10)$$

For $0 < q < 4$, define $\phi(q)$ and $v_\alpha(q)$, $\alpha = 1, 2, 3$, by

$$\begin{aligned}\cos \phi(q) &= \sqrt{q}/2, & 0 < \phi < \pi/2 \\ x_\alpha &= \sin(\phi - v_\alpha)/\sin v_\alpha, & 0 < v_\alpha < \phi.\end{aligned}\quad (11)$$

For integral values of q , for example, we have

$$\begin{aligned}q &= 1, & 0 < v_a < \phi(1) &= \pi/3 \\ q &= 2, & 0 < v_a < \phi(2) &= \pi/4 \\ q &= 3, & 0 < v_a < \phi(3) &= \pi/6.\end{aligned}$$

In terms of these variables, we have

$$e^{K_\alpha} = 1 + \frac{1}{2}[\sqrt{q(4-q)} \cot v_a - q], \quad 0 < q < 4 \quad (12)$$

and the critical point (9) can be written as

$$v_1 + v_2 + v_3 = 2\phi(q). \quad (13)$$

3 The critical free energy

Baxter, Temperley and Ashley [9] considered the random cluster model (7) on the triangular lattice and obtained, among other things, its free energy at the critical point (9). For $0 < q < 4$ they obtained

$$f_{critical}^{RC} = \frac{1}{2} \ln q + \psi(\phi, v_1) + \psi(\phi, v_2) + \psi(\phi, v_3) \quad (14)$$

where

$$\psi(\phi, v) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sinh(\pi - \phi)x \sinh 2(\phi - v)x}{x \sinh \pi x \cosh \phi x} dx, \quad 0 < q < 4. \quad (15)$$

Here, we investigate implications of (14) and (15).

The integral (15) can be evaluated by contour integration. Let C be the contour consisting of the real axis and the half circle at infinity encircling the upper half z plane. In the variable range $0 < \phi < \pi/2, 0 < v < \phi$, we can neglect the contribution from the half-infinite circle and rewrite (15) as

$$\psi(\phi, v) = \frac{1}{2} \int_C \frac{\sinh(\pi - \phi)z \sinh 2(\phi - v)z}{z \sinh \pi z \cosh \phi z} dz, \quad (16)$$

where the integration is along the contour C .

The contour integral can be carried out by using the residue theory. The integrand in (16) has poles at

$$\begin{aligned} \sinh(\pi z_1) = 0 & \quad \text{or} \quad z_1 = ni, & n = 1, 2, \dots, \\ \cosh(\phi z_2) = 0 & \quad \text{or} \quad z_2 = \frac{\pi}{2\phi}(2m+1)i, & m = 0, 1, 2, \dots \end{aligned}$$

The evaluation of the integral thus depends on whether z_1 and z_2 overlap. Overlapping of z_1 and z_2 occurs if $\pi/2\phi$ is a rational number with the denominator an odd integer (M and N have no common integer factors):

$$\frac{\pi}{2\phi} = \frac{M}{N}, \quad M = 1, 2, 3, \dots, \quad N = 1, 3, 5, \dots, \quad N < M. \quad (17)$$

Then there are two cases to consider.

Case 1. There is no overlap between z_1 and z_2 , namely, $\pi/2\phi$ is either irrational or is of the form of (17) but with N even. This includes the $q = 1$ Potts model ($M = 3$, $N = 2$). Then both z_1, z_2 are simple poles, and the residues can be computed straightforwardly. One obtains

$$\begin{aligned} \psi(\phi, v) &= \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\phi) \sin 2n(\phi - v) \\ &\quad + \sum_{m=0}^{\infty} \frac{2}{2m+1} \cot \left[\left(m + \frac{1}{2} \right) \frac{\pi^2}{\phi} \right] \sin \left[(2m+1) \frac{v\pi}{\phi} \right], \end{aligned} \quad (18)$$

where the two terms come from the evaluation of residues at z_1 and z_2 , respectively.

