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# COMPATIBILITY OF FINITE DISCRETE CONDITIONAL DISTRIBUTIONS

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*Abstract:* This paper provides new versions of necessary and sufficient conditions for compatibility of finite discrete conditional distributions, and of the uniqueness for those compatible conditional distributions. We note that the ratio matrix (the matrix $C$ in Arnold and Press (1989)), after interchanging its rows and/or columns, can be rearranged to be an irreducible block diagonal matrix. We find that checking compatibility is equivalent to inspecting whether every block on the diagonal has a rank one positive extension, and that the necessary and sufficient conditions of the uniqueness, if the given conditional densities are compatible, is that the ratio matrix itself is irreducible. We show that each joint density, if it exists, corresponds to a rank one positive extension of the ratio matrix, and we characterize the set of all possible joint densities. Finally, we provide algorithms for checking compatibility, for checking uniqueness, and for constructing densities.

*Key words and phrases:* Compatibility, conditional density, irreducible block diagonal matrix, joint density, marginal density, rank one positive extension matrix, ratio matrix, uniqueness.

## 1. Introduction

Multivariate distributions that are defined by conditional and/or marginal distributions are often used in probability modelling and Bayesian statistics. In particular, there is great interest in determining the joint distribution or marginal distributions when only conditional distributions are specified. The Gibbs sampler and some Markov Chain Monte Carlo methods are important research areas that may involve the characterization of a joint distribution by given conditional distributions (Liu (1996)). When both the conditional distribution functions of $X|Y$ and $Y|X$ are specified, the issue of compatibility is whether there exists a joint distribution function of $(X, Y)$ with these specified functions as conditional distribution functions. It is important to provide some criteria which are simple and easy to use for compatibility checking. In this paper, we concentrate our attention on the compatibility of finite discrete conditional distributions and, for simplicity, we just consider two-dimensional cases.

There are several versions of necessary and sufficient conditions for compatibility given by Arnold and Press (1989) and Arnold, Castillo and Sarabia (2002, 2004). However, an incompatible example (see Example 1 in Section 3) satisfying Arnold and Press' (1989) compatible condition is found. In some cases, the Arnold, Castillo and Sarabia (2004) condition for compatibility checking was found not an easy task and less effective. By investigating the structure of a ratio matrix (Arnold and Press (1989)), we successfully solve both the "existence" and the "uniqueness" problems.

After interchanging rows and/or columns, the ratio matrix can be rearranged to an "irreducible block diagonal matrix." We find that the ranks of all blocks on the diagonal determine the compatibility of finite discrete conditional distributions. More precisely, the necessary and sufficient condition for compatibility is that every block is of rank one. This new method, which needs only some elementary operations of matrices, could provide a simpler and more effective approach. Equivalent criteria and details are presented in Section 3.

When the given conditional distributions are compatible, it is natural to ask whether the associated joint distribution is unique. This issue has been addressed by Amemiya (1975), Gourieroux and Montfort (1979), Nerlove and Press (1986) and Arnold and Press (1989). Arnold and Press (1989) pointed out that the condition for uniqueness is generally difficult to check. In this paper, through the structure of the ratio matrix, we give some quite simple criteria for uniqueness checking. Moreover, we obtain all the associated joint distributions whenever they are not unique.

Other related works such as near compatibility and compatibility of partial conditional distributions can be seen in Arnold, Castillo and Sarabia (2002, 2004). In addition, for the inverse Bayes formula in Bayesian computation refer to Ng (1995, 1997), Tian, Ng and Geng (2003) and Tian and Tan (2003). We do not pursue these topics in this paper.

This paper is organized as follows. In the next section, we introduce some interesting structures of a ratio matrix: the irreducible matrix and irreducible block diagonal matrix. In Section 3, we use this intrinsic structure to derive new necessary and sufficient conditions for compatibility of any two given conditional matrices. In the fourth section, we obtain useful criteria to check whether or not the joint density is unique, and we also characterize all joint densities if two specified conditional densities are compatible. In Section 5, we provide algorithms for checking compatibility and uniqueness. We also present an algorithm to construct joint or marginal densities.

## 2. Some Preliminary Results

We restrict our attention to bivariate random variables that are finite discrete. Two $I \times J$ matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, with nonnegative elements are called conditional matrices if $\sum_{i=1}^{I} a_{ij} = 1$ for all $j$, and $\sum_{j=1}^{J} b_{ij} = 1$ for all $i$. These conditional matrices are said to be compatible if there exists a discrete random vector $(X, Y)$ such that the conditional density $f_{X|Y}(x_i|y_j) = a_{ij}$ and the conditional density $f_{Y|X}(y_j|x_i) = b_{ij}$ for all $(i, j) \in \{1, \ldots, I\} \times \{1, \ldots, J\}$, where $\{x_1, \ldots, x_I\}$ and $\{y_1, \ldots, y_J\}$ are supports of $X$ and $Y$, respectively.

Let $N^A = \{(i, j) \mid a_{ij} > 0\}$ and $N^B = \{(i, j) \mid b_{ij} > 0\}$. Notice that, if $A$ and $B$ are compatible, then $N^A = N^B$ (see Arnold and Press (1989)). The condition $N^A = N^B$ is a required condition for any further discussion of possible compatibility; hereafter, we assume that $N^A = N^B \equiv N$. The matrix $C = [c_{ij}]$, where $c_{ij} = a_{ij}/b_{ij}$ if $(i, j) \in N$, otherwise $c_{ij} = *$ (a symbol to denote the undefined entries), is defined as the ratio matrix of $A$ and $B$. A ratio matrix $C = [c_{ij}]$ is said to be incomplete if some of $c_{ij}$'s are $*$, and complete otherwise. We can extend any incomplete matrix to a complete one by assigning real numbers to substitute all undefined entries. A matrix $\overline{C} = [\bar{c}_{ij}]$ is called a positive extension of a ratio matrix $C = [c_{ij}]$ if all $\bar{c}_{ij} > 0$ and $\bar{c}_{ij} = c_{ij}$ whenever $(i, j) \in N$. Such a positive extension $\overline{C}$ is not necessarily unique, and $\overline{C} = C$ if $C$ is complete. We say that an incomplete ratio matrix $C$ is of rank one if it has a rank one positive extension.

It can be shown that an $I \times J$ incomplete matrix $C$ is of rank one if and only if there exist positive vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ of dimensions $I$ and $J$, respectively, such that $\overline{C} = \boldsymbol{u}\boldsymbol{v}'$ is a rank one positive extension of $C$, since the row or column space of $\overline{C}$ has dimension one.

**Definition 1.** A ratio matrix $C$ is said to be reducible if, after interchanging some rows and/or columns, it can be rearranged as

$$\left(\begin{array}{c|c} T_1 & * \\ \hline * & T_2 \end{array}\right),$$

where entries off the diagonal block matrices $T_1$ and $T_2$ are all "$*$". The matrix $C$ is irreducible if it is not reducible.

By the properties of a ratio matrix, neither entries of $T_1$ nor $T_2$ in Definition 1 are all "$*$" if $C$ is reducible.

