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一維 Emden-Fowler 型半線性波方程式解之存在區間之估計

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中 文 摘 要 ： In this report we will review some results on Emden-Fowler equations and treat the estimates for the life-span of positive solutions of some semi-linear wave equations with initial and boundary values problem in bounded domain. And in the last, we report some

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英文關鍵詞： life-span,　estimates,　some semi-linear wave equations, Emden-Fowler type

SEMILINEAR WAVE EQUATIONS

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On the solution of Emden-Fowler type semilinear wave equation

$$t^2 u_{tt} - u_{xx} = u^p \text{ in } [0,T) \times (a,b).$$

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I. Introduction

II. Review on the Emden-Fowler equation

$$t^2 u_{tt} = u^p, \quad p > 1.$$

III. Some results on semilinear wave equation

$$\Box\, u = u^p in \ [0,T) \times \Omega.$$

IV. Blow-up result on Emden-Fowler type semilinear wave equation

$$t^2 u_{tt} - u_{xx} = u^p \ \ in \ [0,T) \times (r_1, r_2), p > 1.$$

Abstract In this report we will review some results on Emden-Fowler equations and treat the estimates for the life-span of positive solutions of some semi-linear wave equations with initial and boundary values problem in bounded domain. And in the last, we report some recently result on the blow-up phenomenon of Emden-Fowler type semilinear wave equation.

1. Introduction In this report we want to consider the existence and uniqueness of solutions and nonexistence of global solution in time of the Emden-Fowler type semilinear wave equation

$$t^2 u_{tt} - u_{xx} = u^p \text{ in } [0,T) \times (r_1, r_2) \tag{0.1}$$

with boundary value null and initial values $u(0,x) = u_0(x) \in H^2(r_1,r_2) \cap H_0^1(r_1,r_2)$ and $\dot{u}(0,x) = u_1(x)$, where $p > 1, r_1$ and r_2 are real numbers.

II. Review on the Emden-Fowler equation

II.1 Literature The study of the Emden-Fowler equation originates from earlier theories concerning gaseous dynamics in astrophysics around the turn of the 20-th century.

The fundamental problem in the study of stellar structure at that time was to study the equilibrium configuration of the mass of spherical clouds of gas. Under the assumption that the gaseous cloud is under convective equilibrium (first proposed in 1862 by Lord Kelvin): W. Thompson (Lord Kelvin), On the convective equilibrium of temperature in the atmosphere, Manchester Philos. Soc. Proc., 2 (1860-62), pp.170-176; reprint, Math. and Phys. Papers by Lord Kelvin, 3 (1890), pp.255-260.

Lane studied the equation

$$\frac{d}{dt}\left(t^2 \frac{du}{dt}\right) + t^2 u^p = 0, \tag{*}$$

for the cases $p = 1.5$ and 2.5. Equation $(*)$ is commonly referred to as the Lane-Emden equation; see Chandrasekhar: Introduction to the Study of Stellar Structure, Chap. 4. Dover, New York, 1957;

The astrophysicists were interested in the behavior of the solution of $(*)$ which satisfies the initial condition: $u(0) = 1,\ u'(0) = 0$. Special cases of $(*)$, namely, when $p = 1, 5$, have explicit solutions: for $p = 1$,

$$\frac{d}{dt}\left(t^2 \frac{du}{dt}\right) + t^2 u = 0,$$

$$u(0) = 1,\ u'(0) = 0, u = \frac{\sin t}{t}$$

and for $p = 5$,

$$\frac{d}{dt}\left(t^2 \frac{du}{dt}\right) + t^2 u^5 = 0 \ ,$$

$$u(0) = 1,\ u'(0) = 0, u{=}1/\sqrt{1 + \frac{1}{3}t^2}.$$

Many properties for solutions of the Lane-Emden equation was studied by Ritter in a series of 18 papers published during 1878-1889: A. Ritter, Untersuchungen über die Höhe der Atmosphare und die Konstitution gasformiger Weltkörper, 18 articles, Wiedemann Annalen der Physik, 5-20, pp.1878-1883.

The publication of Emden's treatise Gaskugeln marks the end of first epoch in the study of stellar configurations governed by $(*)$. R. Emden, Gaskugeln, Anwendungen der mechanischen Warmentheorie auf Kosmologie und meteorologische Probleme, B. G.Teubner, Leipzig, Germany 1907, Chap. XII.

The mathematical foundation for the study of such an equation and also of the more general equation

$$(**) \qquad \frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right)+t^{\sigma}u^{\gamma}=0,\ t\geq 0,$$

was made by R.H. Fowler in a series of four papers during 1914-1931.

1) The form near infinity of real, continuous solutions of a certain differential equation of the second order, Quart. J. Math., 45 (1914), pp.289-350.

2) The solution of Emden's and similar differential equations, Monthly Notices Roy. Astro. Soc., 91 (1930), pp.63-91.

3) Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations, Proc. London Math. Soc., 13 (1914), pp.341-371.

4) Further studies of Emden's and similar differential equations, Quart. J. Math., 2 (1931),pp.259-288.

We refer the reader to a summary in Bellman's book : R. Bellman, Stability Theory of Differential Equations, McGraw-Hill, New York, 1953, Chap. VII.

The Emden-Fowler equation also arises in the study of gas dynamics and fluid mechanics; see, e.g., the survey article by Conti, Graffi and Sansone, The Italian contribution to the theory of nonlinear ordinary differential equations and to nonlinear mechanics during the years 1951-1961, Qualitative Methods in the Theory of Nonlinear Vibrations, Proc. Internat. Sympos. Nonlinear Vibrations, vol. II, 1961, pp.172-189.

There the solutions of physical interest are bounded nonoscillatory which possess a positive zero. The zero of such a solution corresponds to an equilibrium state in a fluid with spherical distribution of density and under mutual attraction of its particles. The Emden-Fowler equations also appear in the study of relativistic mechanics, nuclear physics and also in the study of chemically reacting systems. One interested in the physical aspects of such studies may wish to consult the related references in the bibliography in the articles and books by:

1) Shevyelo, Problems, methods, and fundamental results in the theory of oscillation of solutions of nonlinear nonautonomous ordinary differential equations, Proc. 2nd All-Union Conf on Theoretical and Appl. Mech., Moscow, 1965, pp.142-157.

2) Das and Coffman, A class of eigenvalues of the fine-structure constant and internal energy obtained from a class of exact solutions of the combined Klein-Gordon-Maxwell -Einstein field equations, J. Math. Phys., 8 (1967), pp.1720-1735.

3) S. Chandrasekhar, Principles of Stellar Dynamics, University of Chicago Press, Chicago, 1942, Chap. V.

The Emden-Fowler equation $(**)$ can be transformed into a first order nonlinear autonomous system, in fact a quadratic system, and information concerning its solutions may be obtained from the associated quadratic systems through phase plane analysis. This approach was in fact first used by Emden in his analysis of the Lane-Emden equation $(*)$. More detailed discussions on this approach we refer to:

1) Coppel, A survey on quadratic systems, J. Differential Equations, 2 (1966), pp.293-304.

2) P. J. Rijnierse, Algebraic solutions of the Thomas-Fermi equation for atoms, Ph. D. thesis, Univ. of St. Andrews, Scotland, 1968.

Progress along Fowler's approach concerning the Emden-Fowler equation (∗∗) may be found in :

1) M. L. J. Hautus, Uniformly asymptotic formulas for the Emden-Fowvler differential equation, J. Math. Anal. Appl., 30 (1970), pp.680-694.

2) R.T.V. Ramnath, On a class of nonlinear differential equations of astrophysics, J. Math. Anal. Appl., 35 (1971), pp.27-47.

Similar analysis concerning the related Thomas-Fermi equation may be found in:

R. V. Ramnath, A new analytical approximation to the Thomas-Fermi model in atomic physics, J. Math. Anal. Appl., 31 (1970), pp.285-296.

N. H. March, The Thomas-Fermi approximation in quantum mechanics, Advances in Phys., 6(1957), pp. 1-101.

The first serious study on the generalized Emden-Fowler equation

$$\frac{d^2u}{dt^2} + a(t)\,|u|^{\gamma}\, sgn\ u = 0, \quad t \geq 0$$

was made by F.V. Atkinson

1) The asymptotic solutions of second order differential equations, Ann. Mat. Pura. Appl., 37 (1954), pp.347-378.

2) On linear perturbation of nonlinear differential equations, Canad. J. Math., 6 (1954), pp.561-571.

