科技部補助專題研究計畫成果報告
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一類耦合系統的同步行為

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中 文 摘 要 ： 在這近幾十年來，耦合系統的同步化行為已經成為相當重要的研究課題。在現有文獻中，用來處理具時間延遲之非線性耦合系統同步化問題的方法仍受到了相當的限制。現有處理非線性藕合系統同步問題的方法大多要求耦合矩陣是與時間無關的、或對稱的，或者要求耦合矩陣之行的總和須為零、或其非對角線元素必須為非負，並且要求耦合函數的斜率必須大於某一個正數。在這個研究中，我們發展出一套可以處理具更一般耦合矩陣形式並具一般形式的耦合函數之具時間延遲的非線性耦合系統的同步化方法。

中 文 關 鍵 詞 ： 同步化、耦合系統、耦合矩陣、耦合函數

英 文 摘 要 ： Synchronization of coupled systems have been important research topics in recent decades. In the literature, the existing methods for the synchronization of nonlinearly coupled systems with delay are still limited. Most of the existing approaches to the synchronization problems of nonlinearly coupled systems require the connectivity matrix to be time-independent, symmetric, with zero row-sums, with nonnegative off-diagonal entries; moreover, these approaches commonly require that the coupling functions have positive lower bounds on their slope. In this project, we develop an approach to the synchronization of nonlinearly coupled systems with delays. Under this approach, the connection matrix could be quite general, and the condition on the slope of the coupling functions, which is commonly imposed on the coupling functions in the existing approaches, is not necessary

英 文 關 鍵 詞 ： Synchronization, Coupled system, Connection matrix, Coupling function

前言

同步化在各種生物系統和物理系統中是關鍵和普遍的現象。因此耦合動力系統的同步現象一直是令人感興趣的研究課題[1-8]。例如, 耦合混沌系統的同步在數學、物理和工程領域一直是重要的研究課題。在這個研究計劃中，我們進行了具時間延遲之非線性藕合系統的全局同步化分析。在我們的同步化理論下，此耦合系統的耦合結構可以是非常的一般化；我們得到與延遲時間相關以及與延遲時間無關的全局同步化條件。並且，我們運用所推得的同步化理論去研究具時間延遲之S型藕合(delayed sigmoidal coupling)FitzHugh–Nagumo neurons的全局同步化。

研究目的與文獻探討

Synchronization is an important and common phenomenon in various biological and physical systems. As a result, the topic of synchronization in coupled dynamical systems has drawn a wide range of ongoing research interest [1-8]. Time delay, which occurs in the propagation of action potentials along the axon, and the transmission of signals across the synapse, is an important factor in the study of coupled neural systems [9]. Thus, such delays have been incorporated into neural network modeling [10,11,12]. Indeed, delay can modify the collective dynamics of neural networks; for example, it can induce synchronization [13] and asynchronization [14].

Some studies on the synchronization of coupled systems have focused on local synchronization, which is concerned with the stability of synchronous manifold, whereas others have studied global synchronization, by showing that all solutions converge to the synchronous manifold. The master-stability-function method, developed by Pecora and Carroll [4], is a well-known approach to the study of local synchronization in coupled chaotic systems. This method computes the eigenvalues of the connectivity matrix, and the Lyapunov exponents of the associated variational equation, in order to determine the stability of the synchronous manifold. It is well known that the largest Lyapunov exponent must be negative for local synchronization to occur. Lyapunov exponents cannot be calculated analytically, and thus this method requires the use of numerical operations [15]. Indeed, methods that rely on the manipulation of connectivity matrix eigenvalues and Lyapunov exponents, such as the master-stability-function method, may be ineffective if the coupling configuration is time-dependent, or has a time-dependent delay, as the stability theory may be invalid for the corresponding linearized system. However, time-delayed and time-dependent connections are more realistic in many real-world networks.

Methodologies for the examination of global synchronization usually involve the notion of Lyapunov functions. For example, Lyapunov's direct method has been applied for studying synchronization in networks in [16-18]. Other works employing Lyapunov functions/functionals include [10, 11, 12, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. In the existing literature, many of the systems considered for tackling synchronization problems exhibit linear diffusive couplings [12, 19, 28, 26, 20, 29]. Some papers including [11, 21, 22, 23, 24, 25, 27, 30, 31] considered nonlinear couplings under the diffusion condition under which the coupling is annihilated in the synchronous state. The existing works on global synchronization of nonlinearly coupled systems commonly impose the slope condition under which the coupling functions have positive lower bounds on their slope. Notably, there is a significant difference between diffusive coupling and general nonlinear coupling. For nonlinear and non-diffusively coupled systems incorporating delay, global synchronization under the Lyapunov function approach often reduces to the situation where every solution converges asymptotically to a unique synchronous equilibrium point [30, 31]. Typically, only delay-independent criteria can be derived under such an approach. Thus, for general nonlinear coupled systems, in particular for those with time delays, finding an effective Lyapunov function that implies synchronization with non-trivial asymptotic dynamics may be a rather challenging and limited method.

Therefore, in this project, we want to develop an approach to the synchronization of nonlinearly coupled network system with delays, under general coupling configuration.

研究方法、結果、討論與計畫成果自評

在我們之前的工作[32]中，我們發展了一個新的方法去探討耦合系統的全局同步化；而其方法的主要要求是其系統的耦合形式滿足 circular coupling 條件。在這個研究中，我們改良[32]的方法並將其運用在具時間延遲之非線性藕合系統上並無須滿足 circular coupling 條件；在這個方法之下，每一個子系統(individual subsystem)可以是非自治系統(non-autonomous)，藕合結構可以是相當地一般性，可以是非線性，與時間相關、非對稱，具時間延遲。我們運用所推得的同步化理論去研究具時間延遲之 S 型藕合(delayed sigmoidal coupling)FitzHugh–Nagumo neurons 的全局同步化。

在這個研究工作中，我們所完成的主要工作為：
An novel approach to synchronization of nonlinearly coupled systems with delay
而這份成果已完成論文撰寫並投至 Physica A，目前正在審查中。

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An novel approach to synchronization of nonlinearly coupled systems with delays

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Abstract

In this investigation, an novel approach to establish the global synchronization of coupled systems is presented. Under this approach, individual subsystems can be non-autonomous; the coupling configuration is rather general, and can be nonlinear, time-dependent, asymmetric, and time delayed. By transforming the problem of synchronizing coupled systems into one of solving corresponding linear systems of algebraic equations, delay-dependent and delay-independent criteria for global synchronization are established. We implement the present approach to nonlinearly coupled FitzHugh–Nagumo neurons under delayed sigmoidal coupling. Two numerical examples are then given to show that oscillatory behavior and multistability can emerge or be suppressed as the coupled neurons synchronize under the synchronization criterion; asynchrony induced by the coupling strength or coupling delay occurs while the coupled neurons do not satisfy the synchronization criterion.