**Lemma 2.** *For any ratio matrix $C$, by interchanging some rows and/or columns, it can be rearranged as an irreducible block diagonal* (*abbreviated as IBD*) *matrix, denoted by*

$$T(C) \equiv \begin{pmatrix} \boxed{T_1} & * & \cdots & \cdots & * \\ * & \boxed{T_2} & * & \cdots & * \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ * & * & \cdots & * & \boxed{T_M} \end{pmatrix},$$

*where diagonal block matrices* $T_1, \ldots, T_M$ $(M \geq 1)$ *are irreducible and entries off these diagonal block matrices are all "$*$". When* $M = 1$*, $C$ itself is irreducible. For convenience, we write* $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ *for* $M \geq 1$.

Interchanging two rows or columns of a matrix simply multiplies it by a permutation matrix (Weisstein (2005)) from the left or right, respectively. The addition and multiplication rules involving "$*$" are defined as follows:

$$0 + * = * + 0 = *, \quad 0 \times * = * \times 0 = 0, \quad \text{and} \quad 1 \times * = * \times 1 = *.$$

With this definition, the irreducible block diagonal matrix of Lemma 2 can be written as $T(C) = ECF$ for products of permutation matrices $E$ on rows and $F$ on columns. Notice that $E^{-1} = E'$ (the transpose of $E$) and $F^{-1} = F'$.

**Lemma 3.** *A ratio matrix $C$ is of rank one if and only if each* $T_m$*,* $1 \leq m \leq M$*, is of rank one, where* $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ *is any IBD matrix of $C$.*

**Proof.** Assume $C$ is of rank one. There exist positive vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\overline{C} = \boldsymbol{u}\boldsymbol{v}'$, where $\overline{C}$ is a rank one positive extension of $C$. By Lemma 2, any IBD matrix $T(C)$ of $C$ is $ECF$ for products of some permutation matrices $E$ and $F$. Clearly, $E\overline{C}F$ is a positive extension of $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$, $E\overline{C}F = E\boldsymbol{u}\boldsymbol{v}'F = (E\boldsymbol{u})(F'\boldsymbol{v})'$. Write $(E\boldsymbol{u})' \equiv \bar{\boldsymbol{u}}' = (\bar{\boldsymbol{u}}'_1, \ldots, \bar{\boldsymbol{u}}'_M)$ and $(F'\boldsymbol{v})' \equiv \bar{\boldsymbol{v}}' = (\bar{\boldsymbol{v}}'_1, \ldots, \bar{\boldsymbol{v}}'_M)$, where the dimensions of $\bar{\boldsymbol{u}}_m$ and $\bar{\boldsymbol{v}}_m$ are the number of rows and the number of columns of $T_m$, respectively. By the property of $E$ and $F$, $\bar{\boldsymbol{u}}_m\bar{\boldsymbol{v}}'_m$ is a positive extension of $T_m$ and therefore, $T_m$ is of rank one for each $1 \leq m \leq M$.

Conversely, assume $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ is any IBD matrix of $C$ with each $T_m$ having rank one, so $T(C) = ECF$ for products of permutation matrices $E$ and $F$. Since $T_m$ is of rank one, there exist a positive extension $\overline{T}_m$ of $T_m$ and positive vectors $\bar{\boldsymbol{u}}_m$ and $\bar{\boldsymbol{v}}_m$ such that $\overline{T}_m = \bar{\boldsymbol{u}}_m\bar{\boldsymbol{v}}'_m$, for each $1 \leq m \leq M$. If $\bar{\boldsymbol{u}}' = (\bar{\boldsymbol{u}}'_1, \ldots, \bar{\boldsymbol{u}}'_M)$ and $\bar{\boldsymbol{v}}' = (\bar{\boldsymbol{v}}'_1, \ldots, \bar{\boldsymbol{v}}'_M)$, $\bar{\boldsymbol{u}}\bar{\boldsymbol{v}}'$ is a positive extension of $T(C)$. Then, $E'(\bar{\boldsymbol{u}}\bar{\boldsymbol{v}}')F' = (E'\bar{\boldsymbol{u}})(F\bar{\boldsymbol{v}})'$ is a rank one positive extension of $E'T(C)F' = C$. Hence, $C$ is of rank one.

An algorithm for finding an IBD matrix of any ratio matrix is given in the Appendix.

## 3. Necessary and Sufficient Conditions for Compatibility

In this section, we present new versions of necessary and sufficient conditions for compatibility. These versions could provide easier and more effective ways to check the compatibility of any specified finite discrete conditional distributions. An incompatible example satisfying Arnold and Press' (1989) compatible condition is given in this section as well. We give a proof of the following Theorem 4 that summarizes Theorem 1 (Theorem 3 of Pérez-Villalta (2000)) and Theorem 2 (implicit in Pérez-Villalta (2000)) of Arnold, Castillo and Sarabia (2004).

**Theorem 4.** *Let $C$ be the ratio matrix of two $I \times J$ matrices $A$ and $B$. Then $A$ and $B$ are compatible if and only if $N^A = N^B$ and there exists a positive extension $\overline{C}$ of rank one.*

**Proof.** Assume $A = [a_{ij}]$ and $B = [b_{ij}]$ are compatible. Then there exist joint and marginal densities, say $f_{X,Y}(x,y)$, $f_X(x)$, $f_Y(y)$, such that $a_{ij} = f_{X,Y}(x_i,y_j)/f_Y(y_j)$ and $b_{ij} = f_{X,Y}(x_i,y_j)/f_X(x_i)$ for $1 \le i \le I$ and $1 \le j \le J$. Clearly, $N^A = N^B$. Take $\overline{C} = (f_X(x_1),\ldots,f_X(x_I))'(1/f_Y(y_1),\ldots,1/f_Y(y_J))$, then $\overline{C}$ is a rank one positive extension of $C$.

Conversely, suppose $\overline{C}$ is a rank one positive extension of $C$. Then $\overline{C}$ can be expressed as $\boldsymbol{uv'}$ for some positive vectors $\boldsymbol{u'} = (u_1,\ldots,u_I)$ and $\boldsymbol{v'} = (v_1,\ldots,v_J)$. Define $g_X(x_i) = u_i/u_+$ for $1 \le i \le I$, where $u_+ = \sum_{i=1}^I u_i$, $g_Y(y_j) = \sum_{i=1}^I b_{ij}u_i/u_+$ for $1 \le j \le J$, and $g_{X,Y}(x_i,y_j) = b_{ij}u_i/u_+$ if $(i,j) \in N$, zero otherwise. Obviously, $g_X$ and $g_Y$ are marginal densities of $g_{X,Y}$. Since $a_{ij}/b_{ij} = \bar{c}_{ij} = u_iv_j$ for $(i,j) \in N$ and $a_{ij} = b_{ij} = 0$ for $(i,j) \notin N$, we have $b_{ij}u_i = a_{ij}/v_j$ for $1 \le i \le I$ and $1 \le j \le J$, which implies that $g_{X|Y}(x_i|y_j) = a_{ij}$ and $g_{Y|X}(y_j|x_i) = b_{ij}$. Therefore, $A$ and $B$ are compatible.