3) On asymptotically linear second order oscillations, J. Rational Mech. Anal., 4 (1955), pp.769-793.

4) On second order nonlinear oscillation, Pacific J. Math., 5 (1955), pp.643-647.

5) On second order differential inequalities, Proc. Roy. Soc. Edinburgh, Sect. A, (1973).

For general reference, we mention the well known texts by:

P. Hartman, Ordinary Differential Equations, John Wiley, New York, 1964.

W.A. Coppel, Stability and Asymptotic behavior of Differential Equations, Heath, Boston, 1965.

R. Bellman, Stability Theory of Differential Equations, McGraw-Hill, New York, 1953.

II-2 Review the result on the positive solution of Emden-Fowler equation $t^2u^{''} = u^p, p > 1$

Consider the transformation $t = e^s$, $u(t) = v(s)$, then $v(0) = u_0$; $v_s(0) = u_1$, the equation $(*)$ can be transformed into the form

$$\begin{cases} v_{ss}(s) - v_s(s) = v(s)^p, & p > 1, \\ v(0) = u_0, \quad v_s(0) = u_1. \end{cases} \tag{1.1}$$

Thus, the local existence of solution u for $(*)$ in $(1, T)$ is equivalent to the local existence of solution v for (1.1) in $(0, \ln T)$. In this report, we have estimated the life-span T^* of positive solution u of $(*)$ under three different cases.

The main results are as follows:

(a) $u_1 = 0, u_0 > 0 :$ $T^* \le e^{k_1}$, $k_1 := s_0 + \frac{2(n+3)}{8-\epsilon}\frac{2}{n-1}v(s_0)^{\frac{1-p}{2}}$, $\varepsilon \in (0,1)$.

(b) $u_1 > 0, u_0 > 0 :$

i) $E(0) \ge 0$, $T^* \le e^{k_2}$, $k_2 := \frac{2}{p-1}\sqrt{\frac{p+1}{2}}u_0^{\frac{1-p}{2}}$. ii) $E(0) < 0$, $T^* \le e^{k_3}$, $k_3 := \frac{2}{p-1}\dfrac{u_0}{u_1}$.

(c) $u_1 < 0, u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right) : u(t) \le (u_0 - u_1 - u_0^p) + (u_1 + u_0^p)t - u_0^p \ln t.$

Notation and Fundamental Lemmas

For a given function v in this work we use the following abbreviations

$$a(s) = v(s)^2, \ E(0) = u_1^2 - \frac{2}{p+1}u_0^{p+1}, J(s) = a(s)^{-\frac{p-1}{4}}.$$

By some calculation we can obtain the following lemma 1 and lemma 2, we omit these argumentations on the proof of lemma 1.

Lemma 2.1. Suppose that $v \in C^2[0, T]$ is the solution of (1.1), then

$$E(s) = E(0), \tag{2.1}$$

$$(p+3)v_s(s)^2 = (p+1)E(0) + a^{''}(s) - a^{'}(s) + 2(p+1)\int_0^s v_s(r)^2\, dr, \tag{2.2}$$

$$J^{''}(s) = \frac{p^2-1}{4}J(s)^{\frac{p+3}{p-1}}\left(E(0) - \frac{a^{'}(s)}{p+1} + 2\int_0^s v_s(r)^2\, dr\right), \tag{2.3}$$

$$\begin{aligned} J^{'}(s)^2 = J^{'}(0)^2 &+ \frac{(p-1)^2}{4}E(0)\left(J(s)^{\frac{2(p+1)}{p-1}} - J(0)^{\frac{2(p+1)}{p-1}}\right) \\ &+ \frac{(p-1)^2}{2}J(s)^{\frac{2(p+1)}{p-1}}\int_0^s v_s(r)^2\, dr. \end{aligned} \tag{2.4}$$

Lemma 2.2. For $u_0 > 0$, the positive solution v of the equation (1.1), we have:

(2.5) i) $u_1 \geq 0$, then $v_s(s) > 0$ for all $s > 0$.

(2.6) ii) $u_1 < 0$, $u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$, then $v_s(s) < 0$ for all $s > 0$.

Proof. i) $v_{ss}(0) = u_1 + u_0^p > 0$, we know that $v_{ss}(s) > 0$ in $[0, s_1)$and $v_s(s)$ is increasing in $[0, s_1)$ for some $s_1 > 0$. Moreover, since $v$and v_s are increasing in $[0, s_1)$,

$$v_{ss}(s) = v_s(s) + v(s)^p > v_s(0) + v(0)^p > 0$$

for all $s \in [0, s_1)$ and $v_s(s_1) > v_s(s) > 0$ for all $s \in [0, s_1)$, we know that there exists a positive number $s_2 > 0$, such that $v_s(s) > 0$ for all $s \in [0, s_1 + s_2)$. Continuing such process, we obtain $v_s(s) > 0$ for all $s > 0$.

ii) According to $v_{ss}(0) = v_s(0) + v(0)^p = u_1 + u_0^p < 0$, there exists a positive number $s_1 > 0$ such that $v_{ss}(s) < 0$ in $[0, s_1)$, $v_s(s)$ is decreasing in $[0, s_1)$; therefore, $v_s(s) < v_s(0) = u_1 < 0$ for all $s \in [0, s_1)$ and $v(s)$ is decreasing in $[0, s_1)$.

Moreover, since v and v_s are decreasing in $[0, s_1)$, $v_{ss}(s) = v_s(s) + v(s)^p < v_s(0) + v(0)^p < 0$ for all $s \in [0, s_1)$ and $v_s(s_1) < v_s(s) < 0$ for all $s \in [0, s_1)$, we know that there exists a positive number $s_2 > 0$, such that $v_s(s) < 0$ for all $s \in [0, s_1 + s_2)$. Continuing such process, we obtain $v_s(s) < 0$ for all $s > 0$.■

Estimates for the life-span of positive solution u of $(*)$ under $u_1 = 0$, $u_0 > 0$.

In this section we want to estimate the life-span of positive solution u of $(*)$ under $u_1 = 0$, $u_0 > 0$. Here the life-span $T^*$of u means that u is the solution of equation $(*)$ and u exists only in $[0, T^*)$ so that the problem $(*)$ possesses the positive solution u in $C^2[0, T^*)$ for $T < T^*$.

Theorem 2.3. For $u_1 = 0$, $u_0 > 0$, the positive solution u of $(*)$ blows up in finite time; that is, there exists a bound number T^* so that

$$u(t)^{-1} \to 0 \quad \text{for} \quad t \to T^*.$$

Proof. By (2.5) and lemma 1

$$a'(s)e^{-s} = 2\int_0^s e^{-r}\left(v_s(r)^2 + v(r)^{p+1}\right)dr \geq 4\int_0^s e^{-r} v_s(r)\, v(r)^{\frac{p+1}{2}}\, dr$$

$$a'(s)e^{-s} \geq \frac{8}{p+3}\left(v(r)^{\frac{p+3}{2}} e^{-r}\,|_{r=0}^{s} + \int_0^s v(r)^{\frac{p+3}{2}} e^{-r} dr\right)$$

$$= \frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}} e^{-s} - v(0)^{\frac{p+3}{2}}\right) + \frac{8}{p+3}\int_0^s v(r)^{\frac{p+3}{2}} e^{-r} dr.$$

Since $a^{'}(s)>0$ for all $s>0$, v is increasing in $(0,\infty)$ and

$$a^{'}(s)e^{-s}\geq\frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}}e^{-s}-v(0)^{\frac{p+3}{2}}\right)+\frac{8}{p+3}v(0)^{\frac{p+3}{2}}\left(1-e^{-s}\right),\tag{3.1}$$

$$a^{'}(s)\geq\frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}}-u_0^{\frac{p+3}{2}}\right).$$

Using $u_1=0$ we obtain

$$v_s(s)\geq v(s)-u_0+\int_0^s v(0)^p\,dr=v(s)-u_0+u_0^p s,$$

$$\left(e^{-s}v(s)\right)_s\geq e^{-s}\left(u_0^p s-u_0\right),\tag{3.2}$$

$$a^{'}(s)\geq\frac{8}{p+3}\left(v(s)^{\frac{p+3}{2}}-u_0^{\frac{p+3}{2}}\right).$$