Keywords: Coupled system, Delay, Synchronization, FitzHugh–Nagumo neuron

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1. Introduction

Synchronization is an important and common phenomenon in various biological and physical systems [1, 2]. As a result, the topic of synchronization in coupled dynamical systems has drawn a wide range of ongoing research interest [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 29]. Time delay, which occurs in the propagation of action potentials along the axon, and the transmission of signals across the synapse, is an important factor in the study of coupled neural systems [6, 31]. Thus, such delays have been incorporated into neural network modeling [8, 12, 14, 20, 30, 29, 31]. Indeed, delay can modify the collective dynamics of neural networks; for example, it can induce synchronization [15] and asynchronization [21, 32].

Some studies on the synchronization of coupled systems have focused on local synchronization, which is concerned with the stability of synchronous manifold, whereas others have studied global synchronization, by showing that all solutions converge to the synchronous manifold. The *master-stability-function method*, developed by Pecora and Carroll [17], is a well-known approach to the study of local synchronization in coupled chaotic systems. This method computes the eigenvalues of the connectivity matrix, and the Lyapunov exponents of the associated variational equation, in order to determine the stability of the synchronous manifold. It is well known that the largest Lyapunov exponent must be negative for local synchronization to occur. Lyapunov exponents cannot be calculated analytically, and thus this method requires the use of numerical operations [9]. Indeed, methods that rely on the manipulation of connectivity matrix eigenvalues and Lyapunov exponents, such as the *master-stability-function method*, may be ineffective if the coupling configuration is time-dependent, or has a time-dependent delay, as the stability theory may be invalid for the corresponding linearized system. However, time-delayed and time-dependent connections are more realistic in many real-world networks.

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Methodologies for the examination of global synchronization usually involve the notion of Lyapunov functions. For example, Lyapunov's direct method has been applied for studying synchronization in networks in [24, 25, 26]. Other works employing Lyapunov functions/functionals include [3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19, 20, 22, 28, 29, 30]. In the existing literature, many of the systems considered for tackling synchronization problems exhibit linear diffusive couplings [3, 4, 14, 17, 18, 20, 22, 23, 27, 30]. Some papers including [5, 7, 10, 11, 12, 13, 19] considered nonlinear couplings under the diffusion condition under which the coupling is annihilated in the synchronous state. The existing works on global synchronization of nonlinearly coupled systems commonly impose the slope condition under which the coupling functions have positive lower bounds on their slope. Notably, there is a significant difference between diffusive coupling and general nonlinear coupling. For nonlinear and non-diffusively coupled systems incorporating delay, global synchronization under the Lyapunov function approach often reduces to the situation where every solution converges asymptotically to a unique synchronous equilibrium point [28, 29, 31, 32]. Typically, only delay-independent criteria can be derived under such an approach. Thus, for general nonlinear coupled systems, in particular for those with time delays, finding an effective Lyapunov function that implies synchronization with nontrivial asymptotic dynamics may be a rather challenging and limited method. In this investigation, we consider the following delayed coupled system:

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t), t) + c \sum_{j\in\mathcal{N}} a_{ij}(t)\mathbf{G}(\mathbf{x}_j(t-\tau(t))),\ i \in \mathcal{N},\ t \geq t_0, \tag{1.1}$$

where $\mathcal{N} := \{1, \ldots, N\}$, $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t)) \in \mathbb{R}^K$, $\mathbf{F} = (F_1, \ldots, F_K)$ is a smooth function, describing the intrinsic dynamics of each subsystem, $c \geq 0$ is the coupling strength, and $a_{ij}(t)$, $i, j \in \mathcal{N}$, are bounded functions of t, satisfying the following condition:

$$\sum_{j\in\mathcal{N}} a_{ij}(t) = \kappa(t), \quad \text{for all } i \in \mathcal{N} \text{ and } t \geq t_0. \tag{1.2}$$

The matrix $A(t) := [a_{ij}(t)]_{1\leq i,j\leq N}$ refers to the connection matrix. The function $\mathbf{G} = (G_1, \ldots, G_K)$ satisfies

$$G_k(\mathbf{x}_j(t-\tau(t))) = g_k(x_{j,k}(t-\tau(t))), \quad \text{for all } i, j \in \mathcal{K} \text{ and } t \geq t_0 \tag{1.3}$$

where $\mathcal{K} := \{1, \ldots, K\}$, g_k is a nondecreasing and differentiable function, and $\tau(t) \in [0, \tau_M]$ stands for the time-dependent transmission delay. For later use, we set

$$\bar{\kappa} = \sup\{|\kappa(t)| : t \geq t_0\},\ \check{\kappa} = \inf\{\kappa(t) : t \geq t_0\},\ \hat{\kappa} = \sup\{\kappa(t) : t \geq t_0\}, \tag{1.4}$$

$$\bar{a}_{ij} = \sup\{|a_{ij}(t)| : t \geq t_0\},\ \bar{a} = \max\{\sum_{j\in\mathcal{N}} \bar{a}_{ij} : i = 1, \ldots, N\},\ \bar{\tau} = \sup\{\tau(t) : t \geq t_0\}. \tag{1.5}$$

System (1.1) is a nonlinearly coupled system if g_k is a nonlinear function for some $k \in \mathcal{K}$; otherwise, it is a linearly coupled system. Systems of neural network and neuronal network in the literature largely admit the form of (1.1) or its similar forms; see [3, 4, 14, 22, 24, 26, 30, 23, 27] for linear coupling case, and [5, 7, 10, 11, 12, 13, 19] for nonlinear coupling case. In the following, $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$ denotes an arbitrary solution of system (1.1), and $(\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t)$ is the corresponding evolution of system (1.1), where $\mathbf{x}_i^t \in C([-\tau_M, 0]; \mathbb{R}^K)$, $i \in \mathcal{N}$, are defined as $\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t+\theta)$ for $\theta \in [-\tau_M, 0]$. System (1.1) is said to attain global (identical) synchronization, if

$$x_{i,k}(t) - x_{j,k}(t) \to 0, \text{ as } t \to \infty, \text{ for all } i, j \in \mathcal{N},\ k \in \mathcal{K},$$

for every solution $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$, where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))$. Recently, the idea of *sequential contracting* has been developed in [21], to establish the global synchronization of coupled systems under circulant coupling. The approach in [21] can apply to system (1.1) with the connection matrix $A(t)$ which is a circulant matrix, this is

$$A(t) = [a_{ij}(t)]_{1\leq i,j\leq N} = \text{circ}(a_1(t), \ldots, a_N(t)), \tag{1.6}$$

for some $a_i(t)$, $i \in \mathcal{N}$. Based on this approach, the authors in [21] established the global synchronization of two FitzHugh–Nagumo neurons under delayed sigmoidal coupling, which gives an analytical support to the numerical finding in [33]. They also considered the global synchronization of an neural network that consists of a ring of K loops, coupled with their nearest neighbors under delayed sigmoidal coupling. The couplings for these two coupled systems considered in [21] do not satisfy the diffusive condition, and do not satisfy the slope condition commonly imposed in the literature; however, their corresponding connection matrices must satisfy condition (1.6). A more detailed discussion of the sequential contracting approach and the other existing synchronization approaches can be found in [21]. In this investigation, we extend the approach developed in [21] to the coupled system (1.1), under a more general coupling configuration; namely, condition (1.6) is not required for our present approach in this paper.

The remainder of this paper is organized as follows. In Section 2, we establish the synchronization of system (1.1). In Section 3, we implement the synchronization criterion derived in Section 2 to coupled FitzHugh–Nagumo neurons under delayed sigmoidal coupling.

