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Global synchronization in nonlinearly coupled delayed memristor-based neural networks with excitatory and inhibitory connections

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Abstract

This investigation establishes the global synchronization of an array of coupled memristor-based neural networks with delays. The coupled networks that are considered can incorporate both the internal delay in each individual network and the transmission delay across different networks. The coupling scheme, which consists of a nonlinear term and a sign term, is rather general. In particular, it can be asymmetric, and admits the coexistence of excitatory and inhibitory connections. Based on an iterative approach, the problem of synchronization is transformed into solving a corresponding linear system of algebraic equations. Subsequently, the respective synchronization criteria, which depend on whether the transmission delay exists, are derived respectively. Three examples are given to illustrate the effectiveness of the theories presented in this paper. The synchronization of the systems in two examples cannot be handled by existing techniques.
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1. Introduction

Over the past few decades, neural networks (NNs) have attracted considerable interest from researchers. Time delays, which occur in the transmission of a signal among neurons, are ubiquitous, and have been incorporated into neural network modeling, cf. [1]. It

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is worth noting that taking delays into account in the coupling terms usually increases the mathematical complexity associated with the dynamics of a nonlinear system. The collective behavior in coupled NNs has received tremendous interest from researchers. In particular, the synchronization of coupled NNs has been extensively investigated by virtue of its wide application in different fields including secret communication [2], pattern recognition [3], and parallel image processing [4].

The memristor, which is the fourth fundamental two-terminal circuit element, was proposed by Chua [5]. The memristor was first realized as a physical device by Hewlett-Packard Laboratories [6]. The term memristor is a contraction for memory resistor, which reflects its property that the value of a memristor, known as memristance, depends on the history of the voltage across the component. It has been discovered that a memristor could mimic the human brain in that it can behave similar to a biological synapse, and thus has potential applications for building brain-like neural computers, cf. [7]. The memristor has been utilized in neural network models known as memristor-based NNs (MNNs). Contrary to conventional NNs in which the connection weights are implemented using resistors, MNNs utilize memristors to function as the synaptic connection weights. MNNs have been widely applied in many scientific fields, such as image processing, pattern recognition, and pseudorandom number generators, and have attracted increasing interests from researchers, see [8–14]. In particular, the synchronization control of coupled MNNs have potential applications including super-dense nonvolatile computer memory and neural synapses, cf. [15,16].

Many existing studies on synchronization of coupled MNNs focused on the synchronization of master-slave MNNs using control schemes. The exponential synchronization of master-slave MNNs with a linear feedback controller was established in [16–19]. The studies in [15,20–22] established the synchronization of master-slave MNNs by imposing feedback controllers that have a linear term and a sign function. Studies on the adaptive synchronization of master-slave MNNs can be found in [20,21,23–25]. The studies in [26,27] established the synchronization of master-slave MNNs with state coupling and output coupling, where Guo et al. [27] also considered dynamic state coupling and dynamic output coupling. The exponential synchronization of master-slave MNNs with periodically intermittent control was investigated in [28]. Studies on the synchronization of stochastic master-slave MNNs can be found in [29,30]. Studies on the lag synchronization of master-slave MNNs can be found in [24,31]. The work in [32] established the existence of periodic solutions and synchronization for master-slave MNNs. The paper [33] studied the general decay synchronization of master-slave MNNs. Master-slave synchronization has promising applications including the control of chaos and chaotic signal masking, cf. [34]. On the other hand, the synchronization of multiple (more than two) networks has promising applications in information processing and cognition behavior of the brain, cf. [35]. The literature contains far fewer studies on the synchronization of multiple MNNs than on master-slave MNNs. In terms of research on the synchronization of multiple MNNs, the studies in [36] and [37] established the global synchronization of multiple MNNs coupled under linear couplings and couplings that have a linear term and a sign function, respectively. The robust synchronization of multiple MNNs with nonidentical uncertain parameters was also studied in [38]. The synchronization of an array of linearly coupled MNNs with impulses and delays was investigated in [39]. The work in [40] established the synchronization of multiple linearly coupled stochastic MNNs with probabilistic delay coupling and impulsive delay.

A survey of existing studies on synchronization of coupled MNNs indicates that many coupled networks that were considered are with linear couplings [16–19,28,36,39,40], or with couplings consisting of a linear term and a sign function term [15,20–22,30,31,37,38]. In addition, the coupled networks that were considered commonly have internal time delays within individual networks, and have no transmission delays across different networks, cf. [15,17–23,25–28,36,38,39]. Thus, the global synchronization of MNNs coupled under general coupling schemes still deserves to be further investigated.