According to (3.2) and $v'(s)>0$, $v(s)^{\frac{p+3}{2}}\geq\left(u_0+u_0^p\left(e^s-1-s\right)\right)^{\frac{p+3}{2}}$ and for all $\epsilon\in(0,1)$, we get that

$$\epsilon v(s)^{\frac{p+3}{2}}\geq\epsilon\left(u_0+u_0^p\left(e^s-1-s\right)\right)^{\frac{p+3}{2}},$$

$$\epsilon v(s)^{\frac{p+3}{2}}-8u_0^{\frac{p+3}{2}}\geq(\epsilon-8)\,u_0^{\frac{p+3}{2}}+\epsilon u_0^{\frac{p(p+3)}{2}}\left(e^s-1-s\right)^{\frac{p+3}{2}}.$$

Now, we want to find a number $s_0>0$ such that $e^{s_0}-s_0=1+\left(\frac{8-\epsilon}{\epsilon}u_0^{\frac{p+3}{2}(1-p)}\right)^{\frac{2}{p+3}}$. This means that there exists a number $s_0>0$ satisfying

$$\epsilon v(s)^{\frac{p+3}{2}}-8u_0^{\frac{p+3}{2}}\geq 0\quad\text{for all}\quad s\geq s_0.$$

From (3.1), it follows that

$$a^{'}(s)\geq\frac{8-\epsilon}{p+3}v(s)^{\frac{p+3}{2}}\quad\text{for all}\quad s\geq s_0.$$

For all $s\geq s_0$, $\epsilon\in(0,1)$, we obtain that

$$2v(s)\,v_s(s)\geq\frac{8-\epsilon}{p+3}v(s)^{\frac{p+3}{2}},\qquad v(s)^{-\frac{p+1}{2}}v_s(s)\geq\frac{8-\epsilon}{2(p+3)},$$

$$\left(v(s)^{\frac{1-p}{2}}\right)_s\leq\frac{8-\epsilon}{2(p+3)}\frac{1-p}{2}.$$

Integrating the above inequality, we conclude that

$$v(s)^{\frac{1-p}{2}}\leq v(s_0)^{\frac{1-p}{2}}-\frac{8-\epsilon}{2(p+3)}\frac{p-1}{2}(s-s_0).$$

Thus, there exists a finite number

$$s_1^*\leq s_0+\frac{2(p+3)}{8-\epsilon}\frac{2}{p-1}v(s_0)^{\frac{1-p}{2}}:=k_1$$

such that $v(s)^{-1}\to 0$ for $s\to s_1^*$, that is,

$$u(t)^{-1}\to 0\quad for\quad t\to e^{k_1},$$

which implies that the life-span T^* of positive solution u is finite and $T^*\leq e^{k_1}$.■

Estimates for the life-span of positive solution u of $(*)$ under $u_1>0$, $u_0>0$

In this section we start to estimate the life-span of positive solution u of $(*)$ under $u_1 > 0$, $u_0 > 0$.

Theorem 2.4. For $u_1 > 0$, $u_0 > 0$, the positive solution u of $(*)$ blows up in finite time; that is, there exists a bound number T^* so that

$$u(t)^{-1} \to 0 \quad \text{for} \quad t \to T^*.$$

Proof. We separate the proof into two parts, $E(0) \geq 0$ and $E(0) < 0$.

i) $E(0) \geq 0$. By (2.1) and (2.5) we have

$$v_s(s)^2 - \frac{2}{p+1} v(s)^{p+1} \geq E(0),$$

$$v_s(s)^2 \geq \frac{2}{p+1} v(s)^{p+1} + E(0), v_s(s) \geq \sqrt{\frac{2}{p+1} v(s)^{p+1} + E(0)},$$

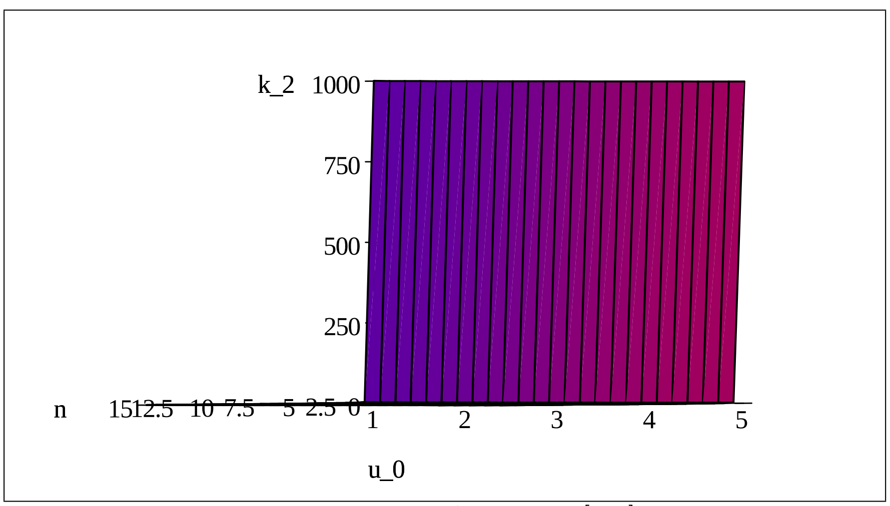
$$v_s(s) \geq \sqrt{\frac{2}{p+1}} v(s)^{\frac{p+1}{2}}, \quad v(s)^{\frac{p+1}{-2}} v_s(s) \geq \sqrt{\frac{2}{p+1}},$$

$$\left(v(s)^{\frac{1-p}{2}}\right)_s \leq \frac{1-p}{2} \sqrt{\frac{2}{p+1}}.$$

Integrating the above inequality, we obtain $v(s)^{\frac{1-p}{2}} \leq u_0^{\frac{1-p}{2}} + \frac{1-p}{2} \sqrt{\frac{2}{p+1}} s$. Thus, there exists a finite time

$$s_2^* \leq \frac{2}{p-1} \sqrt{\frac{p+1}{2}} u_0^{\frac{1-p}{2}} := k_2$$

such that $v(s)^{-1} \to 0$ for $s \to s_2^*$, that is, $u(t)^{-1} \to 0$ for $t \to e^{k_2}$, which means that the life-span T^* of positive solution u is finite and $T^* \leq e^{k_2}$.



Picturte 2 graph of k_2, $\quad u_0 \in [1, 5]$

ii) $E(0) < 0$. From (2.1) and (2.5) we obtain that $a'(s) > 0$, $v_s(s) > 0$ for all $s > 0$ and

$$J'(s) \leq -\frac{p-1}{2} \sqrt{\frac{2}{p+1} + E(0) a(s)^{-\frac{p+1}{2}}},$$

$$J(s) \le J(0) - \frac{p-1}{2}\int_0^s \sqrt{\frac{2}{p+1} + E(0)\,a(r)^{-\frac{p+1}{2}}}\,dr.$$

Since $E(0) < 0$ and $a'(s) > 0$ for all $s > 0$, then

$$J(s) \le a(0)^{-\frac{p-1}{4}} - \frac{p-1}{2}\sqrt{\frac{2}{p+1} + E(0)\,a(0)^{-\frac{p+1}{2}}}\,s.$$

Thus, there exists a finite number

$$s_3^* \le \frac{2}{p-1}a(0)^{-\frac{p-1}{4}}\left(\frac{2}{p+1} + E(0)\,a(0)^{-\frac{p+1}{2}}\right)^{-\frac{1}{2}} := k_3$$

such that $J(s_3^*) = 0$ and $a(s)^{-1} \to 0$ for $s \to s_3^*$, that is, $u(t)^{-1} \to 0$ for $t \to e^{k_3}$. This means that the life-span T^* of u is finite and $T^* \le e^{k_3}$.■

Estimates for the life-span of positive solution u of $(*)$ under $u_1 < 0$

Finally, we estimate the life-span of positive solution u of $(*)$ under $u_1 < 0$ in this section.