2. Synchronization of system (1.1)

In this section, we study the global synchronization of system (1.1), under two assumptions to be introduced in Subsection 2.1; delay-dependent and delay-independent synchrponization criteria are then derived in Subsection 2.2.

2.1. Model assumptions

The first assumption imposed on system (1.1) is as follows.

Assumption (D): All solutions of system (1.1) eventually enter and then remain in some compact set $Q^N := Q \times \cdots \times Q \subset \mathbb{R}^{NK}$, where $Q := [\check{q}_1, \hat{q}_1] \times \cdots \times [\check{q}_K, \hat{q}_K] \subset \mathbb{R}^K$.

Using the set introduced in assumption (D), which we denote as Q, we can define the following set:

$$C_Q := \{(\Phi_1, \ldots, \Phi_N) : \Phi_i = (\phi_{i,1}, \ldots, \phi_{i,K}) \in C([-\tau_M, 0]; \mathbb{R}^K), \phi_{i,k}(\theta) \in [\check{q}_k, \hat{q}_k], \theta \in [-\tau_M, 0], i \in \mathcal{N},\ k \in \mathcal{K}\}. \quad (2.1)$$

Remark 2.1. *(i) The dissipative property, such as assumption (D) for system* (1.1)*, is a basic requirement in studying the synchronization of coupled systems, under which all solutions of the system exist on* $[0, \infty)$*. There is no general methodology to justify this property for a nonlinear system, and concluding such a property is usually case-dependent. One may examine the dissipative property for linearly coupled systems by applying the approaches in [18, 20, 22]. (ii) Any evolution* $(\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t)$ *of system* (1.1) *eventually enter and then remain in* C_Q *under assumption (D).*

To introduce the second assumption, we first consider the configuration of F_k, $k \in \mathcal{K}$, in system (1.1). For each $k \in \mathcal{K}$, we decompose $F_k(\Phi, t) - F_k(\tilde{\Phi}, t)$ as follows:

$$F_k(\Xi, t) - F_k(\tilde{\Xi}, t) = h_k(\xi_k, \tilde{\xi}_k, t) + w_k(\Xi, \tilde{\Xi}, t), \quad (2.2)$$

where $t \geq t_0$, $\Xi = (\xi_1, \ldots, \xi_K)$, and $\tilde{\Xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_K)$. Such a decomposition in (2.2) is always achievable, because the trivial decomposition selects $h_k \equiv 0$. A nontrivial decomposition for the FitzHugh–Nagumo neuron is illustrated in Proposition 3.2. Now, let us introduce the following assumption on h_k and w_k, for $k \in \mathcal{K}$:

Assumption (F): For each $k \in \mathcal{K}$, there exist $\check{\mu}_k, \hat{\mu}_k \in \mathbb{R}$, $\rho_k^w \geq 0$, and $\bar{\mu}_{kl} \geq 0$, for $l \in \mathcal{K} - \{k\}$, such that for any $\Xi = (\xi_1, \ldots, \xi_K), \tilde{\Xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_K) \in Q$, the following two properties hold for all $t \geq t_0$:

(F-i): $\begin{cases} \check{\mu}_k \leq h_k(\xi_k, \tilde{\xi}_k, t)/(\xi_k - \tilde{\xi}_k) \leq \hat{\mu}_k, & \xi_k - \tilde{\xi}_k \neq 0, \\ h_k(\xi_k, \tilde{\xi}_k, t) = 0, & \xi_k - \tilde{\xi}_k = 0, \end{cases}$

(F-ii): $|w_k(\Xi, \tilde{\Xi}, t)| \leq \rho_k^w$, and $|w_k(\Xi, \tilde{\Xi}, t)| \leq \sum_{l \in \mathcal{K} - \{k\}} \bar{\mu}_{kl} |\xi_l - \tilde{\xi}_l|$.

Actually, assumption (F) is commonly satisfied, and $\check{\mu}_k$, $\hat{\mu}_k$, ρ_k^w, and $\bar{\mu}_{kl}$ can be determined by applying the mean-value theorem, provided that $\mathbf{F} = (F_1, \ldots, F_K)$ is sufficiently smooth, and the set Q is given under assumption (D).

2.2. Synchronization criteria

Let us define the following sets of indices:

$$\mathcal{A} := (\mathcal{N} - \{N\}) \times \mathcal{K} \text{ and } \mathcal{A}_{i,k} := \mathcal{A} - \{i\} \times \{k\}, \text{ where } (i, k) \in \mathcal{A}. \quad (2.3)$$

Assume that $(\mathbf{x}_1(t),\ldots,\mathbf{x}_N(t))$, where $\mathbf{x}_i(t) = (x_{i,1}(t),\ldots,x_{i,K}(t))$, is an arbitrary solution of system (1.1). Setting $\mathbf{z}_i(t) = (z_{i,1}(t),\ldots,z_{i,K}(t)) := \mathbf{x}_i(t)-\mathbf{x}_{i+1}(t)$, for $i \in \mathcal{N}-\{N\}$, as seen from (1.1) and (1.3), we have that $(\mathbf{z}_1(t),\ldots,\mathbf{z}_{N-1}(t))$ satisfies the following difference-differential system corresponding to system (1.1):

$$\dot{z}_{i,k}(t) = H_{i,k}(\mathbf{x}_1^t,\ldots,\mathbf{x}_N^t,t),\ (i,k)\in\mathcal{A},\ t\geq t_0, \tag{2.4}$$

where

$$H_{i,k}(\Phi_1,\ldots,\Phi_N,t) := F_k(\Phi_i(0),t) - F_k(\Phi_{i+1}(0),t) + c\sum_{j\in\mathcal{N}}[a_{ij}(t)-a_{(i+1)j}(t)]g_k(\phi_{j,k}(-\tau(t))), \tag{2.5}$$

for $\Phi_j = (\phi_{j,1},\ldots,\phi_{j,K}) \in C([-\tau_M,0];\mathbb{R}^K)$, $j\in\mathcal{N}$. Herein, the roles of $\Phi_j, j\in\mathcal{N}$ are discussed in Remark **??**. Clearly, system (1.1) attains global synchronization if $z_{i,k}(t)\to 0$, as $t\to\infty$, for every $(i,k)\in\mathcal{A}$.