A single MNN model using memristors in the circuit implementation of the neural network connections can be described as follows (cf. [22,28]):

$$\dot{x}_k(t) = -d_k x_k(t) + \sum_{l\in\mathcal{K}} \left[a_{kl}(x_l(t)) f_l(x_l(t)) + b_{kl}(x_l(t-\tau_l)) f_l(x_l(t-\tau_l)) \right] + I_k(t) \tag{1}$$

with

$$a_{kl}(x_l(t)) = \frac{\mathbf{M}_{kl}}{\mathbf{C}_k} \times \mathrm{sgn}_{kl}, \;\; b_{kl}(x_l(t-\tau_l)) = \frac{\tilde{\mathbf{M}}_{kl}}{\mathbf{C}_k} \times \mathrm{sgn}_{kl},$$
$$\mathrm{sgn}_{kl} = \begin{cases} 1, & k \neq l, \\ -1, & k = l, \end{cases}$$

for $t \geq t_0$ and $k \in \mathcal{K} := \{1, \ldots, K\}$, where $x_k(t)$ corresponds to the voltage of capacitor $\mathbf{C}_k$ at time t; $d_k > 0$ represents the neuron self-inhibition rate; $\tau_l \geq 0$, $l \in \mathcal{K}$, denote the internal delays; and f_l, $l \in \mathcal{K}$, are the neuron activation functions which are continuous and satisfy

$$|f_l(\xi) - f_l(\eta)| \leq \bar{\nu}_l^f |\xi - \eta| \text{ for all } \xi, \eta \in \mathbb{R}. \tag{2}$$

Here, $I_k(t)$ denotes the external input to the kth neuron at time t and satisfies

$$|I_k(t)| \leq \bar{I}_k \text{ for all } t \geq t_0. \tag{3}$$

Further, $\mathbf{M}_{kl}$ and $\tilde{\mathbf{M}}_{kl}$ are the memductances of memristors $\mathbf{R}_{kl}$ and $\tilde{\mathbf{R}}_{kl}$, respectively, where $\mathbf{R}_{kl}$ denotes the memristor between $f_l(x_l(t))$ and $x_k(t)$; $\tilde{\mathbf{R}}_{kl}$ represents the memristor between $f_l(x_l(t-\tau_l))$ and $x_k(t)$. $a_{kl}(\cdot)$ and $b_{kl}(\cdot)$ are the synaptic connection weights implemented using memristors without and with delays, respectively. Several types of memristors have been reported in the literature. Here, according to the feature of the memristor and current–voltage characteristic, $a_{kl}(\cdot)$ and $b_{kl}(\cdot)$ are described by

$$a_{kl}(x_l(t)) = \begin{cases} a'_{kl}, & |x_l(t)| < T_l, \\ a''_{kl}, & |x_l(t)| > T_l, \\ a'_{kl} \text{ or } a''_{kl}, & |x_l(t)| = T_l, \end{cases} \quad b_{kl}(x_l(t-\tau_l)) = \begin{cases} b'_{kl}, & |x_{i,l}(t-\tau_l)| < T_l, \\ b''_{kl}, & |x_{i,l}(t-\tau_l)| > T_l, \\ b'_{kl} \text{ or } b''_{kl}, & |x_l(t-\tau_l)| = T_l, \end{cases}$$

in which switching jumps $T_l \geq 0$ and $a'_{kl}, a''_{kl}, b'_{kl}, b''_{kl} \in \mathbb{R}$ are constants relating to memristances, cf. [23,28]. More information about the circuit realization and physical properties of memristors can be found in [18,19,21–23,30,37–40]. In this paper, we consider an array of N coupled MNNs as follows:

$$\dot{x}_{i,k}(t) = -d_k x_{i,k}(t) + \sum_{l\in\mathcal{K}} \left[a_{kl}(x_{i,l}(t)) f_l(x_{i,l}(t)) + b_{kl}(x_{i,l}(t-\tau_l)) f_l(x_{i,l}(t-\tau_l)) \right] + I_k(t) + U_{i,k}(t) \tag{4}$$

with

$$a_{kl}(x_{i,l}(t)) = \begin{cases} a'_{kl}, & |x_{i,l}(t)| < T_l, \\ a''_{kl}, & |x_{i,l}(t)| > T_l, \\ a'_{kl} \text{ or } a''_{kl}, & |x_{i,l}(t)| = T_l, \end{cases} \quad b_{kl}(x_{i,l}(t-\tau_l))$$

$$= \begin{cases} b'_{kl}, & |x_{i,l}(t-\tau_l)| < T_l, \\ b''_{kl}, & |x_{i,l}(t-\tau_l)| > T_l, \\ b'_{kl} \text{ or } b''_{kl}, & |x_{i,l}(t-\tau_l)| = T_l, \end{cases} \tag{5}$$

$$U_{i,k}(t) = \gamma_k \sum_{j\in\mathcal{N}} w_{ij}\Big[h_k\big(x_{j,k}(t-\tau_T)\big) - h_k\big(x_{i,k}(t)\big) + s\big(x_{j,k}(t) - x_{i,k}(t)\big)\Big] \tag{6}$$