Theorem 2.5. For $u_1 < 0$, $u_0 \in \left(0, (-u_1)^{\frac{1}{p}}\right)$ we have:

$$u(t) \le (u_0 - u_1 - u_0^p) + (u_1 + u_0^p)\,t - u_0^p \ln t.$$

and particularly, for $E(0) \ge 0$, then

$$u(t) \le \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}\ln t\right)^{\frac{2}{1-p}}.$$

Proof. i) According to (1.1) and integrating this equation with respect to s, we get

$$v_s(s) = (u_1 - u_0) + v(s) + \int_0^s v(r)^p\,dr.$$

We have v is decreasing and

$$v_s(s) \le (u_1 - u_0) + v(s) + \int_0^s v(0)^p\,dr = (u_1 - u_0) + v(s) + u_0^p s,$$

$$e^{-s}v(s) - u_0 \le (u_1 - u_0)\left(1 - e^{-s}\right) + u_0^p\left(-se^{-s} - e^{-s} + 1\right);$$

that is,

$$u(t) \le (u_0 - u_1) + u_1 t + u_0^p\,(t - 1 - \ln t)$$

$$= (u_0 - u_1 - u_0^p) + (u_1 + u_0^p)\,t - u_0^p \ln t.$$

ii) $E(0) \ge 0$. By (2.1), we have

$$v_s(s)^2 \ge E(0) + \frac{2}{p+1}v(s)^{p+1} \ge \frac{2}{p+1}v(s)^{p+1},$$

$$-v_s(s) \ge \sqrt{\frac{2}{p+1}}v(s)^{\frac{p+1}{2}}, \quad \frac{2}{p-1}\left(v(s)^{\frac{1-p}{2}}\right)_s \ge \sqrt{\frac{2}{p+1}},$$

$$\sqrt{\frac{2}{p+1}}s \leq \frac{2}{p-1}\left(v\left(s\right)^{\frac{1-p}{2}} - v\left(0\right)^{\frac{1-p}{2}}\right),$$

$$v\left(s\right)^{\frac{1-p}{2}} \geq u_0^{\frac{1-p}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}s.$$

Then,

$$v\left(s\right) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}s\right)^{\frac{2}{1-p}}$$

for all $s \geq 0$, that is,

$$u\left(t\right) \leq \left(u_0^{\frac{1-p}{2}} + \frac{p-1}{2}\sqrt{\frac{2}{p+1}}\ln t\right)^{\frac{2}{1-p}} \quad \text{for all} \quad t \geq 1. \blacksquare$$

Reference

[1] M. R. Li, Nichlineare Wellengleichungen 2. Ordnung auf beschränkten Gebieten. PhD-Dissertation Tübingen 1994.

[2] M. R. Li , Estimates for the life-span of solutions of semilinear wave equations. Commun Pure Appl Anal, 2008, 7(2): 417-432.

[3] M. R. Li, On the blow-up time and blow-up rate of positive solutions of semilinear wave equations $\Box u - u^p = 0$. in $1-$dimensional space. Commun Pure Appl Anal, to appear.

[4] M. R. Li , On the semilinear wave equations. Taiwanese J Math, 1998, 2(3): 329-345.

[5] R. Duan, M. R. Li, T. Yang, Propagation of singularities in the solutions to the Boltzmann equation near equilibrium. Math Models Methods Appl Sci, 2008, 18(7): 1093-1114.

[6] M. R. Li, L. Y. Tsai, On a system of nonlinear wave equations. Taiwanese J Math, 2003, 7(4): 555-573.

[7] M. R. Li, L.Y. Tsai, Existence and nonexistence of global solutions of some systems of semilinear wave equations. Nonlinear Analysis, 2003, 54: 1397-1415.

[8] Li Meng Rong, Pai Jente. Quenching problem in some semilinear wave equations. Acta Math Sci,2008, 28B(3): 523-529.

[9] Li Meng Rong. On the generalized Emden-Fowler Equation $u''\left(t\right)u\left(t\right) = c_1 + c_2 u'\left(t\right)^2$ with $c_1 \geq 0$, $c_2 \geq 0$. Acta Math Sci, 2010 30B(4): 1227-1234.

[10] T.H. Shieh, M. R. Li, Numerical treatment of contact discontinuously with multi-gases. J. Comput Appl Math, 2009, 230(2): 656-673.

[11] M. R. Li, Y.J. Lin, T.H. Shieh, The flux model of the movement of tumor cells and health cells using a system of nonlinear heat equations. Journal of Computational Biology, vol. 18, no. 12, 2011, pp.1831-1839.

[12] M. R. Li, T.H. Shieh, C.J. Yue, P. Lee, Y.T. Li, Parabola method in ordinary differential equation. T J M, vol. 15, no 4, 2011, pp.1841-1857.

[13] R. Bellman, Stability Theory of Differential Equations. Yew York: McGraw-Hill, 1953.

[14] M.R. Li, Blow-up results and asymptotic behavior of the Emden-Fowler Equation. Acta Math Scientia, vol.4, pp.703-734 (2007)

III. Some results on semilinear wave equation $\Box\, u = u^p$ in $[0,T)\times\Omega$

We have treated the estimates for the life-span of positive solutions of the semilinear wave equation

$$\Box\, u = u^p \text{ in } [0,T)\times\Omega \tag{3.1}$$

with boundary value null and initial values $u(0,x) = u_0(x) \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\dot{u}(0,x) = u_1(x)$, where $p \in (1, n/n-2]$ and $\Omega \subset \mathbb{R}^n$ is a bouned smooth domain We use the following notations:

$$\cdot := \frac{\partial}{\partial t}, \nabla := \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right), Du{:=}\left(\dot{u}, \nabla u\right), \Box := \frac{\partial^2}{\partial t^2} - \triangle,$$

$$a(t) := \int_\Omega u^2(t,x)\,dx, E(t) := \int_\Omega \left(|Du|^2 - \frac{2}{p+1}u^{p+1}\right)(t,x)\,dx.$$

For a Banach space X and $0 < T \leq \infty$ we set

$$C^k(0,T,X) = \text{ Space of } C^k - \text{ functions} : [0,T) \to X,$$

$$H1 := C^1\left(0,T,H_0^1(\Omega)\right)\cap C^2\left(0,T,L^2(\Omega)\right).$$

Jörgens $[J1]$ published the first exist theorem for global solutions to the wave equation

$$\Box\, u + f(u) \;= 0 \text{ in } [0,T)\times\Omega, \tag{$*$}$$

for $\Omega = R^n, n = 3$ and $f(u) = g\left(u^2\right)u$, his result can be applied to the equation $\Box u + u^3 = 0$; and Browder $[B]$ generalized Jörgens's result to $n > 2$.

For local Lipschitz f , Li [Li2] proved the non-existence of global solution of the initial-boundary value problem of semilinear wave equation $(*)$ in bounded domain $\Omega \subset R^n$ under the assumption

$$\bar{E}(0) = \|Du\|_2^2(0) + 2\int_\Omega f(u)(0,x)\,dx \leq 0,$$

$$\eta f(\eta) - 2(1+2\alpha)\int_0^\eta f(r)\,dr \leq \lambda_1\alpha\eta^2 \quad \forall\eta{\in}\mathbb{R}$$

with $\alpha > 0$, $\lambda_1 := \sup\left\{\|u\|_2 / \|\nabla u\|_2 : u \in H_0^1(\Omega)\right\}$ and $a'(0) > 0$.

There we have a rough estimate for the life-span

$$T \leq \beta_2 := 2\left[1 - \left(1 - k_2 a(0)^{-\alpha}\right)^{1/2}\right]/k_1k_2$$

with

$$k_1 := \alpha a(0)^{-\alpha-1}\sqrt{a'(0)^2 - 4E(0)\,a(0)},$$

$$k_2{:=}\left(-4\alpha^2 E(0)/k_1^2\right)^{\alpha/1+2\alpha}.$$

For $n = 3$ and $f(u) = -u^3$, there exist global solutions of (SL) for small initial data [KP]; but if $E(0) < 0$ and $a'(0) > 0$ then the solutions are only local, i.e. $T < \infty$ [Li2].

John [J2] showed the nonexistence of solutions of the initial-boundary value problem for the wave equation $\Box u = A|u|^p$, $A > 0$,

$$1 < p < 1 + \sqrt{2}, \quad \Omega = \mathbb{R}^3.$$

This problem was considered by Glassey [G] in two dimensional case $n = 2$; for $n > 3$ Sideris [S3] showed the nonexistence of global solutions under the conditions

$$\|u_0\|_1 > 0 \text{ and } \|u_1\|_1 > 0.$$

According to this result Strauss [S1, p.27] guessed that the solutions for the above mentioned wave equation are global for $p \geq p_0(n) = \lambda$ which is the positive root of the quadratic equation

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0$$

and $\Omega = \mathbb{R}^n$. Further literature about blow up one can see [J2], [S3], [Li2] and [Li3] and their reference. Further literature could be fund in [S1] and [R]. Here we extend our results to the equation (3.1) .

Definition and Fundamental Lemmas

There are many definitions of the weak solutions of the initial-boundary problems of the wave equation, we use here as following.