Via $A(t) = [a_{ij}(t)]_{1\leq i,j\leq N}$, we first define $\tilde{A}(t) = [\tilde{a}_{ij}(t)]_{1\leq i,j\leq N}$, where

$$\tilde{a}_{ij}(t) = \begin{cases} a_{ii}(t)-\kappa(t), & \text{if } i=j\in\mathcal{N},\\ a_{ij}(t), & \text{if } i,j\in\mathcal{N} \text{ and } i\neq j.\end{cases} \tag{2.6}$$

We then further define the following matrix $\bar{A}(t)$ derived from $A(t)$, given by

$$\bar{A}(t) = [\alpha_{ij}(t)]_{1\leq i,j\leq (N-1)} := \mathbf{C}\tilde{A}(t)\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} \in \mathbb{R}^{(N-1)\times(N-1)}, \tag{2.7}$$

where

$$\mathbf{C} := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0\\ 0 & 1 & -1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1 & -1\end{pmatrix} \in \mathbb{R}^{(N-1)\times N}.$$

Applying arguments parallel to those for the appendix of [16] shows that $\bar{A}(t)$ in (2.7) is well-defined, and satisfies

$$\mathbf{C}A(t) = \bar{A}(t)\mathbf{C}, \tag{2.8}$$

for all $t\geq t_0$. For later use, we set

$$\bar{\alpha}_{ij} = \sup\{|\alpha_{ij}(t)| : t\geq t_0\},\ \check{\alpha}_{ij} = \inf\{\alpha_{ij}(t) : t\geq t_0\},\ \hat{\alpha}_{ij} = \sup\{\alpha_{ij}(t) : t\geq t_0\}, \tag{2.9}$$

where $\alpha_{ij}(t)$, $1\leq i,j\leq N-1$, are entries of $\bar{A}(t)$, defined in (2.7). Based on (2.8), we derive the following proposition.

Proposition 2.1. *Consider system* (1.1) *which satisfies assumptions (D) and (F). Then, functions $H_{i,k}$ defined in* (2.5)*, $(i,k)\in\mathcal{A}$, can be decomposed as*

$$\begin{aligned} H_{i,k}(\Phi_1,\ldots,\Phi_N,t) &= h_{i,k}(\phi_{i,k}(0),\phi_{i+1,k}(0),t) + \tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t) + w_{i,k}(\Phi_1,\ldots,\Phi_N,t), && (2.10)\\ h_{i,k}(\phi_{i,k}(0),\phi_{i+1,k}(0),t) &= h_k(\phi_{i,k}(0),\phi_{i+1,k}(0),t), && (2.11)\\ \tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t) &= c[\kappa(t)+\alpha_{ii}(t)][g_k(\phi_{i,k}(-\tau(t))) - g_k(\phi_{i+1,k}(-\tau(t))], && (2.12)\\ w_{i,k}(\Phi_1,\ldots,\Phi_N,t) &= w_k(\Phi_i(0),\Phi_{i+1}(0),t) + c\sum_{j\in\mathcal{N}-\{i,N\}}\alpha_{ij}(t)[g_k(\phi_{j,k}(-\tau(t))) - g_k(\phi_{j+1,k}(-\tau(t)))]. && (2.13)\end{aligned}$$

Moreover, for all $(i,k)\in\mathcal{A}$ and all $(\Phi_1,\ldots,\Phi_N)\in C_Q$, where $\Phi_i = (\phi_{i,1},\ldots,\phi_{i,K})$, $i\in\mathcal{N}$, the following three properties hold for all $t\geq t_0$:

(H-i): $\begin{cases} \check{\mu}_k \leq h_{i,k}(\phi_{i,k}(0),\phi_{i+1,k}(0),t)/[\phi_{i,k}(0)-\phi_{i+1,k}(0)] \leq \hat{\mu}_k, & \phi_{i,k}(0)-\phi_{i+1,k}(0)\neq 0,\\ h_{i,k}(\phi_{i,k}(0),\phi_{i+1,k}(0),t) = 0, & \phi_{i,k}(0)-\phi_{i+1,k}(0) = 0,\end{cases}$

(H-ii): $|\tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t)|\leq\rho^h_{ik}$*, and*

$$\begin{cases} \check{\beta}_{ik} \leq \tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t)/[\phi_{i,k}(-\tau(t))-\phi_{i+1,k}(-\tau(t))] \leq \hat{\beta}_{ik}, & \phi_{i,k}(-\tau(t))-\phi_{i+1,k}(-\tau(t))\neq 0,\\ \tilde{h}_{i,k}(\phi_{i,k},\phi_{i+1,k},t) = 0, & \phi_{i,k}(-\tau(t))-\phi_{i+1,k}(-\tau(t)) = 0,\end{cases}$$

(H-iii): $|w_{i,k}(\Phi_1,\ldots,\Phi_N,t)| \le \rho^w_{ik}$, *and*

$$|w_{i,k}(\Phi_1,\ldots,\Phi_N,t)| \le \sum_{(j,l)\in\mathcal{A}_{i,k}} \{\bar{\mu}^{(jl)}_{ik}|\phi_{j,l}(0)-\phi_{j+1,l}(0)| + \bar{\beta}^{(jl)}_{ik}|\phi_{j,l}(-\tau(t))-\phi_{j+1,l}(-\tau(t))|\},$$

for arbitrary ρ^h_{ik} *and* ρ^w_{ik} *which satisfy* $\rho^h_{ik} \ge 2c(\bar{\kappa}+\bar{\alpha}_{ii})\rho^g_k$ *and* $\rho^w_{ik} \ge \rho^w_k + 2c\rho^g_k \times (\sum_{j\in\mathcal{N}-\{i,N\}}\bar{\alpha}_{ij})$*, respectively, and*

$$\check{\beta}_{ik} = \begin{cases} c(\check{\kappa}+\check{\alpha}_{ii})\check{L}_k, & \check{\kappa}+\check{\alpha}_{ii}\ge 0,\\ c(\check{\kappa}+\check{\alpha}_{ii})\hat{L}_k, & \check{\kappa}+\check{\alpha}_{ii}<0,\end{cases} \quad \hat{\beta}_{ik} = \begin{cases} c(\hat{\kappa}+\hat{\alpha}_{ii})\hat{L}_k, & \hat{\kappa}+\hat{\alpha}_{ii}\ge 0,\\ c(\hat{\kappa}+\hat{\alpha}_{ii})\check{L}_k, & \hat{\kappa}+\hat{\alpha}_{ii}<0,\end{cases}$$

$$\bar{\mu}^{(jl)}_{ik} = \begin{cases} \bar{\mu}_{kl}, & i=j,\ k\ne l,\\ 0, & \textit{otherwise},\end{cases} \quad \bar{\beta}^{(jl)}_{ik} = \begin{cases} c\bar{\alpha}_{ij}\hat{L}_k, & j\ne i,\ k=l,\\ 0, & \textit{otherwise},\end{cases}$$

where

$$\rho^g_k := \max\{|g_k(\xi)| : \xi\in[\check{q}_k,\hat{q}_k]\}\ge 0,\ \check{L}_k := \min\{g'_k(\xi) : \xi\in[\check{q}_k,\hat{q}_k]\}\ge 0,\ \hat{L}_k := \max\{g'_k(\xi) : \xi\in[\check{q}_k,\hat{q}_k]\}\ge 0.$$

Herein, $\bar{\kappa}$*,* $\check{\kappa}$*, and* $\hat{\kappa}$ *are defined in* (1.4)*,* C_Q *is defined in* (2.1)*,* $\mathcal{A}_{i,k}$ *is defined in* (2.3)*, functions* h_k *and* w_k *are defined in* (2.2)*,* $\bar{\alpha}_{ij}$*,* $\check{\alpha}_{ij}$*, and* $\hat{\alpha}_{ij}$ *are defined in* (2.9)*, and* $\check{\mu}_k$*,* $\hat{\mu}_k$*,* ρ^w_k*, and* $\bar{\mu}_{kl}$ *are defined in assumption (F).*