For later use the expression (18) can be further simplified by introducing in the first term the identity

$$\tan x \sin 2(x - y) = \cos 2y - \cos 2(x - y) + \tan x \sin 2y$$

with $x = n\phi$, $y = nv$. This leads to

$$\begin{aligned} \psi(\phi, v) &= \ln \left[\frac{\sin(\phi - v)}{\sin v} \right] + \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\phi) \sin 2nv \\ &\quad + \sum_{n=0}^{\infty} \frac{2}{2n+1} \cot \left[\left(n + \frac{1}{2} \right) \frac{\pi^2}{\phi} \right] \sin \left[(2n+1) \frac{v\pi}{\phi} \right] \end{aligned} \quad (19)$$

after making use of the summation identity (1.441.2 of [13])

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln \left[2 \sin \left(\frac{x}{2} \right) \right], \quad 0 < x < 2\pi.$$

Case 2. There is overlapping between z_1 and z_2 . This occurs when (17) holds with $N = \text{odd}$ so that when

$$2m + 1 = N, 3N, 5N \dots \quad (20)$$

we have $z_2 = Mi, 3Mi, 5Mi, \dots$ which coincide with z_1 where they form double poles. Note that when $N = 1$ every point in z_2 is a double pole. This includes the $q = 2, 3$ ($M = 2, 3, N = 1$) Potts models. The integral (16) now consists of three terms, $\psi_2(N\pi/2M, v) = R_{S1} + R_D + R_{S2}$, where R_{S1} is the sum of residues from simple poles in z_1 , R_D the residues from double poles, and R_{S2} the residues from simple poles in z_2 , if any.

To compute R_{S1} , we note that the forming of double poles excludes points $Mi, 3Mi, 5Mi, \dots$ which divide the remaining $z_1 = ni$ into sections $n = \{1, M-1\}, \{M+1, 3M-1\}, \{3M+1, 5M-1\}, \dots$. Then we can write

$$R_{S1} = R_{S1a} + R_{S1b} \quad (21)$$

where R_{S1a} is the sum of the first $M-1$ residues and R_{S1b} is the sum of the rest. From we have already computed in the first term in (18), we obtain

$$\begin{aligned} R_{S1a} &= \sum_{k=1}^{M-1} \tan(k\phi) \left[\frac{\sin[2k(\phi-v)]}{k} \right], \quad \phi = \frac{N\pi}{2M} \\ R_{S1b} &= \sum_{k=-(M-1)}^{M-1} \sum_{n=1}^{\infty} \tan[(2nM+k)\phi] \left[\frac{\sin[2(2nM+k)(\phi-v)]}{2nM+k} \right]. \end{aligned}$$

Expanding the sine function, we can rewrite R_{S1a} as

$$R_{S1a} = \sum_{k=1}^{M-1} \tan(k\phi) \left[\sin(2k\phi) \int_{2v}^{\pi/2k} \sin kx dx - \cos(2k\phi) \int_0^{2v} \cos kx dx \right]. \quad (22)$$

To evaluate R_{S1b} we use $4M\phi = 2N\pi$ and proceed similarly as in (22), and obtain

$$\begin{aligned} R_{S1b} &= \sum_{k=-(M-1)}^{M-1} \tan(k\phi) \sum_{n=1}^{\infty} \left[\sin(2k\phi) \int_{2v}^{\pi/2|k|} \sin[(2nM+k)x] dx \right. \\ &\quad \left. - \cos(2k\phi) \int_0^{2v} \cos[(2nM+k)x] dx \right], \quad \phi = \frac{N\pi}{2M} \end{aligned}$$

where one verifies that the integrated contributions at the upper limit $\pi/2|k|$ cancel out exactly after combining terms from k and $-k$. Here the $k = 0$ term vanishes due to the $\tan(k\phi)$ factor.

Expanding out the sine and cosine functions as before, terms odd in k in the summand are canceled and terms even in k give the same contribution for $k > 0$ and $k < 0$. Thus we have

$$R_{S1b} = 2 \sum_{k=1}^{M-1} \tan(k\phi) \sum_{n=1}^{\infty} \left[\sin(2k\phi) \int_{2v}^{\pi/2k} \sin(2nMx) \cos(kx) dx + \cos(2k\phi) \int_0^{2v} \sin(2nMx) \sin(kx) dx \right]. \quad (23)$$

Write

$$\begin{aligned} \sum_{n=1}^{\infty} \sin(2nMx) &= \lim_{\mathcal{N} \rightarrow \infty} \sum_{n=1}^{\mathcal{N}} \sin(2nMx) \\ &= \lim_{\mathcal{N} \rightarrow \infty} \frac{\cos(Mx) - \cos[(2\mathcal{N} + 1)Mx]}{2 \sin Mx}, \end{aligned} \quad (24)$$

where we have used the summation identity 1.342.1 of [13] in writing down the last step. Substituting (24) into (23), terms with the factor $\cos[(2\mathcal{N} + 1)Mx]$ inside an integral vanish in the $\mathcal{N} \rightarrow \infty$ limit. Thus, one obtains

$$R_{S1b} = \sum_{k=1}^{M-1} \tan(k\phi) \left[\sin(2k\phi) \int_{2v}^{\pi/2k} \cos(kx) \cot(Mx) dx + \cos(2k\phi) \int_0^{2v} \sin(kx) \cot(Mx) dx \right]. \quad (25)$$