When $A$ and $B$ are compatible, we can find a positive extension $\overline{C} = \boldsymbol{uv'}$ for some positive vectors $\boldsymbol{u'} = (u_1,\ldots,u_I)$ and $\boldsymbol{v'} = (v_1,\ldots,v_J)$. The marginal density of $X$ corresponding to $\overline{C}$ is given by $f_X(x_i) = u_i/u_+$ for $1 \le i \le I$, then the joint density is $f_{X,Y}(x_i,y_j) = b_{ij}u_i/u_+$, and the marginal density of $Y$ is $f_Y(y_j) = \sum_{i=1}^I b_{ij}u_i/u_+$ for $1 \le j \le J$. Note that the marginal density of $Y$ can also be obtained as $f_Y(y_j) = v/v_j$ for $1 \le j \le J$, where $1/v = \sum_{j=1}^J 1/v_j$, $f_{X,Y}(x_i,y_j) = va_{ij}/v_j$ for $(i,j) \in N$, and $f_X(x_i) = v\sum_{j=1}^J a_{ij}/v_j$ for $1 \le i \le I$.

The rank one requirement of $\overline{C}$ in Theorem 4 is also equivalent to each of the following three statements. The first two are Theorem 3 and Theorem 4, respectively, of Arnold, Castillo and Sarabia (2004), and the last is Theorem 2.1 of Arnold and Gokhale (1998).

1. There exists a positive extension $\overline{C}$ such that $\bar{c}_{ij}\bar{c}_{IJ} = \bar{c}_{iJ}\bar{c}_{Ij}$ for $1 \le i \le I-1$ and $1 \le j \le J-1$.
2. There exists a positive extension $\overline{C}$ such that $\bar{c}_{ij}\bar{c}_{++} = \bar{c}_{i+}\bar{c}_{+j}$ for $1 \le i \le I$ and $1 \le j \le J$, where $\bar{c}_{i+} = \sum_{j=1}^J \bar{c}_{ij}$, $\bar{c}_{+j} = \sum_{i=1}^I \bar{c}_{ij}$, and $\bar{c}_{++} = \sum_{i=1}^I\sum_{j=1}^J \bar{c}_{ij}$.
3. There exist positive vectors $\boldsymbol{\tau'} = (\tau_1,\ldots,\tau_I)$ and $\boldsymbol{\eta'} = (\eta_1,\ldots,\eta_J)$ such that $\eta_j a_{ij} = \tau_i b_{ij}$ for $1 \le i \le I$ and $1 \le j \le J$.

To check the compatibility of specified finite discrete conditional distributions using Theorem 4, we only have to see if we can find a rank one positive extension of the ratio matrix. Alternatively, if we can find a positive extension, where all rows (or all columns) are proportional, then the given specified finite discrete conditional distributions are compatible. However, during the extension process, if there exist two rows (or two columns) which cannot be proportional, then the given specified finite discrete conditional distributions are incompatible.

The following example shows that the method of Theorem 4 might be easier than that used by Arnold, Castillo and Sarabia (2004) for compatibility checking. This example also points out that Arnold and Press' compatible condition (Theorem 3.1 of Arnold and Press (1989)) is insufficient.

**Example 1.** Given two conditional matrices:

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{5}{6} \end{pmatrix}.$$

Then, the corresponding ratio matrix is

$$C = \begin{pmatrix} 1 & 2 & 2 & * \\ 1 & 2 & * & 2 \\ * & * & 2 & \frac{4}{5} \end{pmatrix}.$$

By Theorem 3.1 of Arnold and Press (1989), one only need check that $N^A = N^B$ and $c_{11} \times c_{22} = c_{12} \times c_{21} = 2$ to conclude that $A$ and $B$ are compatible. However, by either the method used by Arnold, Castillo and Sarabia (2004) or the method of Theorem 4, $A$ and $B$ are incompatible.

By the method of Arnold, Castillo and Sarabia (2004), first, we have to extend $C$ to

$$D = \begin{pmatrix} 1 & 2 & 2 & d_{14} \\ 1 & 2 & d_{23} & 2 \\ d_{31} & d_{32} & 2 & \frac{4}{5} \end{pmatrix}.$$

Next, set up a system of equations (as Eq. (16) of Arnold, Castillo and Sarabia (2004)) to see if there are any values of $d_{14}$, $d_{23}$, $d_{31}$, and $d_{32}$ to make $D$ rank one. In this case, we must solve six equations, namely

$$1 \times \frac{4}{5} = d_{14} \times d_{31}, \quad 2 \times \frac{4}{5} = d_{14} \times d_{32}, \quad 2 \times \frac{4}{5} = d_{14} \times 2,$$

$$1 \times \frac{4}{5} = 2 \times d_{31}, \quad 2 \times \frac{4}{5} = 2 \times d_{32}, \quad d_{23} \times \frac{4}{5} = 2 \times 2.$$

No solution exists, and it can be concluded that $A$ and $B$ are incompatible.

By the method of Theorem 4, in order to make the first two rows proportional, we have to set $\bar{c}_{14} = \bar{c}_{23} = 2$, and the matrix turns out to be

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ * & * & 2 & \frac{4}{5} \end{pmatrix}.$$

However, no matter what the values of $\bar{c}_{31}$ and $\bar{c}_{32}$ are, the last two columns will never be proportional. From Theorem 4, we conclude that $A$ and $B$ are incompatible.

In the next theorem, we use Lemma 3 to present another version of necessary and sufficient conditions for compatibility. This new version plays an essential role in understanding whether the joint density of the compatible conditional densities is unique or not. By Lemma 2, any ratio matrix can be rearranged to an irreducible block diagonal matrix $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ for $M \geq 1$. If $M > 1$, this theorem provides a more effective method to check compatibility than that of Theorem4, especially when the ratio matrix has a large size or contains many $*$'s.

**Theorem 5.** *Let $C$ be the ratio matrix of conditional matrices $A$ and $B$ and $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ be any IBD matrix of $C$. Then $A$ and $B$ are compatible if and only if $N^A = N^B$ and each $T_m$, $1 \leq m \leq M$, is of rank one.*

**Proof.** From Theorem 4, the necessary and sufficient conditions of compatibility are $N^A = N^B$ and $C$ is of rank one. By Lemma 3, $C$ is of rank one if and only if each $T_m$, $1 \leq m \leq M$, is of rank one. The proof is complete.

One advantage of Theorem 5 is that the compatibility checking only needs to focus on whether all $T_m$'s are of rank one or not. Since the size of each $T_m$ is smaller than that of $C$, this can be more effective. When $C$ is large, Theorem 5 can significantly reduce computing time and, once we find one $T_m$ not to be of rank one, $A$ and $B$ are incompatible.