Using $\check{\mu}_k, \hat{\mu}_k, \check{\beta}_{ik}, \hat{\beta}_{ik}, \bar{\mu}^{(jl)}_{ik}$, and $\bar{\beta}^{(jl)}_{ik}$, introduced in Proposition 2.1, and $\bar{\tau}$, defined in (1.5), we define

$$\eta_{ik} := -\hat{\mu}_k - \hat{\beta}_{ik} + \bar{\beta}_{ik}\bar{\tau}(\check{\mu}_k+\hat{\mu}_k+\check{\beta}_{ik}+\hat{\beta}_{ik}),\ \tilde{\eta}_{ik} := -\hat{\mu}_k - \bar{\beta}_{ik},\ \bar{L}^{(jl)}_{ik} := \bar{\mu}^{(jl)}_{ik} + \bar{\beta}^{(jl)}_{ik}, \tag{2.14}$$

where

$$\bar{\beta}_{ik} := \max\{|\check{\beta}_{ik}|, |\hat{\beta}_{ik}|\}. \tag{2.15}$$

The parameter η_{ik} can be regarded as delay-dependent, as it is related to $\bar{\tau}$, defined in (1.5), whereas the parameters $\tilde{\eta}_{ik}$ and $\bar{L}^{(jl)}_{ik}$ are delay-independent. Via η_{ik}, $\tilde{\eta}_{ik}$, and $\bar{L}^{(jl)}_{ik}$, defined in (2.14), we define two matrices that are delay-dependent and delay-independent, respectively, to obtain the delay-dependent and delay-independent synchronization criteria for system (1.1) (see Theorems 2.1 and 2.2). The delay-dependent matrix is defined as follows:

$$\mathbf{M} := D_{\mathbf{M}} - L_{\mathbf{M}} - U_{\mathbf{M}} = [M^{(kl)}]_{1\le k,l\le K}, \tag{2.16}$$

where $D_{\mathbf{M}}$, $-L_{\mathbf{M}}$, and $-U_{\mathbf{M}}$ represent the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\mathbf{M}$, respectively; $M^{(kl)} = [m^{(kl)}_{ij}]_{1\le i,j\le N-1}$, $k,l\in\mathcal{K}$, are blocks of $(N-1)\times(N-1)$ matrices, defined by

$$m^{(kl)}_{ij} = \begin{cases} \eta_{ik}, & \text{if } i=j\in\mathcal{N}-\{N\} \text{ and } k=l\in\mathcal{K},\\ -\bar{L}^{(jl)}_{ik}, & \text{otherwise}.\end{cases}$$

Notably, matrix $\mathbf{M}$ in (2.16) is delay-dependent, because the diagonal entries η_{ik} are delay-dependent. The proof of Theorem 2.1 explains the formulation of matrix $\mathbf{M}$. We now introduce the delay-independent matrix:

$$\tilde{\mathbf{M}} := D_{\tilde{\mathbf{M}}} - L_{\tilde{\mathbf{M}}} - U_{\tilde{\mathbf{M}}} = [\tilde{M}^{(kl)}]_{1\le k,l\le K}, \tag{2.17}$$

where $D_{\tilde{\mathbf{M}}}$, $-L_{\tilde{\mathbf{M}}}$, and $-U_{\tilde{\mathbf{M}}}$ represent the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\tilde{\mathbf{M}}$, respectively; $\tilde{M}^{(kl)} = [\tilde{m}^{(kl)}_{ij}]_{1\le i,j\le N-1}$, $k,l\in\mathcal{K}$, are blocks of $(N-1)\times(N-1)$ matrices, defined by

$$\tilde{m}^{(kl)}_{ij} = \begin{cases} \tilde{\eta}_{ik}, & \text{if } i=j\in\mathcal{N}-\{N\} \text{ and } k=l\in\mathcal{K},\\ -\bar{L}^{(jl)}_{ik}, & \text{otherwise}.\end{cases}$$

Let us now introduce two conditions needed for criteria for the synchronization of system (1.1), as follows:

Condition (S1): $\hat{\mu}_k + \hat{\beta}_{ik} < 0$ and $\bar{\beta}_{ik}\bar{\tau} < 3\rho^h_{ik}(\hat{\mu}_k+\hat{\beta}_{ik})/[(\hat{\mu}_k+\check{\mu}_k+\hat{\beta}_{ik}+\check{\beta}_{ik})(3\rho^h_{ik}+\rho^w_{ik})]$, for $(i,k)\in\mathcal{A}$,

Condition (S2): $\bar{\beta}_{ik} < -\hat{\mu}_k/(1+\rho^w_{ik}/\rho^h_{ik})$, for $(i,k)\in\mathcal{A}$.

The parameters in conditions (S1) and (S2) are defined in Proposition 2.1, (1.5), (2.14), and (2.15). Condition (S1) is related to $\bar{\tau}$, and is thus delay-dependent, whereas condition (S2) is independent of time delay. We note that each $\eta_{ik} > 0$ (resp., $\tilde{\eta}_{ik} > 0$), cf. (2.14), under condition (S1) (resp., (S2)).

Theorem 2.1. *Consider system* (1.1) *which satisfies assumptions (D) and (F). Then, the system globally synchronizes if condition (S1) holds, and the Gauss-Seidel iterations for the linear system:*

$$\mathbf{M}\mathbf{v} = \mathbf{0}, \tag{2.18}$$

converge to zero, the unique solution of (2.18)*; or equivalently,*

$$\lambda_{\text{syn}} := \max_{1\le\sigma\le K\times(N-1)} \{|\lambda_\sigma| : \lambda_\sigma : \text{ eigenvalue of } (D_{\mathbf{M}} - L_{\mathbf{M}})^{-1} U_{\mathbf{M}}\} < 1. \tag{2.19}$$

where the matrices $\mathbf{M}$*,* $D_{\mathbf{M}}$*,* $L_{\mathbf{M}}$*, and* $U_{\mathbf{M}}$ *are defined in* (2.16)*.*

Using arguments similar to those for Theorem 2.1, but using Proposition 5.2 (given in the appendix) instead of Proposition 5.1, we can derive the delay-independent criterion for the global synchronization of system (1.1).

Theorem 2.2. *Consider system* (1.1) *which satisfies assumptions (D) and (F). Then, the system globally synchronizes if condition (S2) holds, and the Gauss-Seidel iterations for the linear system:*

$$\tilde{\mathbf{M}}\mathbf{v} = \mathbf{0}, \tag{2.20}$$

converge to zero, the unique solution of (2.20)*; or equivalently,*

$$\tilde{\lambda}_{\text{syn}} := \max_{1\le\sigma\le K\times(N-1)} \{|\lambda_\sigma| : \lambda_\sigma : \text{ eigenvalue of } (D_{\tilde{\mathbf{M}}} - L_{\tilde{\mathbf{M}}})^{-1} U_{\tilde{\mathbf{M}}}\} < 1.$$

where the matrices $\tilde{\mathbf{M}}$*,* $D_{\tilde{\mathbf{M}}}$*,* $L_{\tilde{\mathbf{M}}}$*, and* $U_{\tilde{\mathbf{M}}}$ *are defined in* (2.17)*.*