Finally, combining (22) and (25), we obtain

$$R_{S1} = \sum_{k=1}^{M-1} \tan(k\phi) \left[\sin(2k\phi) \int_{2v}^{\pi/2k} \frac{\cos(M-k)x}{\sin(Mx)} dx - \cos(2k\phi) \int_0^{2v} \frac{\sin(M-k)x}{\sin(Mx)} dx \right], \quad \phi = \frac{N\pi}{2M}. \quad (26)$$

Now we compute R_D . Residues from double poles at $z_2 = (2m + 1)Mi$ are

$$R_D = \pi i \sum_{m=0}^{\infty} \lim_{z \rightarrow z_2} \frac{d}{dz} \left[(z - z_2)^2 \left(\frac{\sinh(\pi - \phi)z \sinh 2(\phi - v)z}{z \sinh \pi z \cosh \phi z} \right) \right]. \quad (27)$$

where $z_2 = (2m + 1)Mi$, $\phi = N\pi/2M$, $N = \text{odd}$.

Define number u and integer p by

$$Mv = p\pi/2 + u, \quad \text{where } 0 < u < \pi/2, \quad p = 0, 1, 2, \dots, \quad p < N. \quad (28)$$

Here for technical reasons we exclude points $u = 0$. At these points each of R_{S1} , R_D and R_{S2} diverge. But in this case $\psi(\phi, v)$ can be computed by taking an appropriate large M, N limit of (19). (See discussions in Section 5 below.)

Then using the identities

$$\begin{aligned} \lim_{z \rightarrow z_2} \frac{z - z_2}{\sinh \pi z} &= \frac{1}{\pi}(-1)^M, & \lim_{z \rightarrow z_2} \frac{z - z_2}{\cosh \phi z} &= \frac{2M}{N\pi i}(-1)^{m+(N-1)/2} \\ \sinh(\pi - \phi)z_2 &= i(-1)^{M-m+(N+1)/2}, & \sinh 2(\phi - v)z_2 &= i(-1)^p \sin 2(2m + 1)u \\ \cosh(\pi - \phi)z_2 &= 0, & \cosh 2(\phi - v)z_2 &= (-1)^{p+1} \cos 2(2m + 1)u \end{aligned}$$

it is straightforward to obtain from (27)

$$R_D = \frac{2(-1)^p}{MN\pi} \sum_{m=0}^{\infty} \frac{\sin[2(2m + 1)u]}{(2m + 1)^2} + \frac{4(-1)^p(\phi - v)}{N\pi} \sum_{m=0}^{\infty} \frac{\cos[2(2m + 1)u]}{2m + 1}. \quad (29)$$

The first summation in (29) can be carried out by using the summation identity 3.1 established below, and the second summation carried out using 1.442.2 of [13], namely,

$$\sum_{k=0}^{\infty} \frac{\cos 2(2k + 1)x}{2k + 1} = \frac{1}{2} \ln \cot x, \quad 0 < x < \frac{\pi}{2}. \quad (30)$$

This leads to the simple expression

$$R_D = \frac{2(-1)^p}{MN\pi} \text{Ti}_2(\tan u) + \frac{(-1)^p(N - p)}{MN} \ln \cot u. \quad (31)$$

Identity 3.1.

$$\sum_{k=0}^{\infty} \frac{\sin 2(2k + 1)x}{(2k + 1)^2} = \text{Ti}_2(\tan x) + x \ln \cot x, \quad 0 < x < \frac{\pi}{2}. \quad (32)$$

Proof. We have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\sin 2(2k + 1)x}{(2k + 1)^2} &= \int_0^{2x} \sum_{k=0}^{\infty} \frac{\cos(2k + 1)y}{2k + 1} dy \\ &= \frac{1}{2} \int_0^{2x} \ln \left(\cot \frac{y}{2} \right) dy \\ &= - \int_0^{\tan x} \frac{\ln t}{1 + t^2} dt, \quad 0 < x < \frac{\pi}{2}, \end{aligned} \quad (33)$$

where we have made use of (30) and the change of variable $t = \tan(y/2)$. This leads to (32) after using the identity (4.29) of [8], namely,

$$\text{Ti}_2(\tan x) = x \ln(\tan x) - \int_0^{\tan x} \frac{\ln t}{1+t^2} dt, \quad x < \frac{\pi}{2}. \quad (34)$$

Q.E.D.