Definition 3.1: For $p \in (1, n/n-2]$, $u \in H1$ is called a positive weakly solution of equation (3.1), if

$$\int_0^t \int_\Omega (\dot{u}\dot{\varphi} - \nabla u \cdot \nabla \varphi + u^p \varphi)(r, x)\, dx dr = 0$$

for all $\varphi \in H1$ and $\int_0^t \int_\Omega u(r, x)\, \psi(r, x)\, dx dr \geq 0$ for each positive $\psi \in C_0^\infty([0, T) \times \Omega)$.

Remark 3.2: 1) This definition 3.1 is resulted from the multiplication with φ to the equation (3.1) and integration in Ω from 0 to t.

2) From the local Lipschitz functions $u^p, p \in [1, n/n-2]$, the initial-boundary value problem (0.1) possesses a unique solution in $H1$ [Li1].

Hereto we use the notations for $q \in [1, 2n/n-2]$:

$$\frac{1}{C_\Omega} \quad : \quad = \eta_1 = \sup\left\{\|u\|_2 / \|Du\|_2 : u \in H_0^1(\Omega)\right\},$$

$$\lambda_q \quad = \quad \sup\left\{\|u\|_q / \|Du\|_2 : u \in H_0^1(\Omega) \cap L_q(\Omega)\right\}.$$

In this report we need the following lemmas

Lemma 3.3: Suppose that $u \in H1$ is a weakly positive solution of (3.1) with $E(0) = 0$ for $p \in [1, n/n-2]$, then for $a(0) > 0$ we have:

(i) $a \in C^2(\mathbb{R}^+)$ and $E(t) = E(0) \quad \forall t \in [0, T)$.

(ii) $a'(t) > 0 \quad \forall t \in [0, T)$, provided $a'(0) > 0$.

(iii) $a'(t) > 0 \quad \forall t \in (0,T)$, if $a'(0) = 0$.

(iv) For $a'(0) < 0$, there exists a constant $t_0 > 0$ with

$$a''(t) = \frac{2(p+3)}{p+1} \int_{\Omega} u^{p+1}(t,x)\,dx - 4 \int_{\Omega} |\nabla u|^2 (t,x)\,dx.$$

Lemma 3.4: Suppose that u is a positive weakly solution in $H1$ of equation (3.1) with $u(0,\cdot) = 0, \dot{u}(0,\cdot) = 0$ in $L^2(\Omega)$. For $p \in [1, n/n-2]$, we have $u \equiv 0$ in $H1$.

According to Lemma 3.4, we discuss the following theme

(3) $E(0) = 0, a(0) > 0$ and $a'(0) \geq 0$ or $a'(0) < 0$.

(4) $E(0) < 0, a(0) > 0$ and $a'(0) \geq 0$ or $a'(0) < 0$.

Estimates for the Life-Span of the Solutions of (3.1) under Null-Energy

In this section we focus on the case that $E(0) = 0$, $p \in [1, n/n-2]$ and divide it into two parts

(i) $a(0) > 0, a'(0) \geq 0$

(ii) $a(0) > 0, a'(0) < 0$

Estimates for the Life-span of the Solutions of (3.1) under $a'(0) \geq 0$

Theorem 3.5.1. *Suppose that $u \in H1$ is a positive weakly solution of equation (3.1) with $a'(0) \geq 0$ and $E(0) = 0$. Then the Life-span of u is finite, further*

$$T \leq \alpha_1 := k_2^{-1} \sin^{-1}\left(\frac{k_2}{k_1 a(0)^{\frac{P-1}{4}}}\right) \tag{3.2}$$

with

$$k_1 \ : \ = \frac{p-1}{4} \cdot a(0)^{-\frac{p-1}{4}} \sqrt{a'(0)\,a^{-2}(0) + 4C_\Omega^2},$$
$$k_2 \ : \ = \frac{p-1}{2} C_\Omega.$$

If $T = \alpha_1$, then $a(t)^{-1} \to 0, t \to T$.

Furthermore, we have also the estimate for $a(t)$:

$$a(t) \geq \left(\frac{k_2}{k_1}\right)^{\frac{4}{p-1}} \left(\sin(k_2\alpha_1 - k_2 t)\right)^{-\frac{4}{p-1}} \quad \forall t \in [0,T). \tag{3.3}$$

This means that the blow-up rate of u is $\dfrac{4}{p-1}$ in the sin-growth.

Proof: By Lemma 3.3 (i) we have

$$a''(t) - (p+3) \int_{\Omega} \dot{u}^2(t,x)\,dx \geq (p-1)\, C_\Omega^2 a(t) \quad \forall t \geq 0. \tag{3.4}$$

Set $J(t) := a(t)^{-\frac{p-1}{4}}$. Then

$$J''(t) = \frac{p-1}{4} a(t)^{-\frac{p-1}{4}-2} \left[-aa''(t) + \frac{p+3}{4} a'(t)^2 \right]. \tag{3.5}$$

Using Lemma 3.3 (ii), (iii) and (3.5), we find that $a'(t) > 0$ for each $t \geq 0$, and also $J'(t) < 0$ for each $t > 0$.

Hereto, for each $t \geq 0$, we obtain

$$aa''(t) - \frac{p+3}{4} a'(t)^2 \geq aa''(t) - (p+3)\, a(t) \int_\Omega \dot{u}^2(t,x)\, dx$$

$$\geq (p-1)\, C_\Omega^2 a(t)^2 .$$

Thereby, $J''(t) \leq -\frac{1}{4}(p-1)^2 C_\Omega^2 J(t) < 0 \ \ \forall t \geq 0$ and $J(t) < J(0) + J'(0)\, t$. According to $J'(0) < 0$, there exists a constant $T^* > 0$ with $J(T^*) = 0$. Since we have $J'(t) < 0$ for each $t \geq 0$, by (3.4) we find

$$J'J''(t) \geq -\frac{1}{4}(p-1)^2 C_\Omega^2 JJ'(t) \geq 0 \ \ \forall t \in [0, T^*),$$

$$J'(t)^2 \geq k_1^2 - k_2^2 J(t)^2 \geq 0,$$

and $J'(t) \leq -\sqrt{k_1^2 - k_2^2 J(t)^2} \ \ \forall t \in [0, T)$. Therefore we obtain $\int_{J(t)}^{J(0)} \frac{dr}{\sqrt{k_1^2 - k_2^2 r^2}} = -\int_{J(0)}^{J(t)} \frac{dr}{\sqrt{k_1^2 - k_2^2 r^2}} \geq t$ and

$$\sin^{-1}\left(\frac{k_2}{k_1} J(0) \right) - \sin^{-1}\left(\frac{k_2}{k_1} J(t) \right) \geq k_2 t \ \ \forall t \in [0, T). \tag{3.6}$$

From (3.6), it follows $T \leq k_2^{-1} \sin^{-1}\left(\frac{k_2}{k_1} J(0) \right) = k_2^{-1}\alpha_1$ and herewith

$$J(t) \leq \frac{k_1}{k_2} \sin(k_2\alpha_1 - k_2 t) \ \ \forall t \in [0, T). \tag{3.7}$$

and $a(t) \geq \left(\frac{k_1}{k_2} \right)^{-\frac{4}{p-1}} (\sin(k_2\alpha_1 - k_2 t))^{-\frac{4}{p-1}} \ \ \forall t \in [0, T)$. If $T = \alpha_1$, by (3.7), then $J(t) \to 0$ as $t \to T$, this means that

$$a(t)^{-1} \to 0, \ t \to T.$$

Remark) The theorem 3.5.1 is a extension of my own Satz 2 in [Li1]. And the local existence and uniqueness of solutions of equation (3.1) in $H1$ are known [Li2].

2) For special cases: i) For $n = 2$, $p > 1$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (3.1) is bounded by $T \leq \alpha_1$.

ii) For $n = 3$, $p = 2$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (3.1) is bounded $T \leq \alpha_2 := 2C_\Omega^{-1} \sin^{-1}\left(2C_\Omega \left(a'(0)^2\, a(0)^{-2} + 4C_\Omega^2 \right)^{-\frac{1}{2}} \right)$.

If $T = \alpha_2$, then $a^{-1}(t) \to 0, t \to T$.

iii) For $n=3$, $p=3, E(0)=0$ the life-span of the positive solution $u\in H1$ of equation (3.1) is bounded $T\leq\alpha_3:=C_\Omega^{-1}\sin^{-1}\left(2C_\Omega\left(a'(0)^2a(0)^{-2}+4C_\Omega^2\right)^{-\frac{1}{2}}\right)$.