**Example 2.** Consider the matrices $A$ and $B$ with ratio matrix $C$:

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2} & * & * & * & * & * \\ * & 2 & * & * & 3 & 2 \\ * & * & 2 & * & 1 & * \\ * & \frac{1}{3} & * & * & * & * \\ 1 & * & * & 2 & * & * \end{pmatrix}.$$

One of the IBD matrices of $C$ can be expressed as $T(C) = \mathrm{Diag}(T_1, T_2)$, where

$$T_1 = \begin{pmatrix} 2 & * & 2 & 3 \\ * & * & \frac{1}{3} & * \\ * & 2 & * & 1 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & * \end{pmatrix}.$$

The detailed transformation is given in Example 3 of Section 5. Consider the positive extensions

$$\overline{T}_1 = \begin{pmatrix} 2 & 6 & 2 & 3 \\ \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & 2 & \frac{2}{3} & 1 \end{pmatrix} \text{ and } \overline{T}_2 = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

We see that $T_1$ and $T_2$ are of rank one so $A$ and $B$ are compatible. We subsequently find a joint distribution of $(X, Y)$ with $A$ and $B$ as its conditional matrices.

| $f_{X,Y}(x_i, y_j)$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | total |
|---|---|---|---|---|---|---|---|
| $x_1$ | $\frac{1}{12}$ | 0 | 0 | 0 | 0 | 0 | $\frac{1}{12}$ |
| $x_2$ | 0 | $\frac{1}{6}$ | 0 | 0 | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{6}{12}$ |
| $x_3$ | 0 | 0 | $\frac{1}{12}$ | 0 | $\frac{1}{12}$ | 0 | $\frac{2}{12}$ |
| $x_4$ | 0 | $\frac{1}{12}$ | 0 | 0 | 0 | 0 | $\frac{1}{12}$ |
| $x_5$ | $\frac{1}{12}$ | 0 | 0 | $\frac{1}{12}$ | 0 | 0 | $\frac{2}{12}$ |
| total | $\frac{2}{12}$ | $\frac{3}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{2}{12}$ | $\frac{3}{12}$ | 1 |

The detailed steps for finding a corresponding joint distribution for Example 2 are given in Section 5.

## 4. Necessary and Sufficient Conditions for the Uniqueness

When the given conditional distributions are compatible, the next question to ask is whether or not the associated joint distribution is unique. We show below that it is not always unique.

Arnold and Press (1989, p.155) discussed the necessary and sufficient condition for uniqueness based on the irreducibility concept of Markov Chains (called MC-irreducibility). They also stated that the statements (iv)−(vi) in Theorem 6 are equivalent. However, they also pointed out that their criterion is difficult to check in practice. Using the concept of irreducibility for the ratio matrix in this paper, we can provide an alternative necessary and sufficient condition that is easier to check for uniqueness.

**Theorem 6.** *Let $C$ be the ratio matrix of two $I \times J$ compatible conditional matrices $A$ and $B$. The following statements are equivalent:* (i) $M = 1$ *for any IBD matrix* $T(C) = Diag(T_1, \ldots, T_M)$ *of* $C$*;* (ii) $C$ *is irreducible;* (iii) $C$ *has a unique rank one positive extension;* (iv) *the joint distribution with ratio matrix* $C$ *is unique;* (v) *the Markov chain corresponding to* $BA'$ *is MC-irreducible;* (vi) $\sum_{i=1}^{I}(BA')^{k+i} > 0$ *for some* $k$*.*

**Proof.** (i)⇔(ii) by Lemma 2. That (iv)⇔(v)⇔(vi) can be seen in Arnold and Press (1989, p.155). (ii)⇒(iii) For $J = 1$, $\overline{C}$ is unique for any $I$ since $\overline{C} = C$. We will complete the proof using induction on $J \geq 2$ for any $I$.

For $J = 2$, partition $C$ as $C = (C_1 \vdots C_2)$. When $C_1$ is complete, $\overline{C}_1 = C_1$. By Theorem 4, $\overline{C}_2$ must be proportional to $C_1$. Since $C$ is irreducible, at least one entry of $C_2$ is positive, which implies $\overline{C}_2$ is uniquely determined. When $C_1$ is incomplete it can be rearranged, by multiplying by $E$ (a product of permutation matrices on rows), as

$$EC = \left( \begin{array}{c|c} T_1 & Z_1 \\ \hline Q & Z_2 \end{array} \right),$$

where all entries of $T_1$ are positive and $Q$ is a $p \times 1$ ($p \geq 1$) matrix whose entries are all $*$. Since $EC$ is obtained by permuting rows of an irreducible $C$, $EC$ is also irreducible. This implies that $Z_2$, whose entries are all positive, is complete and $Z_1$, whose entries cannot all be $*$, contains at least one positive entry. By Theorem 4, through the proportionality of columns of $C$, $\overline{Z}_1$ is proportional to $\overline{T}_1 = T_1$ and $\overline{Q}$ is proportional to $\overline{Z}_2 = Z_2$. Both $\overline{Z}_1$ and $\overline{Q}$ are uniquely determined since the proportionality constant, the ratio of any positive entry of $Z_1$ and the entry of its corresponding row of $T_1$, is fixed. Therefore, the rank one positive extension of $EC$ is unique. It follows that

$$E' \left( \begin{array}{c|c} T_1 & \overline{Z}_1 \\ \hline \overline{Q} & Z_2 \end{array} \right)$$

is the unique rank one positive extension of $C$.

Assuming now that the induction hypothesis holds for $2 \le J \le n$, we are going to prove it for $J = n+1$. Let $C$ be an irreducible ratio matrix with $J = n+1$. Partition $C$ as $C = (C_1 \vdots C_2)$, where $C_1$ and $C_2$ are $I \times n$ and $I \times 1$, respectively. By interchanging rows of $C$ and columns of $C_1$, $C$ can be transformed into

$$ECF = \left(\begin{array}{ccc|c} \boxed{T_1} & & & Z_1 \\ \hline & \ddots & & \vdots \\ \hline & & \boxed{T_M} & Z_M \\ \hline & Q & & Z_{M+1} \end{array}\right),$$

where $T_1, \ldots, T_M$ $(M \ge 1)$ are irreducible matrices and $Q$ is a $p \times n$ matrix whose entries are all $*$. Here $Q$ may not exist, then neither does $Z_{M+1}$. It can be seen that the following proof also holds when $Q$ in this case.

By the induction hypothesis, $T_m$ has a unique rank one positive extension $\overline{T}_m$ for $1 \le m \le M$. From Theorem 4, $\overline{Z}_m$ is proportional to columns of $\overline{T}_m$. The irreducibility of $C$ implies that each $Z_m$ must have at least one positive entry. These particular entries serve to unite all proportionality constants between $\overline{Z}_m$ and columns of $\overline{T}_m$. Hence $\overline{Z}_m$ is uniquely determined for $1 \le m \le M$. Again, $\overline{Z}_{M+1} = Z_{M+1}$ by the irreducibility of $C$. Therefore, the entire column $(\overline{Z}_1', \ldots, \overline{Z}_M', \overline{Z}_{M+1}')'$ is uniquely determined. Since the columns are proportional by Theorem 4, $ECF$ has a unique rank one positive extension $\overline{ECF}$. It follows that $E'\overline{ECF}F'$ is the unique rank one positive extension of $C$. Now, the induction hypothesis holds for $J = n + 1$.