To compute R_{S_2} , the sum of residues of simple poles in z_2 , we need to exclude from $z_2 = [(2m+1)M/N]i$ the double poles $Mi, 3Mi, 5Mi, \dots$. As a result, the remaining simple poles in z_2 are divided into sections $m = \{0, (N-3)/2\}, \{(N+1)/2, (3N-3)/2\}, \{(3N+1)/2, (5N-3)/2\}, \dots$. The situation is similar to that of R_{S_1} , and R_{S_2} can be similarly computed. Omitting the details we arrive at the expression

$$R_{S_2} = -\frac{2M}{N} \sum_{k=1}^{(N-1)/2} \cot\left(\frac{2kM\pi}{N}\right) \int_0^{2v} \frac{\sin(2kMx/N)}{\sin(Mx)} dx, \quad N = 1, 3, 5, \dots. \quad (35)$$

Note that $R_{S_2} = 0$ when $N = 1$, as every point in z_2 is a double pole.

Finally, combining the above results, we obtain the expression

$$\psi\left(\frac{N\pi}{2M}, v\right) = R_{S_1} + R_D + R_{S_2}, \quad M = 1, 2, 3, \dots \quad N = 1, 3, 5, \dots \quad (36)$$

where R_{S_1} is given in (26), R_D by (31), and R_{S_2} by (35).

4 Special cases

In this section we consider the special cases of $q = 1, 2, 3$ Potts models.

For $q = 1$, we have $\phi = \pi/3, M = 3, N = 2$, so we use (19). For $\phi = \pi/3$, we have $\cot[(n+1/2)\pi^2/\phi] = 0$ and the third term in (19) vanishes identically. It is then a simple matter to use the summation identity 4.1 we establish below to obtain

$$\psi\left(\frac{\pi}{3}, v\right) = \ln \frac{\sqrt{3} \cot v + 1}{2}. \quad (37)$$

It follows that the free energy is

$$f^{RC} = K_1 + K_2 + K_3, \quad q = 1$$

after making use of (12) and (14). This result agrees with the physical expectation that, since there is only one spin state in the $q = 1$ Potts model, the partition function is $Z = \exp[N_s(K_1 + K_2 + K_3)]$.

Identity 4.1.

$$\sum_{n=1}^{\infty} \frac{1}{n} \tan(n\pi/3) \sin 2nv = \ln \frac{\sqrt{3} \cot v + 1}{\sqrt{3} \cot v - 1}, \quad 0 < v < \pi/6. \quad (38)$$

Proof. Since $\tan(n\pi/3) = 0$ for $n = 3 \times$ integers, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\pi/3) \sin 2nv &= \sqrt{3} \sum_{n=0}^{\infty} \left[\frac{\sin 2(3n+1)v}{3n+1} - \frac{\sin 2(3n+2)v}{3n+2} \right] \\ &= \sqrt{3} \sum_{n=0}^{\infty} \left[\int_0^{2v} [\cos(3n+1)x - \cos(3n+2)x] dx \right] \\ &= \sqrt{3} \lim_{\mathcal{N} \rightarrow \infty} \sum_{k=0}^{\mathcal{N}-1} \int_0^{2v} [\cos(3k+1)x - \cos(3k+2)x] dx \end{aligned}$$

The summation can be carried out by using the formula (1.341.3 of [13])

$$\sum_{k=0}^{n-1} \cos(x + ky) = \cos \left(x + \frac{n-1}{2}y \right) \sin \frac{ny}{2} \csc \frac{y}{2}.$$

This yields

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\pi/3) \sin 2nv &= \sqrt{3} \lim_{\mathcal{N} \rightarrow \infty} \int_0^{2v} \frac{(1 - \cos 3\mathcal{N}x) \sin(x/2)}{\sin(3x/2)} dx \\ &= \sqrt{3} \int_0^{2v} \frac{dx}{2 \cos x + 1} - \sqrt{3} \lim_{\mathcal{N} \rightarrow \infty} \int_0^{2v} \frac{\cos 3\mathcal{N}x}{2 \cos x + 1} dx. \end{aligned} \quad (39)$$

The second term in (39) vanishes and the first term can be evaluated by using the formula (Cf. 2.551.3 of [13])

$$\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{b^2 - a^2}} \ln \left[\frac{\sqrt{b^2 - a^2} \tan(\frac{x}{2}) + a + b}{\sqrt{b^2 - a^2} \tan(\frac{x}{2}) - a - b} \right], \quad a^2 < b^2.$$

This leads to (38). Q.E.D.