3. Synchronization of coupled FitzHugh–Nagumo neurons

This section implements the present approach, developed in Section 2, to a network of FitzHugh–Nagumo neurons. The dynamics of an isolated FitzHugh–Nagumo neuron is described by the nonlinear differential equations [34]:

$$\begin{cases} \dot{u}(t) = f_u(u(t), v(t)) := -u^3(t) + (a+1)u^2(t) - au(t) - v(t), \\ \dot{v}(t) = f_v(u(t), v(t)) := bu(t) - \gamma v(t), \end{cases} \tag{3.1}$$

where $a, b, \gamma > 0$. We consider coupled FitzHugh–Nagumo neurons under delayed sigmoidal coupling, as follows:

$$\begin{cases} \dot{u}_i(t) = f_u(u_i(t), v_i(t)) + c\sum_{j\in\mathcal{N}} a_{ij}(t) g(u_j(t-\tau)), \\ \dot{v}_i(t) = f_v(u_i(t), v_i(t)), \end{cases} \tag{3.2}$$

for $t \ge t_0$ and $i \in \mathcal{N}$, where $a_{ij}(t)$ satisfies (1.2), and the coupling function g is in the following class:

$$\{g \in C^1 : g(0) = 0,\ \delta := g'(0) > g'(\xi) > 0,\ |g(\xi)| < \rho,\ \text{for } \xi \ne 0\}. \tag{3.3}$$

Moreover, we assume that system (3.2) is with $c\bar{a} > 0$, cf. (1.5); otherwise, there exists no connection between neurons in system (3.2). Notably, system (3.2) is in the form of (1.1). In referring to the notation in (1.1), $\mathbf{F} = (F_1, F_2)$ and $\mathbf{G} = (G_1, G_2)$ now satisfy

$$F_1(\Xi, t) = f_u(\xi_1, \xi_2),\ F_2(\Xi, t) = f_v(\xi_1, \xi_2),\ G_1(\Xi) = g_1(\xi_1) = g(\xi_1),\ G_2(\Xi) = g_2(\xi_2) = 0, \tag{3.4}$$

for $\Xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $t \ge t_0$, where f_u, f_v, and g are defined in (3.1) and (3.3). Now, we show that system (3.2) satisfies assumption (D) via the following iteratively constructed functions. We define, for $k \in \mathbb{N}$,

$$P^{(k)}(\xi) := -\xi^4 + (a+1)\xi^3 - a\xi^2 + c\bar{a}\rho^{(k-1)}|\xi|, \tag{3.5}$$

where $\rho^{(0)} := \rho$, and

$$\rho^{(k)} := \max\{|g(\xi)| : \xi \in [-\sqrt{\gamma^2 + b\bar{q}^{(k)}}/\gamma,\ \sqrt{\gamma^2 + b\bar{q}^{(k)}}/\gamma]\},\ \text{with } \bar{q}^{(k)} := \max\{|\xi| : P^{(k)}(\xi) = 0\}. \tag{3.6}$$

Herein, the parameters $\bar{a}$, a, b, c, γ, ρ and function g were introduced in (1.5) and (3.1)–(3.3).

Proposition 3.1. *For each $k \in \mathbb{N}$, all solutions of system* (3.2) *eventually enter and then remain in $Q^{(k)} \times \cdots \times Q^{(k)} \subset \mathbb{R}^{2N}$, where*

$$Q^{(k)} := [-\sqrt{\gamma^2 + b\bar{q}^{(k)}}/\gamma, \sqrt{\gamma^2 + b\bar{q}^{(k)}}/\gamma] \times [-b\bar{q}^{(k)}/\gamma, b\bar{q}^{(k)}/\gamma] \subset \mathbb{R}^2.$$

Throughout this section, thanks to Proposition 3.1, we consider that system (3.2) satisfies assumption (D) with

$$Q = Q^{(k)} = [-\rho_1^*, \rho_1^*] \times [\rho_2^*, \rho_2^*] =: Q^*, \text{ where } \rho_1^* := \sqrt{\gamma^2 + b\bar{q}^{(k)}}/\gamma,\ \rho_2^* := b\bar{q}^{(k)}/\gamma, \tag{3.7}$$

for some fixed $k \in \mathbb{N}$. Indeed, $\bar{q}^{(k)}$, $k \in \mathbb{N}$, are strictly decreasing with respect to k. Thus, for larger k, $Q^{(k)}$ provides a smaller attracting region for the dynamics of system (3.2), and hence relaxes the conditions in our synchronization criterion.

Proposition 3.2. *System* (3.2) *satisfies assumption (F) with $\check{\mu}_1 = -3(\rho_1^*)^2 - 2(a+1)\rho_1^* - a$, $\hat{\mu}_1 = (a^2 - a + 1)/3 > 0$, $\check{\mu}_2 = \hat{\mu}_2 = -\gamma$, $\bar{\mu}_{12} = 1$, $\bar{\mu}_{21} = b$, $\rho_1^w = 2\rho_2^*$, and $\rho_2^w = 2b\rho_1^*$, where ρ_i^*, $i = 1, 2$, are defined in* (3.7).

Next, we derive delay-dependent synchronization criterion for system (3.2) based on Propositions 3.1 and 3.2, and Theorem 2.1.

Theorem 3.1. *System* (3.2) *achieves global synchronization, if*

$$(a^2 - a + 1)/3 + c(\hat{\kappa} + \hat{\alpha}_{ii})\tilde{L} < 0 \text{ and } \tau < \tau_i^*, \text{ for } i = 1, \ldots, N-1, \tag{3.8}$$

and the Gauss-Seidel iterations for the linear system (2.18) *converge to zero, the unique solution of* (2.18)*, where the entries of matrix* **M** *in* (2.18)*, cf.* (2.16)*, are defined by*

$$m_{ij}^{(kl)} = \begin{cases} -(a^2 - a + 1)/3 - c(\hat{\kappa} + \hat{\alpha}_{ii})\tilde{L} - \tau\tilde{\beta}_i, & \text{if } i = j,\ (k,l) = (1,1), \\ \gamma, & \text{if } i = j,\ (k,l) = (2,2), \\ -\tilde{\mu}_{ik}^{(jl)} - \tilde{\beta}_{ik}^{(jl)}, & \text{otherwise}, \end{cases} \tag{3.9}$$

with

$$\tilde{\mu}_{ik}^{(jl)} := \begin{cases} 1, & i = j,\ (k,l) = (1,2), \\ b, & i = j,\ (k,l) = (2,1), \\ 0, & \text{otherwise}, \end{cases} \quad \tilde{\beta}_{ik}^{(jl)} := \begin{cases} \beta_{ij}^*, & j \neq i,\ k = l = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Herein,

$$\tilde{L} := \min\{g'(\xi) : \xi \in [-\rho_1^*, \rho_1^*]\},\ \beta_{ij}^* := c\bar{\alpha}_{ij}\delta,\ \tilde{\rho}_i^h := 2c(\bar{\kappa} + \bar{\alpha}_{ii})\max\{|g(\xi)| : \xi \in [-\rho_1^*, \rho_1^*]\}, \tag{3.10}$$

$$\tilde{\rho}_i^w := 2\rho_2^* + 2c(\sum_{j \in \mathcal{N} - \{i,N\}} \bar{\alpha}_{ij})\max\{|g(\xi)| : \xi \in [-\rho_1^*, \rho_1^*]\}, \tag{3.11}$$

$$\tilde{\beta}_i := c(\check{\kappa} + \check{\alpha}_{ii})\delta\{-3(\rho_1^*)^2 - 2(a+1)\rho_1^* - a + (a^2 - a + 1)/3 + c[(\hat{\kappa} + \hat{\alpha}_{ii})\tilde{L} + (\check{\kappa} + \check{\alpha}_{ii})\delta]\}, \tag{3.12}$$

$$\tau_i^* := -3\tilde{\rho}_i^h[(a^2 - a + 1)/3 + c(\hat{\kappa} + \hat{\alpha}_{ii})\tilde{L}]/[\tilde{\beta}_i(3\tilde{\rho}_i^h + \tilde{\rho}_i^w)], \tag{3.13}$$