(iii)$\Rightarrow$(iv) Assume $C$ has a unique rank one positive extension $\overline{C}$. Suppose $f_{X,Y}$ and $g_{X,Y}$ are joint densities with $A$ and $B$ as their conditional matrices. For $(i, j) \in N$, we have $f_X(x_i)/f_Y(y_j) = c_{ij} = g_X(x_i)/g_Y(y_j)$. Let matrices

$$\overline{C}_f \equiv \begin{pmatrix} f_X(x_1) \\ \vdots \\ f_X(x_I) \end{pmatrix} \left(\frac{1}{f_Y(y_1)}, \ldots, \frac{1}{f_Y(y_J)}\right),$$

$$\overline{C}_g \equiv \begin{pmatrix} g_X(x_1) \\ \vdots \\ g_X(x_I) \end{pmatrix} \left(\frac{1}{g_Y(y_1)}, \ldots, \frac{1}{g_Y(y_J)}\right).$$

Clearly, $\overline{C}_f$ and $\overline{C}_g$ are rank one positive extensions of $C$. Since the rank one positive extension matrix of $C$ is unique, we then have $f_X(x_i)/f_Y(y_j) =$

$g_X(x_i)/g_Y(y_j)$ for $1 \le i \le I$ and $1 \le j \le J$. Hence,

$$\frac{1}{f_Y(y_j)} = \sum_{i=1}^{I} \frac{f_X(x_i)}{f_Y(y_j)} = \sum_{i=1}^{I} \frac{g_X(x_i)}{g_Y(y_j)} = \frac{1}{g_Y(y_j)}, \quad 1 \le j \le J.$$

That is, $f_Y = g_Y$. In addition, $f_{X,Y}(x_i, y_j) = a_{ij} f_Y(y_j) = a_{ij} g_Y(y_j) = g_{X,Y}(x_i, y_j)$ for $1 \le i \le I$ and $1 \le j \le J$. Therefore, the joint distribution of $A$ and $B$ is unique.

(iv)$\Rightarrow$(ii) Assume $C$ is reducible, then there exist invertible matrices $E$ and $F$ such that

$$ECF = \left( \begin{array}{c|c} T_1 & * \\ \hline * & T_2 \end{array} \right).$$

Since $A$ and $B$ are compatible, there exist positive vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\overline{C} = \boldsymbol{u}\boldsymbol{v}'$. Clearly, $E\overline{C}F = (E\boldsymbol{u})(F'\boldsymbol{v})'$ is a rank one positive extension of $ECF$. Write $(E\boldsymbol{u})' \equiv (\boldsymbol{e}_1', \boldsymbol{e}_2')$ and $(F'\boldsymbol{v})' \equiv (\boldsymbol{f}_1', \boldsymbol{f}_2')$, where the dimensions of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the number of rows of $T_1$ and $T_2$, respectively, and the dimensions of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are to the number of columns of $T_1$ and $T_2$, respectively. Next, we want to construct another rank one positive extension of $C$. Consider $(2\boldsymbol{e}_1', \boldsymbol{e}_2')'((1/2)\boldsymbol{f}_1', \boldsymbol{f}_2')$, which is a rank one positive extension of $ECF$. Then $E'(2\boldsymbol{e}_1', \boldsymbol{e}_2')'((1/2)\boldsymbol{f}_1', \boldsymbol{f}_2')F'$ is also a rank one positive extension of $C$.

The entries of $E\boldsymbol{u} = (\boldsymbol{e}_1', \boldsymbol{e}_2')'$ are just those of $\boldsymbol{u}$ except for order, since $E$ is a product of permutation matrices. Therefore, the marginal density of $X$ constructed by Theorem 4 using $\boldsymbol{u}$ is the same as that using $E\boldsymbol{u}$. However the marginal density of $X$ using $(\boldsymbol{e}_1', \boldsymbol{e}_2')'$ or $(2\boldsymbol{e}_1', \boldsymbol{e}_2')'$ is different. This implies that we have two joint densities, which contradicts the uniqueness of the joint density.

From Theorem 4, we can obtain at least one joint density if $\overline{C}$ is a rank one positive extension of a ratio matrix $C$. When $C$ is irreducible, from Theorem 6, $\overline{C}$ is unique and there is only one joint density. However, if $C$ is reducible, Theorem 6 shows that $\overline{C}$ is not unique and there is more than one joint density. Whether there is only one or more than one joint density associated with any given $\overline{C}$ is not clear at this point. The following theorem clarifies this issue.

**Theorem 7.** *Let $C$ be the ratio matrix of two $I \times J$ compatible conditional matrices $A$ and $B$, $\mathfrak{C}$ be the set of all rank one positive extensions of $C$, and $\mathfrak{F}$ be the set of all possible joint densities having $A$ and $B$ as their conditional matrices. Then the mapping $H : \mathfrak{C} \to \mathfrak{F}$ defined by $H(\overline{C}) = f$ is bijective, where $\overline{C} = \boldsymbol{u}\boldsymbol{v}'$ and $f(x_i, y_j) = b_{ij} u_i / u_+$ for $(i, j) \in N$.*

**Proof.** First, we claim that the mapping $H$ is well-defined. If $\overline{C} = \boldsymbol{u}\boldsymbol{v}' = \boldsymbol{s}\boldsymbol{t}'$, we show $f_X(x_i)$ defined by $u_i/u_+$ is the same as that defined by $s_i/s_+$. By the property of rank one, we conclude that $\boldsymbol{u} = r\boldsymbol{s}$ for some positive $r$. Therefore, $u_i/u_+ = s_i/s_+$.

Next we claim that $H$ is onto. For every $f \in \mathfrak{F}$, let $f_X$ and $f_Y$ be its marginal densities. We then have

$$\frac{f_X(x_i)}{f_Y(y_j)} = \frac{f(x_i,y_j)/f_Y(y_j)}{f(x_i,y_j)/f_X(x_i)} = \frac{a_{ij}}{b_{ij}} = c_{ij}, \quad (i,j) \in N.$$

Therefore, the $I \times J$ matrix $[f_X(x_i)/f_Y(y_j)]$ is a rank one positive extension of $C$. This implies that the matrix $[f_X(x_i)/f_Y(y_j)] \in \mathfrak{C}$ and $H([f_X(x_i)/f_Y(y_j)]) = f$. In other words, the mapping $H$ is onto.

Finally, we claim that $H$ is one-to-one. Assume $H(\overline{C}_1) = H(\overline{C}_2) = f$. Write $\overline{C}_1 = \boldsymbol{u}\boldsymbol{v}'$ and $\overline{C}_2 = \boldsymbol{s}\boldsymbol{t}'$. Then, $u_i/u_+ = f_X(x_i) = s_i/s_+$ for $1 \le i \le I$. For each $1 \le j \le J$, there must exist an index $i_j$ such that $(i_j, j) \in N$ and $u_{i_j} v_j = (\overline{C}_1)_{i_j j} = c_{i_j j} = (\overline{C}_2)_{i_j j} = s_{i_j} t_j$. Since $u_{i_j} = r s_{i_j}$, $v_j = (1/r) t_j$ for each $1 \le j \le J$, where $r = u_+/s_+$. Therefore $\overline{C}_1 = \boldsymbol{u}\boldsymbol{v}' = (r\boldsymbol{s})(\boldsymbol{t}/r)' = \boldsymbol{s}\boldsymbol{t}' = \overline{C}_2$, completing the proof.