If $T=\alpha_3$, then $a(t)^{-1}\to 0, t\to T$.

iv) For $a'(0)=0$, we have $\alpha_1=\dfrac{\pi}{p-1}C_\Omega$.

v) For $|\Omega|\to\infty$, we have also $\alpha_1\to\dfrac{1}{p-1}\dfrac{a(0)}{a'(0)}$; as $|\Omega|\to 0$, then $\alpha_1\to\dfrac{2}{p-1}\sin^{-1}\left(\dfrac{1}{4}C_\Omega^{-1}\right)$.

Estimates for the Life-span of the Solutions of equation (3.1) under $a'(0)<0$

Theorem 3.5.2 *Suppose that $u\in H1$ is a positive weakly solution of the initial-boundary value problem equation (3.1) with $a(0)>0$, $E(0)=0$ and $a'(0)<0$. Then the life span of u is bounded:*

$$T\leq\alpha_5:=\frac{\pi}{(p-1)C_\Omega}-\frac{a'(0)}{p-1}\left(\frac{2\lambda_{p+1}^{p+1}}{p-1}\right)^{\frac{2}{p-1}}.$$

If $T=\alpha_5$, then $a(t)^{-1}\to 0, T\to\alpha_5$. Further, we have the estimate for the blow-up rate of $a(t)$ in the neighborhood of α_5:

$$a(t)\geq a(t_0)\left[\sin\left(\frac{(p-1)C_\Omega}{2}(\alpha_5-t)\right)\right]^{-\frac{4}{p-1}}$$

for $\forall t\in[t_0,T), t_0\leq t_1$ with $t_1:=\dfrac{-1}{p-1}\left(\dfrac{p+1}{2\lambda_{p+1}^{p+1}}\right)^{-\frac{2}{p-1}}a'(0)$.

Proof. By Lemma 3.3 (iv), we have $a'(t)>0\ \ \forall t>t_1$. Similar to the proof of theorem 3.5.1, we get $J''(t)\leq-\frac{(p-1)^2}{4}C_\Omega^2J(t)<0\ \ \forall t>t_1$ and there exists a constant $t_0\leq t_1$ with $a'(t)>0\ \ \forall t>t_0$, $a'(t_0)=0$; thus $J'(t)<0\ \ \forall t>t_0$, $J'(t_0)=0$. And

$$J'(t)^2\leq\frac{(p-1)^2}{4}C_\Omega^2\left(J(t_0)^2-J(t)^2\right)\quad\forall t\geq t_0$$

and herewith $J'(t)\leq-\frac{p-1}{2}C_\Omega\sqrt{J(t_0)^2-J(t)^2}\ \ \forall t\geq t_0$,

$$\begin{aligned}t-t_0&\leq\frac{2}{(p-1)C_\Omega}\int_{J(t)}^{J(t_0)}\frac{dr}{\sqrt{J(t_0)^2-r^2}}\\&=\frac{2}{(p-1)C_\Omega}\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{J(t)}{J(t_0)}\right)\right)\quad\forall t\geq t_0.\end{aligned}$$

Therefore we conclude that $T \leq t_0 + \dfrac{\pi}{(p-1)\,C_\Omega} \leq \alpha_5$, $\frac{(p-1)C_\Omega}{2}(t-\alpha_5) \leq -\sin^{-1}\left(\dfrac{J(t)}{J(t_0)}\right)$ $\forall t \geq t_0$ and

$$J(t) \leq J(t_0)\sin\left(\frac{p-1}{2}C_\Omega(\alpha_5 - t)\right) \quad \forall t \geq t_0.$$

Theorem 3.5.3: *Suppose that u is a positive weakly solution of equation (3.1) with $a(0) > 0$, $E(0) = 0$ and*

$$(i) - \frac{1}{2}r_1 a(0) < a'(0) < 0 \quad (ii)\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)} \leq e^{2r_1 t_1},$$

where $r_1 := \sqrt{2(p-1)}C_\Omega$. Then the life-span of u is bounded:

$$T \leq \alpha_6 := \frac{\pi}{(p-1)\,C_\Omega} + \frac{1}{2\gamma_1}\ln\left(\frac{\gamma_1 a(0) - 2a'(0)}{\gamma_1 a(0) + 2a'(0)}\right) \leq \alpha_5.$$

And there is a constant $t_4 > 0$ with

$$(iii) \quad t_4 \quad \leq \quad t_3 := \frac{1}{2r_1}\ln\left(\frac{r_1 a(0) - 2a'(0)}{r_1 a(0) + 2a'(0)}\right),$$

$$(iv) \quad a(t) \quad \geq \quad a(t_4)\left[\sin\left(\frac{p-1}{2}C_\Omega(\alpha_6 - t)\right)\right]^{-\frac{4}{p-1}}.$$

Remark 3.5.3: In theorem 3.5.1 we have no restriction (i) or (ii) under theorem 3.5.2. It seems that theorem 3.5.1 is better as theorem 3.5.2, yet under the suppositions (i) , (ii) theorem 3.5.2 is better then theorem 3.5.1.

Proof of Theorem 3.5.2: Similar to the proof of Lemma 3.3 (iii) for all $t \geq 0$ it yields

$$2aa'(t) = b'(t) \geq \left(a'(0) + \frac{r_1}{2}a(0)\right)a(0)\,e^{r_1 t} + \left(a'(0) - \frac{r_1}{2}a(0)\right)a(0)\,e^{-r_1 t},$$

herewith $a'(t) > 0$ $\forall t > t_3$ and there exists a constant $t_4 \leq t_3$ with

$$a'(t_4) = 0, a'(t) > 0 \;\; \forall t > t_4.$$

Using the same steps in the proof of Theorem 3.5.1 we obtain

$$t - t_4 \leq \frac{2}{(p-1)\,C_\Omega}\int_{J(t)}^{J(t_4)}\frac{dr}{\sqrt{J(t_4)^2 - r^2}} = \frac{2}{(p-1)\,C_\Omega}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{J(t)}{J(t_4)}\right)\right) \quad \forall t \geq t_4.$$

Thus we get the assertions in Theorem 3.5.2.

Estimates for the Life-Span of the Solutions of equation (3.1) under Negative Energy

In this chapter we suppose the energy $E(0)$ is negative and consider the following cases:

(i) $a(0) > 0, a'(0) > 0$ (ii) $a(0) > 0, a'(0) = 0$ (iii) $a(0) > 0, a'(0) < 0$.

Fundamental Lemmas

In this section we use the following lemmas and those argumentations of proof to lemmas are not true for positive energy, so under positive energy we need another method to show the results.

Lemma 3.6.1: *Suppose that $u \in H1$ is a positive weakly solution of equation (3.1) with $a(0) > 0$ and $E(0) < 0$. Then*

(i) for $a'(0) \geq 0$, we have $a'(t) > 0$ $\forall t > 0$.

(ii) for $a'(0) < 0$, there exists a constant $t_5 > 0$ with $a'(t) > 0$ $\forall t > t_5$, $a'(t_5) = 0$ and $t_5 \leq t_6 := \dfrac{-a'(0)}{(p-1)\left(\delta^2 - E\right)}$, where δ is the positive root of the equation $\dfrac{2}{p+1}\lambda_{p+1}^{p+1} \cdot r^{p+1} - r^2 + E(0) = 0$.

Proof: By Sobolev inequality it yields

$$a''(t) \leq 2E(0) + 2\|\nabla u\|_2^2(t)\left(\frac{p+3}{p+1}\lambda_{p+1}^{p+1}\|\nabla u\|_2^{p-1}(t) - 2\right) \quad \forall t \geq 0.$$

We have also

$$a''(t) \geq -(p+1)E(0) + (p-1)\|\nabla u\|_2^2(t) \quad \forall t \geq 0,$$

herewith follows $-E(0) \leq \|\nabla u\|_2^2(t)\left(\dfrac{2}{p+1}\lambda_{p+1}^{p+1}\|\nabla u\|_2^{p-1}(t) - 1\right) \quad \forall t \geq 0.$

According to $E(0) < 0$ there exists the positive root δ to the equation

$$\frac{2}{p+1}\lambda_{p+1}^{p+1} \cdot r^{p+1} - r^2 + E(0) = 0.$$

and $\delta > \left(\frac{p+1}{2}\right)^{\frac{2}{p-1}} \cdot \lambda_{p+1}^{-\frac{2(p+1)}{p-1}}$, so it is $\|\nabla u\|_2^2(t) \geq \delta^2$ $\forall t \geq 0$, hereby $a''(t) \geq -(1 \pm p)E(0) + (p-1)\delta^2$ $\forall t \geq 0$ and

$$a'(t) \geq a'(0) + (p-1)\left(\delta^2 - \frac{p+1}{p-1}E(0)\right) \cdot t \quad \forall t \geq 0.$$

Thus, (i) under lemma 3.6.1 follows.