ρ_i^* is defined in (3.7)*, and $\check{\kappa}$, $\hat{\kappa}$, $\bar{\kappa}$, $\bar{\alpha}_{ij}$, $\check{\alpha}_{ii}$, and $\hat{\alpha}_{ii}$ are defined in* (1.4) *and* (2.9)*, respectively.*

4. Conclusion

Over the past few decades, synchronization in coupled systems has been a subject of intensive research. However, the existing analytical tools for studying the global synchronization of coupled systems with nonlinear or delayed coupling are rather limited. There are still synchronization problems for some coupled systems which cannot be solved with the use of existing methodologies. Moreover, delay-dependent synchronization criteria are rare in coupled systems under delayed and nonlinear coupling in the literature. Without relying on the existence of certain Lyapunov functions, our approach in this investigation has provided a new and effective alternative for the examination of global synchronization of coupled systems. We present such an approach by connsidering coupled systems (1.1) under assumptions (D) and (F). The connection matrix $A(t)$ of system (1.1) could be time-dependent, asymmetric, and with negative off-diagonal entries; hence can be free from the commonly imposed conditions in the existing works, cf. Remark **??**. The unique condition imposed on $A(t)$ is condition (1.2), under which every solution of system (1.1) that has evolved from the synchronous set will remain on the set; it is indeed a basic condition for studying the identical synchronization of coupled systems. The diffusive condition, such as (1.2) with $\kappa(t) \equiv 0$ for system (1.1), is not imposed on $A(t)$, alothough system (1.1) can be recast into a system satisfying the diffusive condition, as follows:

$$\dot{\mathbf{x}}_i(t) = \tilde{\mathbf{F}}_c(\mathbf{x}_i^t, t) + c \sum_{j \in \mathcal{N}} \tilde{a}_{ij}(t)\mathbf{G}(\mathbf{x}_j(t - \tau(t))),$$

where $\tilde{a}_{ij}(t)$ is defined in (2.6) and $\tilde{\mathbf{F}}_c(\mathbf{x}_i^t, t) := \mathbf{F}(\mathbf{x}_i(t), t) + c\kappa(t)\mathbf{G}(\mathbf{x}_i(t - \tau(t)))$; notably, $\sum_{j\in\mathcal{N}} \tilde{a}_{ij}(t) = 0$, $t \geq t_0$. For such a recast system, function $\tilde{\mathbf{F}}_c$ now involves some coupling terms, including the coupling strength c and the coupling delay $\tau(t)$, and it may exhibit much more complicated property than function $\mathbf{F}$. It should be noticed that, a synchronization theory should depend on both the properties of the intrinsic dynamics of each subsystem and the coupling terms. We implement the present approach to coupled FitzHugh–Nagumo neurons under delayed sigmoidal coupling.

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5. Appendix. Scalar delay-differential equation.

We denote by t_0 the initial time and by $\tau_M > 0$ the upper bound of delay magnitude. Let $w(t)$ be a bounded continuous function for $t \geq t_0$, and let $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : C([-\tau_M, 0]; \mathbb{R}) \times C([-\tau_M, 0]; \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Let $x_t, y_t \in C([-\tau_M, 0]; \mathbb{R})$ for $t \geq t_0$, and set $x(t + \theta) = x_t(\theta)$, $y(t + \theta) = y_t(\theta)$ for $\theta \in [-\tau_M, 0]$. We assume that $x(t)$ and $y(t)$ eventually enter and then remain in some compact interval $[\check{q}, \hat{q}]$; namely, $x(t)$ and $y(t)$ lie in $[\check{q}, \hat{q}]$ for all $t \geq \tilde{t}_0$, for some $\tilde{t}_0 \geq t_0$. We suppose that $z(t) = x(t) - y(t)$ satisfies the following scalar equation:

$$\dot{z}(t) = h(x(t), y(t), t) + \tilde{h}(x_t, y_t, t) + w(t), \; t \geq t_0. \tag{5.14}$$

Set $|w|^{\max}(T) := \sup\{|w(t)| : t \geq T\}$ and $|w|^{\max}(\infty) := \lim_{T\to\infty} |w|^{\max}(T)$. Then, we introduce the following conditions:

Condition (H$_0$): There exist $\hat{\mu}, \check{\mu}, \check{\beta}, \hat{\beta} \in \mathbb{R}$, $\rho^h > 0$, and $0 \leq \bar{\tau} \leq \tau_M$, such that for each $\phi, \psi \in \{\varphi \in C([-\tau_M, 0]; \mathbb{R}) : \varphi(\theta) \in [\check{q}, \hat{q}], \theta \in [-\tau_M, 0]\}$, the following properties hold for all $t \geq t_0$:

(H$_0$-i): $\begin{cases} \check{\mu} \leq h(\phi(0), \psi(0), t)/[\phi(0) - \psi(0)] \leq \hat{\mu}, & \phi(0) - \psi(0) \neq 0, \\ h(\phi(0), \psi(0), t) = 0, & \phi(0) - \psi(0) = 0, \end{cases}$

(H$_0$-ii): $|\tilde{h}(\phi, \psi, t)| \leq \rho^h$, and there exists a $\tau = \tau(\phi, \psi, t) \in [0, \bar{\tau}]$, such that

$$\begin{cases} \check{\beta} \leq \tilde{h}(\phi, \psi, t)/[\phi(-\tau) - \psi(-\tau)] \leq \hat{\beta}, & \phi(-\tau) - \psi(-\tau) \neq 0, \\ \tilde{h}(\phi, \psi, t) = 0, & \phi(-\tau) - \psi(-\tau) = 0. \end{cases}$$

Condition (A1): $\hat{\mu} + \hat{\beta} < 0$ and $\bar{\beta}\bar{\tau} < 3\rho^h(\hat{\mu} + \hat{\beta})/[(\hat{\mu} + \check{\mu} + \hat{\beta} + \check{\beta})(3\rho^h + |w|^{\max}(\tilde{t}_0))]$, where $\bar{\beta} := \max\{|\check{\beta}|, |\hat{\beta}|\}$.

Condition (A2): $0 \leq \bar{\beta} < -\hat{\mu}/[1 + |w|^{\max}(\tilde{t}_0)/\rho^h]$.