From Theorem 7, the cardinal number of the set of all possible joint densities is the same as that of all rank one positive extensions. This holds both for any irreducible ratio matrix, there is one in this case, and for any reducible ratio matrix, they are uncountable. In addition, Theorem 7 also shows that all possible joint densities can be constructed by Theorem 4.

## 5. Algorithms

We use Lemma 3 to present a different version of Theorem 7. This new version provides a more effective method to find all possible rank one positive extensions of a reducible ratio matrix $C$ or, equivalently, to find all possible joint densities or marginal densities for given $C$, especially when $C$ is large.

**Theorem 8.** *Assume $C$ is the ratio matrix of $I \times J$ compatible conditional matrices $A$ and $B$. Let $T(C) = ECF = \mathrm{Diag}(T_1, \ldots, T_M)$ be an IBD matrix of $C$ for some products of permutation matrices $E$ and $F$, and $\overline{T}_m = \boldsymbol{u}_m\boldsymbol{v}_m'$ be the rank one positive extension of $T_m$ for $1 \le m \le M$. For any $\boldsymbol{k} = (k_1, \ldots, k_M)$ with $k_m > 0$, let $\boldsymbol{u}_{\boldsymbol{k}}' = (k_1\boldsymbol{u}_1', \ldots, k_M\boldsymbol{u}_M')$, $E'\boldsymbol{u}_{\boldsymbol{k}} = \boldsymbol{p}_{\boldsymbol{k}}$, and $f_{\boldsymbol{k}}(x_i, y_j) = b_{ij}p_{\boldsymbol{k}i}/p_{\boldsymbol{k}+}$ for $(i,j) \in N$, where $\boldsymbol{p}_{\boldsymbol{k}}' = (p_{\boldsymbol{k}1}, \ldots, p_{\boldsymbol{k}I})$ and $p_{\boldsymbol{k}+} = \sum_{i=1}^{I} p_{\boldsymbol{k}i}$. Then the family $\{f_{\boldsymbol{k}} \mid \boldsymbol{k} = (k_1, \ldots, k_M),\ k_m > 0,\ \text{for } 1 \le m \le M\}$ is the set of all possible joint densities of $(X,Y)$ for given $C$.*

**Proof.** Let $\boldsymbol{v}_{\boldsymbol{k}}' = (\boldsymbol{v}_1'/k_1, \ldots, \boldsymbol{v}_M'/k_M)$ and $\boldsymbol{q}_{\boldsymbol{k}}' = (\boldsymbol{v}_1'/k_1, \ldots, \boldsymbol{v}_M'/k_M)F'$. Since $\boldsymbol{u}_{\boldsymbol{k}}\boldsymbol{v}_{\boldsymbol{k}}'$ is a rank one positive extension of $T(C) = ECF$, $E'\boldsymbol{u}_{\boldsymbol{k}}\boldsymbol{v}_{\boldsymbol{k}}'F' = \boldsymbol{p}_{\boldsymbol{k}}\boldsymbol{q}_{\boldsymbol{k}}'$ is a rank one positive extension of $C$. As $\boldsymbol{k}$ runs over all possible vectors, the set of all possible rank one positive extensions of the ratio matrix $C$ is $\{\boldsymbol{p}_{\boldsymbol{k}}\boldsymbol{q}_{\boldsymbol{k}}' \mid \boldsymbol{k}\}$. By Theorem7, the family $\{f_{\boldsymbol{k}} \mid \boldsymbol{k}\}$ is the set of all possible joint densities of $(X,Y)$ for given $C$.

If $\boldsymbol{k}^* = \boldsymbol{k}/(\sum_{m=1}^{M} k_m)$, then $f_{\boldsymbol{k}^*} = f_{\boldsymbol{k}}$. Hence, the family of all possible joint densities can be expressed as

$$\left\{ f_{\boldsymbol{k}} \mid \boldsymbol{k} = (k_1, \ldots, k_M),\ \sum_{m=1}^{M} k_m = 1,\ k_m > 0, \ \text{ for } \ 1 \le m \le M \right\}.$$

In the following, we provide algorithms for checking compatibility and uniqueness, and for constructing joint and marginal densities. The first algorithm is used to check the compatibility of two conditional densities.

**Compatibility Algorithm**

Step 1. Construct the conditional matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ where $a_{ij} = f_{X|Y}(x_i|y_j)$ and $b_{ij} = f_{Y|X}(y_j|x_i)$.

Step 2. Let $N^A = \{(i,j) \mid a_{ij} > 0\}$ and $N^B = \{(i,j) \mid b_{ij} > 0\}$. If $N^A \ne N^B$, stop and claim that $\{f_{X|Y}(x_i|y_j)\}$ and $\{f_{Y|X}(y_j|x_i)\}$ are incompatible.

Step 3. If $N^A = N^B$, compute the ratio matrix $C = [c_{ij}]$, where $c_{ij} = a_{ij}/b_{ij}$ if $(i,j) \in N = N^A = N^B$, otherwise $c_{ij} = *$.

Step 4. Obtain an IBD matrix $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ of $C$ using the IBD Matrix Algorithm of the Appendix.

Step 5. Check whether each $T_m$ is of rank one or not. See the ROPE Algorithm in the Appendix. If any $T_m$ is not of rank one, then stop and claim that $\{f_{X|Y}(x_i|y_j)\}$ and $\{f_{Y|X}(y_j|x_i)\}$ are incompatible. Otherwise, claim that $\{f_{X|Y}(x_i|y_j)\}$ and $\{f_{Y|X}(y_j|x_i)\}$ are compatible.

For two compatible conditional densities, the second algorithm is used to check the uniqueness of the joint density.

**Uniqueness Algorithm**

Step 1. Obtain an IBD matrix $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ of $C$ using the IBD Matrix Algorithm in the Appendix.

Step 2. Claim that the joint density is unique if $M = 1$. Otherwise, claim that more than one joint density exists.

Assume compatible conditional densities. The third algorithm is used to construct the unique joint density, or all joint densities. Let $T(C) = \mathrm{Diag}(T_1, \ldots, T_M)$ be an IBD matrix of $C$.

**Density Construction Algorithm**

Step 1. Obtain an IBD matrix $T(C) = \mathrm{Diag}(T_1, \ldots, T_M) = ECF$ of $C$ using the IBD Matrix Algorithm in the Appendix. Output $M$.

Step 2. Find a rank one positive extension $\overline{T}_m$ $(= \boldsymbol{u}_m \boldsymbol{v}'_m)$ of each $T_m$, $1 \le m \le M$, using the ROPE Algorithm in the Appendix.

Step 3. Input an $M$-dimensional vector $\boldsymbol{k} = (k_1, \ldots, k_M)$ with $k_m > 0$ and $\sum_{m=1}^{M} k_m = 1$. Define $\boldsymbol{u}'_{\boldsymbol{k}} = (k_1\boldsymbol{u}'_1, \ldots, k_M\boldsymbol{u}'_M)$ and $\boldsymbol{p}_{\boldsymbol{k}} = (p_{\boldsymbol{k}1}, \ldots, p_{\boldsymbol{k}I})' = E'\boldsymbol{u}_{\boldsymbol{k}}$. One corresponding joint density is obtained as $f_{\boldsymbol{k}}(x_i, y_j) = b_{ij}p_{\boldsymbol{k}i}/p_{\boldsymbol{k}+}$ for $(i,j) \in N$, where $p_{\boldsymbol{k}+} = \sum_{i=1}^{I} p_{\boldsymbol{k}i}$.