For $q = 2$ we have $\phi = \pi/4$, $M = 2$, $N = 1$. By comparing our result with that of the Ising model, the critical free energy leads to the integration formula (5) as we shall now show.

First, using (36) we obtain

$$\psi\left(\frac{\pi}{4}, v\right) = \frac{1}{2} \ln[(\cot v)(\cot 2v)] + \frac{1}{\pi} \text{Ti}_2(\tan 2v), \quad q = 2. \quad (40)$$

Therefore from (14) the critical free energy is

$$f^{Potts} = \frac{1}{2} \ln 2 + \sum_{\alpha=1}^3 \left[\frac{1}{2} \ln[(\cot v_\alpha)(\cot 2v_\alpha)] + \frac{1}{\pi} \text{Ti}_2(\tan 2v_\alpha) \right], \quad q = 2. \quad (41)$$

On the other hand, the $q = 2$ Potts model is completely equivalent to an Ising model. Consider an Ising model on the same triangular lattice of N_s site with anisotropic interactions $K_\alpha/2$, $\alpha = 1, 2, 3$. We have the equivalence

$$\begin{aligned} Z_{N_s}^{Ising} &= \sum_{\sigma=\pm 1} \prod_E e^{(K_\alpha/2)\sigma_i\sigma_j} \\ &= e^{-N_s(K_1+K_2+K_3)/2} Z_N^{Potts} \Big|_{q=2} \end{aligned} \quad (42)$$

after making use of the identity $\sigma_i\sigma_j = 2\delta_{Kr}(\sigma_i, \sigma_j) - 1$. It follows that their critical free energies are related by

$$\begin{aligned} f^{Ising} &= f^{Potts} \Big|_{q=2} - \frac{1}{2}(K_1 + K_2 + K_3) \\ &= f^{Potts} \Big|_{q=2} - \frac{1}{2} \ln[(\cot v_1)(\cot v_2)(\cot v_3)] \\ &= \frac{1}{2} \ln 2 + \sum_{\alpha=1}^3 \left[\frac{1}{2} \ln(\cot 2v_\alpha) + \frac{1}{\pi} \text{Ti}_2(\tan 2v_\alpha) \right] \end{aligned} \quad (43)$$

after substituting with (41) and using (12) for $q = 2$, namely,

$$e^{K_\alpha} = \cot v_\alpha, \quad \text{or} \quad \sinh K_\alpha = \cot 2v_\alpha. \quad (44)$$

In fact, the Ising free energy is known at all temperatures [14] to be

$$\begin{aligned} f^{Ising} &= \ln 2 + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \left[\cosh K_1 \cosh K_2 \cosh K_3 + \sinh K_1 \sinh K_2 \sinh K_3 \right. \\ &\quad \left. - \sinh K_1 \cos \theta - \sinh K_2 \cos \phi - \sinh K_3 \cos(\theta + \phi) \right] d\theta d\phi. \end{aligned} \quad (45)$$

It can be verified that the critical condition (9) is equivalent to

$$\begin{aligned} & \cosh K_1 \cosh K_2 \cosh K_3 + \sinh K_1 \sinh K_2 \sinh K_3 \\ & = \sinh K_1 + \sinh K_2 + \sinh K_3, \end{aligned} \quad (46)$$

so we can rewrite (45) as

$$f^{I\text{sing}} = \ln 2 + \frac{1}{2}I(a, b, c) \quad (47)$$

with $a = \cot 2v_1, b = \cot 2v_2, c = \cot 2v_3$ and $I(a, b, c)$ defined in (1). It can also be verified that the critical condition $v_1 + v_2 + v_3 = \pi/2$ is the same as

$$ab + bc + ca = 1. \quad (48)$$

Equating (43) with (47), we obtain

$$I(a, b, c) = -\ln 2 + \ln(abc) + \frac{2}{\pi} \left[\text{Ti}_2(a^{-1}) + \text{Ti}_2(b^{-1}) + \text{Ti}_2(c^{-1}) \right]. \quad (49)$$