For $a'(t_0) < 0$, we have $a'(t_0) \geq 0 > a'(0)$, therefore (ii) in lemma 3.6.1 is proved.

Estimates for the Life-Span of the Solutions of equation (3.1) under $E(0) < 0$, $a'(0) \geq 0$.

Theorem 3.6.2: *Suppose that $u \in H1$ is a positive weakly solution of equation (3.1) with $E(0) < 0$ and $a'(0) \geq 0$. Then the life-span of u is bounded:*

$$T \leq \alpha_5 := k_0^{-1}k_2^{-1} \cdot \sin\left(k_2 a(0)^{-\frac{p-1}{4}}\right) \tag{3.8}$$

where $k_0 := \frac{p-1}{2}a(0)^{-\frac{p+1}{4}}\sqrt{\dfrac{1}{4}a(0)^{-1}a'(0)^2 + \frac{p-1}{p+1}\left(\delta^2 - \frac{p+1}{p-1}E(0)\right)}$, $k_2 := \left(\frac{k_1}{k_0}\right)^{\frac{p-1}{p+1}}$, $k_1 := \frac{p-1}{2}\sqrt{\frac{\delta^2 - \frac{p-1}{p+1}E(0)}{p+1}}$. *Further we have*

$$a(t) \geq k_2^{\frac{4}{p-1}}\left(\sin\left[k_0 k_2(\alpha_5 - t)\right]\right)^{-\frac{4}{p-1}} \quad \forall t \in [0, T]. \tag{3.9}$$

Remark 3.6.2: 1) We can good estimate the rate of the singularity of $a(t)$ and the life-span of u, but we can not get them contemporaneously:

$$T \le \alpha_6 := k_0^{-1} k_2^{-1} \frac{\tan^{-1}\left(k_2 a(0)^{-\frac{p-1}{4}}\right)}{\sqrt{1 - k_2^2 a(0)^{-\frac{p-1}{2}}}} \tag{3.10}$$

(3.11)

$$a(t) \ge k_2^{\frac{4}{p-1}} \left\{\tan\left\{\tan^{-1}\left[\left(k_2 a(0)^{-\frac{p-1}{4}}\right) - k_0 k_2 \sqrt{1 - k_2^2 a(0)^{-\frac{p-1}{2}}}\, t\right]\right\}\right\}^{-\frac{4}{p-1}}$$

for each $t \in [0, T]$.

2) For $k_2 \cdot a(0)^{-\frac{p-1}{4}} = 1$, that is,

$$4(2-p)\left(\delta^2 - E(0)\right) a(0) = (p+1)\, a'(0)^2, \quad n \ge 4, \tag{3.12}$$

then we can get a better estimate for the life-span of u:

$$T \le k_0^{-1} k_2^{-1} \frac{p-1}{2(p+1)} \sqrt{\pi} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{p+1}\right)}{\Gamma\left(\frac{p}{p+1}\right)} := \alpha_7. \tag{3.13}$$

Proof of Theorem 3.6.2: By lemma 3.6.1 we have $a'(t) > 0 \;\; \forall t > 0$ and $J'(t) < 0 \;\; \forall t > 0$. Similar to the proofs in above

$$a''(t) - (p+3) \int_\Omega u^2(t,x)\, dx \ge (p-1)\left(\delta^2 - \frac{p+1}{p-1} E(0)\right) \quad \forall t \ge 0.$$

And

$$J''(t) \le -\frac{(p-1)^2}{4}\left(\delta^2 - \frac{p+1}{p-1} E(0)\right) J(t)^{\frac{p+3}{p-1}} < 0 \;\; \forall t > 0.$$

So there exists a finite number $T > 0$ such that $J(T) = 0$, and herewith

$$2J'J''(t) \ge -\frac{(p-1)^2}{2}\left(\delta^2 - \frac{p+1}{p-1}E(0)\right) J(0)^{\frac{2(p+1)}{p-1}} - \frac{(p-1)^3}{4(p+1)}\left(\delta^2 - \frac{p+1}{p-1}E(0)\right) J(t)^{\frac{2(p+1)}{p-1}}$$

According to the definitions of k_0, k_1 and k_2 and the fact that $J'(t) < 0$ for each $t > 0$, it follows

$$J'(t) \le -k_0 \sqrt{1 - (k_2 J(t))^{\frac{2(p+1)}{p-1}}} \quad \forall t > 0.$$

Using $J'(t) < 0 \quad \forall t > 0$ we find that $g(t)$ is a monotone increasing function and $0 \le J'(0)^2 = g(0) \le g(t) \le g(T) = k_0^2 \;\; \forall t \in [0, T]$, therefore

$$k_2 J(t) \le 1 \;\forall t \in [0, T].$$

Hence,

$$k_2 k_0 t \le \int_{k_2 \cdot J(t)}^{k_2 \cdot J(0)} \frac{dr}{\sqrt{1 - r^{\frac{2(p+1)}{p-1}}}} \quad \forall t \in [0, T]. \tag{3.14}$$

Because $p \in \left[1, \frac{n}{n-2}\right]$, we get $\frac{2(p+1)}{p-1} \geq 4$ and $1 - r^{\frac{2(p+1)}{p-1}} \geq 1 - r^4 \;\; \forall r \in [0,1]$. Therefore we obtain

$$k_0 k_2 t \leq \int_{k_2 J(t)}^{k_2 J(0)} \frac{dr}{\sqrt{1+r^2}\sqrt{1-r^2}} \leq \int_{k_2 J(t)}^{k_2 J(0)} \frac{1}{\sqrt{1+(k_2 J(t))^2}} \frac{dr}{\sqrt{1-r^2}}$$

$$= \frac{\sin^{-1}\left(k_2 J(0) - \sin^{-1}(k_2 J(t))\right)}{\sqrt{1+(k_2 J(t))^2}} \quad \forall t \in [0,T].$$

Hence the estimate (3.8) follows. Further, we have

$$\sin^{-1}(k_2 J(t)) + k_0 k_2 t \leq k_0 k_2 \alpha_5 \quad \forall t \in [0,T],$$

$$J(t) \leq k_2^{-1} \sin[k_0 k_2 (\alpha_5 - t)] \quad \forall t \in [0,T].$$

Herewith we obtain the assertion (3.9).

Proof of Remark 3.6.2: Using the inequality (3.14) we obtain

$$k_0 k_2 t \leq \frac{1}{\sqrt{1-(k_2 J(0))^2}} \int_{k_2 J(t)}^{k_2 J(0)} \frac{dr}{\sqrt{1+r^2}}$$

$$= \frac{1}{\sqrt{1-(k_2 J(0))^2}} \left[\tan^{-1}(k_2 J(0)) - \tan^{-1}(k_2 J(t))\right].$$

From this, the estimates (3.10) and (3.11) follows.
Under the inequality (3.12) we use (3.14), then it yields

$$k_0 k_2 t \leq \int_0^1 \frac{dr}{\sqrt{1-r^{\frac{2(p+1)}{p-1}}}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p-1}{2(p+1)}\right)}{\Gamma\left(\frac{p-1}{2(p+1)} + \frac{1}{2}\right)}.$$

Thus the estimate (3.13) is proved.

Reference:

[B] Browder, F.E: On Non-linear Wave Equations. M.Z.80. pp249-264(1962).

[G] Glassey, R.: Finite-time Blow-up for solutions of Nonlinear Wave Equations. M.Z. 177. pp.323-340 (1981).

[J1] Jörgens,K.: Das Anfangswertproblem im Großen für eine Klasse nichtlinearer Wellengleichungen. M.Z.77. pp.295-307(1961).

[J2]John, F.: Blow-up for Quasilinear Wave Equations in Three Space Dimensions. Comm.Pure. Appl. Math. 36 pp.29-51 (1981).

[J3]John, F.: Delayed Singularity Formation in solutions of Nonlinear Wave Equations in higher Dimensions. Comm. Pure Appl. Math. 29. pp.649-682 (1976).

[H] Haraux, A.: Nonlinear Evolution Equations -Global Behavior of Solutions. Lecture Notes in Math. Springer (1981).