Step 4. Compute marginal densities $f_{X,\boldsymbol{k}}(x_i) = \sum_{j=1}^{J} f_{\boldsymbol{k}}(x_i, y_j)$, $1 \le i \le I$, and $f_{Y,\boldsymbol{k}}(y_j) = \sum_{i=1}^{I} f_{\boldsymbol{k}}(x_i, y_j)$, $1 \le j \le J$. Go to Step 3 for another solution of joint density and marginal densities as needed, otherwise stop.

The above algorithms solve the problems of compatibility and uniqueness. We also provide a tool to construct joint and marginal densities whenever the specified conditional densities are compatible. Next, we demonstrate these algorithms by using the following example.

**Example 3.** Consider two conditional matrices $A$ and $B$ and their corresponding conditional ratio matrix $C$ in Example 2. We use the Compatibility Algorithm to check whether $A$ and $B$ are compatible or not. Since $N^A = N^B$, we use Step 4 to obtain an IBD matrix of $C$, $T(C) = ECF = \mathrm{Diag}(T_1, T_2)$, where

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 2 & * & 2 & 3 \\ * & * & \frac{1}{3} & * \\ * & 2 & * & 1 \end{pmatrix},$$

and $T_2 = \begin{pmatrix} 1 & 2 \\ 1/2 & * \end{pmatrix}$. Following Step 5, we find $T_1$ and $T_2$ are both of rank one so $A$ and $B$ are compatible.

Since $M = 2$, by the Uniqueness Algorithm, the joint density function, whose conditional matrices are $A$ and $B$, is not unique.

We use the Density Construction Algorithm to construct all possible solutions of joint density functions. Following Step 2, the rank one positive extensions of $T_1$ and $T_2$ are

$$\begin{aligned} \overline{T}_1 &= \begin{pmatrix} 2 & 6 & 2 & 3 \\ \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & 2 & \frac{2}{3} & 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}\Big), \\ \overline{T}_2 &= \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Big(\frac{1}{2}, 1\Big). \end{aligned}$$

Let $\boldsymbol{k} = (k_1, k_2)$ and define $\boldsymbol{p_k} = E'(6k_1, k_1, 2k_1, 2k_2, k_2)' = (k_2, 6k_1, 2k_1, k_1, 2k_2)'$, where $k_1 > 0$, $k_2 > 0$, and $k_1 + k_2 = 1$. Then all possible solutions of the joint distribution of $(X, Y)$ are

| $f_{X,Y}(x_i, y_j)$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | total |
|---|---|---|---|---|---|---|---|
| $x_1$ | $\frac{k_2}{9k_1+3k_2}$ | 0 | 0 | 0 | 0 | 0 | $\frac{k_2}{9k_1+3k_2}$ |
| $x_2$ | 0 | $\frac{2k_1}{9k_1+3k_2}$ | 0 | 0 | $\frac{k_1}{9k_1+3k_2}$ | $\frac{3k_1}{9k_1+3k_2}$ | $\frac{6k_1}{9k_1+3k_2}$ |
| $x_3$ | 0 | 0 | $\frac{k_1}{9k_1+3k_2}$ | 0 | $\frac{k_1}{9k_1+3k_2}$ | 0 | $\frac{2k_1}{9k_1+3k_2}$ |
| $x_4$ | 0 | $\frac{k_1}{9k_1+3k_2}$ | 0 | 0 | 0 | 0 | $\frac{k_1}{9k_1+3k_2}$ |
| $x_5$ | $\frac{k_2}{9k_1+3k_2}$ | 0 | 0 | $\frac{k_2}{9k_1+3k_2}$ | 0 | 0 | $\frac{2k_2}{9k_1+3k_2}$ |
| total | $\frac{2k_2}{9k_1+3k_2}$ | $\frac{3k_1}{9k_1+3k_2}$ | $\frac{k_1}{9k_1+3k_2}$ | $\frac{k_2}{9k_1+3k_2}$ | $\frac{2k_1}{9k_1+3k_2}$ | $\frac{3k_1}{9k_1+3k_2}$ | 1 |

Note that Example 2 is a special solution when $k_1 = k_2 = 1/2$.

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## Appendix

Two algorithms are presented here. The goal of the first is to find an irreducible block diagonal matrix of any $I \times J$ specified ratio matrix $C$.

## IBD Matrix Algorithm

Step 0. Let $m = 1$.

Step 1. Let $E_m = \mathbf{I}_I$ and $F_m = \mathbf{I}_J$, and $J_{hb} = 1$, where $\mathbf{I}_k$ is the identity matrix of dimension $k$. Interchange rows of $C$ such that all positive entries of the first column of $C$ are on the top $I_1^*$ rows. Let $I_{hb} = I_1^*$. We call the block formed by the first $I_{hb}$ rows and the first $J_{hb}$ columns the "hot block", and the rearranged matrix as $C_T$. Note $C_T = E_T C$, where $E_T$ is a product of some permutation matrices on rows. Let $E_m = E_T E_m$. If $I_{hb} = I$ then go to Step 5. Interchange columns so that all columns whose entries in the first $I_{hb}$ rows are all $*$, say $J_1^*$ columns, are to the farthest right. Now $C_T = C_T F_T$, where $F_T$ is a product of some permutation matrices on columns. Let $F_m = F_m F_T$. If $J_1^* = 0$, then go to Step 5. If $J_1^* + J_{hb} = J$, then go to Step 4. Otherwise let $J_{hb} = J - J_1^*$.

Step 2. Interchange rows so that all rows whose entries in the first $J_{hb}$ columns are $*$, say $I_2^*$ rows, are at the very bottom. Note that new $C_T$ is again obtained by multiplying $C_T$ of the previous step by some new $E_T$ from the left. Let $E_m = E_T E_m$. If $I_2^* = 0$, then go to Step 5. If $I_2^* + I_{hb} = I$, then go to Step 4. Otherwise, let $I_{hb} = I - I_2^*$. Interchange columns so that all columns whose entries in the first $I_{hb}$ rows are all $*$, say $J_2^*$ columns, are on the farthest right. New $C_T$ is again obtained by multiplying the previous $C_T$ by some new $F_T$ from the right. Let $F_m = F_m F_T$. If $J_2^* = 0$, then go to Step 5. If $J_2^* + J_{hb} = J$, then go to Step 4. Otherwise, let $J_{hb} = J - J_2^*$.

Step 3. Go to Step 2. This step can be only executed $I$ or $J$ times.