Proposition 5.1. *If $z(t)$ satisfies* (5.14)*, then $z(t)$ converges to interval $[-\bar{m}, \bar{m}]$ as $t \to \infty$, under conditions (H$_0$) and (A1). Moreover,*

$$0 \leq \bar{m} \leq |w|^{\max}(\infty)/[-\hat{\mu} - \hat{\beta} + \bar{\beta}\bar{\tau}(\check{\mu} + \hat{\mu} + \check{\beta} + \hat{\beta})].$$

Proposition 5.2. *If $z(t)$ satisfies* (5.14)*, then $z(t)$ converges to interval $[-\tilde{m}, \tilde{m}]$ as $t \to \infty$, under conditions (H$_0$) and (A2). Moreover,*

$$0 \leq \tilde{m} \leq |w|^{\max}(\infty)/(-\hat{\mu} - \bar{\beta}).$$

Propositions 5.1 and 5.2 can be derived by arguments parallel to those for Propositions 2.3 and 2.4 in [21], respectively. The assertion in Proposition 2.3 (resp., 2.4) in [21] uses t_0 instead of $\tilde{t}_0$ in condition (A1) (resp., (A2)). From the arguments for Proposition 2.3 (resp., 2.4) in [21], it can seen that t_0 in condition (A1) (resp., (A2)) for Proposition 2.3 (resp., 2.4) in [21] can be replaced by $\tilde{t}_0$ to weaken the condition, which then implies Proposition 5.1 (resp., 5.2).

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科技部補助專題研究計畫出席國際學術會議心得報告

日期：104 年 11 月 4 日

計畫編號	MOST 103-2115-M-004 -001 -		
計畫名稱	一類耦合系統的同步行為		
出國人員姓名	曾睿彬	服務機構及職稱	國立政治大學應用數學系， 助理教授
會議時間	2015 年 8 月 10 日 至 2015 年 8 月 13 日	會議地點	中國大陸，北京
會議名稱	(中文) (英文) The 8th International Congress on Industrial and Applied Mathematics		
發表題目	(中文) (英文) A novel synchronization approach of coupled systems and its applications		

参加會議經過

此次参加的會議是於中國大陸北京所舉辦的 The 8th International Congress on Industrial and Applied Mathematics (ICIAM2015)（期間：2015,8/10-8/13）。The International Congress on Industrial and Applied Mathematics(ICIAM)是應用數學領域的重要國際會議，在 International Council for Industrial and Applied Mathematics 的主持之下每四年舉行一次。此會議所包含的議題相當的廣泛；共分成 28 個 sections；分別為 A1: Linear Algebra, A2: Real and Complex Analysis, A3: Ordinary Differential Equations, A4: Partial Differential Equations, A5: Discrete Mathematics, A6: Numerical Analysis, A7: Computational Science, A8: Computer Science, A9: Probability and Statistics, A10: Control and Systems Theory, A11: Optimization and Operations Research, A12: Information, Communication, Signals, A13: Applied, Algebraic, and Computational Geometry, A14: Imaging Science, A15: Fluids, 16: Physics and Statistical Mechanics, A17: Geophysical, Atmospheric & Oceanographic Science, A18: Chemistry, Chemical Engineering, A19: Life Science and Medicine, A20: Social Science, A21: Finance and Management Science, A22: Education in the Mathematical and Computational, A23: Science, Simulation and Modeling, A24: Materials Science and Solid Mechanics, A25: Applications of the Mathematical and Computational Sciences in Industry, A26: Dynamical Systems and Nonlinear Analysis, A27: Other Mathematical Topics and their Applications, A28: General。 而大會議程主要分為 Prize lectures, Public Lectures, Invited lectures, Minisymposia, Iudustrial Minisymposia, Contributed Papers, Poster sessions, satellite meetings, Embedded Meeting。在這次的會議，我所参加的部分是 Dynamical Systems and Nonlinear Analysis 這個 section 的 Poster sessions，我報告的題目為 A novel synchronization approach of coupled systems and its applications，而 Poster standing 的時間是安排於會議的第二天(8/11)中午 12:00-13:00。在會議的海報時間之外，依時間安排我也聽了數個較感興趣的演講；其中，有幾個演講是有關於同步化的研究，算是與我目前的研究主題較相關的演講，而這些演講的內容也令我收獲良多。

計畫主持人參加會議之照片

與會心得:

這個會議算是相當地大型的會議，所討論的議題相當的廣泛。我所聽的演講大多是我較熟悉的 Ordinary Differential Equations 或是Dynamical Systems and Nonlinear Analysis這些與我的研究領域較接進的section的演講，不過有許多的演講都為我不熟悉的研究題目，即使是我較熟悉的同步化問題，我也有聽到了一些我之前較沒有接觸到的研究問題。因此藉這次參加這個會議的機會，我接觸到許多不同的研究課題，增廣了的見聞，對我長期的研究發展有相當的幫助。這次參加會議與我同行的還有台灣師大數學系陳賢修教授與高雄師大數學系的李俊憲教授。他們所報告的的題目分別為Travelling Waves in A Continuum Coupled Hindmarsh-Rose Type Model與Dynamics of A Network-based SIS Epidemic Model with Nonmonotone Incidence Rate, 我也用藉這次參加會議的機會與這兩位教授交流討論，也對彼此的研究工作有進一步的了解。

發表論文全文或摘要:

The investigation presents a novel approach to establish the global synchronization of coupled systems of differential equations. Under this approach, the problem of synchronizing coupled systems is transformed into one of solving corresponding linear systems of algebraic equations; moreover, the coupling configuration of the coupled systems can be quite general. The framework established in this investigation can accommodate a wide range of coupled systems, such as chaotic oscillators, neuronal models, and neural networks.

建議:

非常感謝科技部給我研究計劃補助，讓我有機會可以參加這個國際間相當知名的應用數學領域的會議。藉這次參加會議，我接觸、認識了一些學者，也對他們的研究有初步的了解，也接觸到一些不同的研究課題，增廣自已的見聞，覺得自己又學到了許多東西，收益良多。在參加這個大型的會議的過程，我可以感受到國際學術之間的交流的力量與重要性。很希望貴部未來持

續多補助國內學者經費參加國際會議，去了解目前國際間所重視的研究課題，也可持續多補助國內的相關學術研究單位爭取舉辦大型的國際研討會，相信這對台灣的學術研究發展會有很大的幫助。

攜回資料名稱與內容：
與會名牌、註冊收據、大會議程電子檔

科技部補助計畫衍生研發成果推廣資料表

日期:2015/11/27

科技部補助計畫	計畫名稱: 一類耦合系統的同步行為
	計畫主持人: 曾睿彬
	計畫編號: 103-2115-M-004-001- 學門領域: 動態系統

無研發成果推廣資料

103年度專題研究計畫研究成果彙整表

計畫主持人：曾睿彬			計畫編號：103-2115-M-004-001-				
計畫名稱：一類耦合系統的同步行為							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數（含實際已達成數）	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	1	0%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
其他成果（無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）		研究成果已完成論文撰寫並投至國際期刊，目前正在審查中。					

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

科技部補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以100字為限）

   　□實驗失敗

   　□因故實驗中斷

   　□其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：□已發表 ■未發表之文稿 □撰寫中 □無

   專利：□已獲得 □申請中 ■無

   技轉：□已技轉 □洽談中 ■無

   其他：（以100字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以500字為限）