[K] Klainerman,S.: Global Existence for Nonlinear Wave Equations. Comm.Pure Appl.Math.33. pp.43-101(1980).

[KP] Klainerman,S., Ponce,G.: Global, Samll Amplitude Solutions to Nonlinear Evolution Equations. Comm.Pure Appl.Math.36. pp.133-141(1983).

[Li1] Li,M.R.: On the Semi-Linear Wave Equations (I). Taiwanese Journal of Math. Vol. 2, No. 3, pp. 329-345, Sept. 1998

[Li2] Li,M.R.: Estimates for the Life-Span of the Solutions of some Semilinear Wave Equations. ACPAA. Vol.7, No.2, pp.417-432.(2008)

[Li3] Li, M.R.: Nichtlineare Wellengleichungen 2. Ordnung auf beschränkten Gebieten. PhD-Dissertation Tübingen 1994.

[R] Racke,R.: Lectures on nonlinear Evolution Equations: Initial value problems. Aspects of Math. Braunschweig Wiesbaden Vieweg(1992).

[S1] Strauss,W.A.: Nonlinear Wave Equations. A.M.S.Providence(1989). Dimensions. J. Differential Equations52.p.378-406(1984)

[S2] Segal,I.: Nonlinear Semigroups. Ann. Math. (2)78.p.339-364(1963).

[S3] Sideris, T.: Nonexistence of global solutions to Semilinear Wave Equations in high dimensions.J. Differential Equations 52. p.303-345 (1982).

[W1] von Wahl,W.: Klassische Lösungen nichtlinearer Wellengleichungen im Großen. M.Z.112.p.241-279(1969).

[W2] von Wahl,W.: Klassische Lösungen nichtlinearer gedämpfter Wellengleichungen im Großen. Manuscripta.Math.3.p7-33(1970).

IV. Blow-up result on Emdenm-Fowler type semilinear wave equation $t^2u_{tt}-u_{xx}=u^p$ in $[0,T)\times(a,b)$, $p>1$

Under some tranformations one can get the existence of solutions to the Emdenm-Fowler type semilinear wave equation

$$t^2u_{tt}-u_{xx} = u^p \; in \; [0,T)\times(a,b), \quad p>1, \tag{4.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x)=u_1(x),$$

for suitable conditions. There are at least three methods on this existence result, the simplest is taking the transform $s=e^t$, $u(t,x)=v(s,x)$ then $u_t=t^{-1}v_s$, $t^2u_{tt}=-v_s+v_{ss}$, the equation (4.1) can be transformed into the following type

$$\begin{aligned} v_{ss}-v_{xx} &= v_s+v^p \; in \; \left[1,e^T\right)\times(a,b), \\ v(1,x) &= u_0(x), v_s(1,x)=u_1(x). \end{aligned}$$

By the similar way to disscuse the existence of solutions for nonlinear wave equation in bounded domain. Second method is the Laplace transform method on t. Third is the method of Laplace-Fourier transform, taking Laplace tranceform on t and Fourier transform on space x. In this report we focus on the Blow-up property of the solution u.

After some tedious argumentations, we can obtain the following results:

4.1 Life-Span of the Solutions of (4.1) under Null-Energy

Theorem 4.1.1. *Suppose that $u\in H1$ is a positive weakly solution of equation (4.1) with $a'(0)\ \geq 0$ and $E(0)\ =0$. Then the life-span of u is finite and the blow-up rate of u is smaller than* $\dfrac{4}{p-1}$.

Theorem 4.1.2. *Suppose that $u\in H1$ is a positive weakly solution of the initial-boundary value problem equation (4.1) with $a(0)\ >0$, $E(0)=0$ and $a'(0)\ <0$. Then the life span of u is bounded and the blow-up rate of $a(t)$ in the neighborhood of α_5 is smaller than* $\dfrac{4}{p-1}$.

4.2 Life-Span of the Solutions of (4.1) under Negative-Energy

Theorem 4.2: *Suppose that $u\in H1$ is a positive weakly solution of equation (4.1) with $E(0)\ <0$ and $a'(0)\ \geq 0$. Then the life-span of u is bounded and the blow-up rate of $a(t)$ in the neighborhood of α_5 is smaller than* $\dfrac{4}{p-1}$.

The above results are maybe not the best but it is not easy to achieve.

二零一三年訪問香港中文大學數學科學所副所長 辛周平 教授
香港城市大學數學系系主任/講座教授 楊彤
心得報告

八月 二 日 ~ 八月十四日 抵港及私人行程 拜訪友人

八月十五日 ~ 八月二十二日 於中文大學數學科學所與副所長辛周平教授見面並準備演講內容的第一部份 關於 Emden-Fowler 方程正解問題

On the positive solution for the Emden-Fowler equation $t^2 u'' = u^p,\ p > 1.$

八月二十二日 ~ 八月二十八日 準備演講內容的第二部份 關於半線性波方程解的性質 On the Life-span of solutions to semilinear wave equation

八月二十八日 ~ 八月三十日 準備演講內容的第三部份 關於一維有界域上 Emden-Fowler 型態半線性波方程解 並發表演講

”關於 Emden-Fowler 型態半線性波方程解”

八月三十一日 ~ 九月 一 日 參觀香港中文大學數學系與沙田鄉萬佛寺

九月 二 日 ~ 九月 六 日 訪問香港城市大學數學系系主任/講座教授楊彤 此期間楊彤教授常有事開會 指導其博士生與碩士班學生 無暇討論
九月 七 日 ~ 九月 八 日 楊彤教授不在香港 繼續研究一維有界域上 Emden-Fowler 型態半線性波方程解的性質
九月 九 日 ~ 九月 十一 日 與楊彤教授討論一維有界域上 Emden-Fowler 型態半線性波方程解的性質 並獲得初步結果
九月 十二 日 楊彤教授赴日本 不在香港 繼續研究一維有界域上 Emden-Fowler 型態半線性波方程解的性質 並獲得更進一步結果

九月 十三 日 準備返台

科技部補助計畫衍生研發成果推廣資料表

日期:2014/09/19

科技部補助計畫	計畫名稱: 一維 Emden-Fowler 型半線性波方程式解之存在區間之估計
	計畫主持人: 李明融
	計畫編號: 102-2115-M-004-002-　　學門領域: 偏微分方程

無研發成果推廣資料

102 年度專題研究計畫研究成果彙整表

計畫主持人：李明融	計畫編號：102-2115-M-004-002-
計畫名稱：一維 Emden-Fowler 型半線性波方程式解之存在區間之估計	

成果項目			量化：實際已達成數（被接受或已發表）	量化：預期總達成數(含實際已達成數)	量化：本計畫實際貢獻百分比	單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
國內	論文著作	期刊論文	0	0	100%	篇	no
		研究報告/技術報告	0	0	100%		no
		研討會論文	2	0	100%		航太年會兩篇共同作者文章
		專書	0	0	100%		no
	專利	申請中件數	0	0	100%	件	no
		已獲得件數	0	0	100%		no
	技術移轉	件數	0	0	100%	件	no
		權利金	0	0	100%	千元	no
	參與計畫人力（本國籍）	碩士生	1	0	100%	人次	陳仁發
		博士生	2	0	100%		李詠玄/姜林宗叡
		博士後研究員	0	0	100%		no
		專任助理	0	0	100%		no
國外	論文著作	期刊論文	3	0	100%	篇	Numerical Heat Transfer/Abstract and Applied Analysis/Mathematical and Computational Applications
		研究報告/技術報告	0	0	100%		no
		研討會論文	0	0	100%		no
		專書	0	0	100%	章/本	no
	專利	申請中件數	0	0	100%	件	no
		已獲得件數	0	0	100%		n0
	技術移轉	件數	0	0	100%	件	n0
		權利金	0	0	100%	千元	n0
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	n0
		博士生	0	0	100%		no
		博士後研究員	0	0	100%		no
		專任助理	0	0	100%		n0

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	北京師範大學數學系研究傑出獎座教授將與我一起合作關於 Euler 方程之文

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

科技部補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   □達成目標

   ■未達成目標（請說明，以 100 字為限）

   　　□實驗失敗

   　　□因故實驗中斷

   　　■其他原因

   說明：

   結果尚未發表出來　正在進行投稿中

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：□已發表 ■未發表之文稿 □撰寫中 □無

   專利：□已獲得 □申請中 ■無

   技轉：□已技轉 □洽談中 ■無

   其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

   對一維 Emden-Fowler 型半線性波方程式解之存在區間之估計應有相當的貢獻