Step 4. Let $T_m$ be the hot block, $I_m = I_{hb}$, and $J_m = J_{hb}$. $E_m = \begin{pmatrix} \mathbf{I}_{I_1+\cdots+I_{m-1}} & \mathbf{0} \\ \mathbf{0} & E_m \end{pmatrix} \times E_{m-1}$ and $F_m = F_{m-1} \begin{pmatrix} \mathbf{I}_{J_1+\cdots+J_{m-1}} & \mathbf{0} \\ \mathbf{0} & F_m \end{pmatrix}$. Delete the first $I_{hb}$ rows and the first $J_{hb}$ columns from the previous $C_T$, and call the remaining matrix $C$. Replace $I$ by $I - I_{hb}$, $J$ by $J - J_{hb}$, $m$ by $m+1$, go to Step 1.

Step 5. Let $M = m$, $T_M = C_T$, $I_M = I$, $J_M = J$. $T_1, \ldots, T_M$ are the diagonal blocks. If $m = 1$, then $E = E_m$ and $F = F_m$. Otherwise $E = \begin{pmatrix} \mathbf{I}_{I_1+\cdots+I_{M-1}} & \mathbf{0} \\ \mathbf{0} & E_T \end{pmatrix} E_{M-1}$ and $F = F_{M-1} \begin{pmatrix} \mathbf{I}_{J_1+\cdots+J_{M-1}} & \mathbf{0} \\ \mathbf{0} & F_T \end{pmatrix}$.

To obtain another IBD matrix of the ratio matrix $C$, one can either rearrange the order of those irreducible blocks $T_m$ for $1 \le m \le M$, or rearrange rows and/or columns within each $T_m$ of the output from the above IBD Matrix Algorithm.

The following algorithm can be used to check whether a given irreducible matrix $T$ has a rank one positive extension, and to find one if it does.

**Rank One Positive Extension (ROPE) Algorithm**

Step 0. Let $I$ and $J$ be the row and column dimensions of $T$, respectively. Let $F$ be the $J \times J$ identity matrix.

Step 1. Interchange rows of $T$ such that all positive entries of the first column of $T$ are on the top $I_1$ rows. Let $I_{hb} = I_1$ and $J_{hb} = 1$. We call the block formed by the first $I_{hb}$ rows and the first $J_{hb}$ columns the "hot block," and the rearranged matrix $T$. Note the new matrix $T$ is obtained by multiplying the previous $T$ by a matrix $E_T$ from the left, where $E_T$ is a product of some permutation matrices on rows. Let $E = E_T$.

Step 2. If $I_{hb} = I$, then let $J_{hb} = J$ and go to Step 3. Except the first $J_{hb}$ columns, interchange columns so that all columns having any positive entries in their first $I_{hb}$ rows, say $J_1$ columns, are next to the hot block. Let $J_{hb} = J_{hb} + J_1$. Note that this new $T$ is again obtained by multiplying the previous $T$ by some new $F_T$ from the right. Let $F = FF_T$.

Step 3. Within the new hot block, for any column not in the previous hot block having more than one positive entry in it, calculate the ratio of every positive entry in that column to the entry in its corresponding row of the first column. If these ratios are not the same, then stop and claim no rank one positive extension exists. Otherwise go to Step 4.

Step 4. Within the hot block, calculate the ratio of one positive entry for every column, not in the previous hot block, to the entry in its corresponding row of the first column. Then, within the hot block, substitute all $*$ entries in that column by products of this ratio and entries in their corresponding rows of the first column. If $I_{hb} = I$ or $J_{hb} = J$, go to Step 8. Otherwise go to Step 5.

Step 5. If $J_{hb} = J$, then let $I_{hb} = I$ and go to Step 6. Except the first $I_{hb}$ rows, interchange rows so that all rows having any positive entries in their first $J_{hb}$ columns, say $I_2$ rows, are just below the hot block. Let $I_{hb} = I_{hb} + I_2$. Note that this new $T$ is again obtained by multiplying the previous $T$ by some new $E_T$ from the left. Let $E = E_T E$.

Step 6. Within the new hot block, for any row not in the previous hot block having more than one positive entry in it, calculate the ratio of every positive entry in that row to the entry in its corresponding column of the first row. If these ratios are not the same, then stop and claim no rank one positive extension exists. Otherwise go to Step 7.

Step 7. Within the hot block, calculate the ratio of one positive entry in every row not in the previous hot block and the entry in its corresponding column of the first row. Then substitute all $*$ entries in that row by products of the ratio and entries in their corresponding columns of the first row. If $I_{hb} = I$ or $J_{hb} = J$ then go to Step 8; otherwise go back to Step 2.

Step 8. Let $T = E'TF' = [t_{ij}]$, and let $\boldsymbol{u}' = (u_1, \ldots, u_I)$ and $\boldsymbol{v}' = (v_1, \ldots, v_J)$, where $u_i = t_{i1}$ for $1 \le i \le I$, and $v_1 = 1$, $v_j = t_{1j}/t_{11}$ for $2 \le j \le J$.

The input of the ROPE Algorithm can be either $T$ or $T'$, and the output is the unique rank one positive extension of $T$.

## References

Amemiya, T. (1975). Qualitative response models. *Ann. Economic Social Measurement* **4**, 363-372.

Arnold, B. C. and Gokhale, D. V. (1998). Distributions of the most nearly compatible with given families of conditional distributions. *Test* **7**, 377-390.

Arnold, B. C. and Press, S. J. (1989). Compatible conditional distributions. *J. Amer. Statist. Assoc.* **84**, 152-156.

Arnold, B. C., Castillo, E. and Sarabia, J. M. (2002). Exact and near compatibility of discrete conditional distributions. *Comput. Statist. Data Anal.* **40**, 231-252.

Arnold, B. C., Castillo, E. and Sarabia, J. M. (2004). Compatibility of partial or complete conditional probability specifications. *J. Statist. Plann. Inference* **123**, 133-159.

Gourieroux, C. and Montfort, A. (1979). On the characterization of a joint probability distribution by conditional distributions. *J. Econometrics* **10**, 115-118.

Liu, J. S. (1996). Discussion of "Statistical inference and Monte Carlo algorithms" by G. Casella. *Test* **5**, 305-310.

Nerlove, M. and Press, S. J. (1986). Multivariate log-linear probability models in econometrics. In *Advances in Statistical Analysis and Statistical Computing* (Edited by R. S. Mariano), 117-171. JAI Press, Greenwich, CT.

Ng, K. W. (1995). Explict formulas for unconditional pdf. research report, No. 82 (revised). Department of Statistics, University of Hong Kong.

Ng, K. W. (1997). Inversion of Bayes formula: Explict formulae for unconditional pdf. In *Advance in the Theory and Practice in Statistics* (Edited by N. L. Johnson and N. Balakrishnan), 571-584. Wiley, New York.

Pérez-Villalta, R. (2000). Variables finitas condicionalmente especificadas. *Questioó* **24**, 425-448.

Tian, G.-L., Ng, K. W. and Geng, Z. (2003). Bayesian computation for contingency tables with incomplete cell-counts. *Statist. Sinica* **13**, 189-206.

Tian, G.-L. and Tan, M. (2003). Exact statistical solutions using the inversion Bayes formulae. *Statist. Probab. Lett.* **62**, 305-315.

Weisstein, E. W. (2005). "Permutation Matrix." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/PermutationMatrix.html`

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