For the general integral $I(A, B, C)$ we can always define variables $a = AS, b = BS, c = CS$ with $S = 1/\sqrt{AB + BC + CA}$ to satisfy (48). Thus, one has

$$\begin{aligned} I(A, B, C) &= -\ln S + I(a, b, c) \\ &= -\ln(2S) + \ln(abc) + \frac{2}{\pi} \left[\text{Ti}_2(a^{-1}) + \text{Ti}_2(b^{-1}) + \text{Ti}_2(c^{-1}) \right] \\ &= -\ln(2S) + \frac{2}{\pi} \left[\text{Ti}_2(a) + \text{Ti}_2(b) + \text{Ti}_2(c) \right], \end{aligned} \quad (50)$$

where use has been made of the identity (Cf. (2.6) of [8])

$$\text{Ti}_2(y^{-1}) = \text{Ti}_2(y) - \frac{\pi}{2} \ln y, \quad y > 0.$$

The last line in (50) establishes the integration formula (5). It is readily checked that (50) yields the previous known values of $I(2, 2, 0)$ and $I(2, 2, 2)$.

For $q = 3$ we have $\phi = \pi/6, M = 3, N = 1$, so we again use (36). It is straightforward to obtain

$$\begin{aligned} \psi\left(\frac{\pi}{6}, v\right) &= \frac{1}{6} \ln \frac{2 \cos 2v + \sqrt{3}}{2 \cos 2v - \sqrt{3}} + \frac{3}{2} \ln \frac{\sqrt{3} + 1}{\sqrt{3} - 1} - \ln \frac{\sqrt{3} \cot 2v + 1}{\sqrt{3} \cot 2v - 1} \\ &+ \frac{1}{12} \ln \frac{2 - \sqrt{3}}{2 + \sqrt{3}} + \frac{1}{3} \ln \cot 3v + \frac{2}{3\pi} \text{Ti}_2(\tan 3v), \quad v < \frac{\pi}{6}. \end{aligned} \quad (51)$$

5 Summary and acknowledgments

We have evaluated explicitly the free energy of the random cluster model at its critical point for $0 < q < 4$ using an exact result of Baxter, Temperley and Ashley. It is found that the critical free energy is given by (18) if $\pi/2\phi(q) = \pi/2 \cos^{-1}(\sqrt{q}/2)$ is irrational or a rational number with an even denominator, and by (36) if $\pi/2\phi$ is a rational number with an odd integer in the denominator. Special cases of our results lead to new summation identities (32) and (38), and a new integration formula (5) for $I(A, B, C)$, which do not appear to have previously appeared in print.

It is instructive to see how the critical free energy passes from (36) to (18) as q varies and $\pi/2\phi$ changes from rational to irrational. Indeed, any irrational $\pi/2\phi$ can be reached by taking an appropriate $M, N \rightarrow \infty$ limit of $\pi/2\phi = N/M$. In this limit we have $R_D = 0$ by (31). It can be verified that R_{S1} and R_{S2} in (26) and (35) can be rewritten in equivalent forms

$$\begin{aligned}
 R_{S1} &= \sum_{n=1}^{M-1} \frac{1}{n} \tan(n\phi) \sin[2n(\phi - v)] \\
 &+ \sum_{n=1}^{M-1} \tan(n\phi) \left[\sin(2n\phi) \int_{2v}^{\pi/2n} \cot(Mx) \cos(nx) dx \right. \\
 &\quad \left. + \cos(2n\phi) \int_0^{2v} \cot(Mx) \sin(nx) dx \right], \\
 R_{S2} &= 2 \sum_{m=0}^{(N-3)/2} \cot \left[\left(m + \frac{1}{2} \right) \frac{\pi x}{\phi} \right] \left\{ \frac{\sin[(2m+1)v\pi/\phi]}{2m+1} \right. \\
 &\quad \left. - \frac{M}{N} \int_0^{2v} \cot(Mx) \sin \left[\left(m + \frac{1}{2} \right) \frac{\pi x}{\phi} \right] dx \right\}. \quad (52)
 \end{aligned}$$

In the large M, N limit, the integrals in (52) vanish. Then R_{S1} becomes the first term in (18), R_{S2} becomes the second term in (18) and one recovers (18) from (36). Similarly (36) can be reached by taking an appropriate limit of (18) or (19). Particularly, in the case of the points $u = 0$ excluded in (28), $\psi(\phi, v)$ can be compute by taking an appropriate large M, N limit of (19).

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