for $t \geq t_0$ and $(i,k) \in \mathcal{A}_x := \{(j,l) : j \in \mathcal{N},\ l \in \mathcal{K}\}$, where $\mathcal{A}_x$ is the subscript set of $x_{i,k}$ and $\mathcal{N} := \{1, \ldots, N\}$; $(x_{i,1}(t), x_{i,2}(t), \ldots, x_{i,K}(t))^T$ is the state of the ith MNN; $U_{i,k}(t)$ represents the coupling term for $x_{i,k}$ at time t, in which $\gamma_k \geq 0$ signifies the coupling strength; $w_{ij} \in \mathbb{R}$ is the coupling coefficient from the jth network to the ith network; h_k is a smooth and nondecreasing function; $\tau_T \geq 0$ represents the transmission delay across different networks; $s(\cdot)$ is defined by

$$s(\xi) = \bar{s} \times \text{sign}(\xi) := \bar{s} \times \begin{cases} 1, & \xi > 0, \\ -1, & \xi < 0, \\ 0, & \xi = 0, \end{cases} \tag{7}$$

for some $\bar{s} \geq 0$. For later use, from Eq. (5), we define

$$\begin{cases} \hat{a}_{kl} := \max\{a'_{kl}, a''_{kl}\}, \check{a}_{kl} := \min\{a'_{kl}, a''_{kl}\}, \hat{b}_{kl} := \max\{b'_{kl}, b''_{kl}\}, \check{b}_{kl} := \min\{b'_{kl}, b''_{kl}\}, \\ \bar{a}_{kl} := \max\{|a'_{kl}|, |a''_{kl}|\}, d^a_{kl} := \hat{a}_{kl} - \check{a}_{kl}, \bar{b}_{kl} := \max\{|b'_{kl}|, |b''_{kl}|\}, d^b_{kl} := \hat{b}_{kl} - \check{b}_{kl}. \end{cases} \tag{8}$$

In the literature, many investigations considered coupled MNNs that are in the form of system (4) or its similar form. Among these studies, many memristor types considered are in the form of Eq. (5), or its similar forms, see [15–17,19,21–23,26,28,40]. The approach presented in this paper can be applied to other types of memristors other than those mentioned above, such as those considered in [20,27,30,36–39]. The coupling terms $U_{i,k}(t)$ in system (4) are with a sign function sign(·). The concept of including the sign function in the coupling scheme to synchronize coupled MNNs was reported in many studies, see [15,20–22,27,29–32,37,38]. The studies in [15,20–22,37,38] considered controllers or coupling functions in the form of $U_{i,k}(t)$ in Eq. (6), with linear function h_k and without transmission delay ($\tau_T = 0$). The matrix $W := [w_{ij}]_{1\leq i,j\leq N}$ is referred to as a connection matrix. The connection from the jth network to the ith network is excitatory if $w_{ij} > 0$, and inhibitory if $w_{ij} < 0$. In the literature, the connection matrices considered for synchronization of multiple MNNs commonly satisfy the condition that all nonzero off-diagonal entries have the same signs, cf. [36–40]. It is worth noting that such a condition is not necessary for the approach presented in this paper.

It is known that MNNs, such as system (4), are differential equations with discontinuous right-hand sides, in which solutions in the conventional sense may not exist, cf. [15,19,20,28,36]. Based on the Filippov regularization approach, differential equations with discontinuous right-hand sides can be transformed into differential inclusions, cf. [1,41]. By utilizing the theories of set-valued maps of differential inclusions [41–43], many studies investigated the dynamics of MNNs under the framework of a Filippov solution [41]; for examples, see [15–17,19–23,26,28,29,32,36,40]. We denote in this paper, $Ix = \{ax : a \in I\}$, $I + J = \{a + b : a \in I, b \in J\}$, and $I - J = \{a - b : a \in I, b \in J\}$ for sets $I, J \subseteq \mathbb{R}$ and a real number $x \in \mathbb{R}$. By considering the solutions of the system (4) in the Filippov sense, system

(4) can be written as the following differential inclusions:

$$\dot{x}_{i,k}(t) \in -d_k x_{i,k}(t) + \sum_{l\in\mathcal{K}} \left[co[a_{kl}(x_{i,l}(t))] f_l(x_{i,l}(t)) + co[b_{kl}(x_{i,l}(t-\tau_l))] f_l(x_{i,l}(t-\tau_l)) \right] + I_k(t) + U_{i,k}(t) \tag{9}$$

for almost everywhere (a. e.) $t \geq t_0$ and all $(i, k) \in \mathcal{A}_x$, where each $x_{i,k}(t)$ is an absolutely continuous function on any compact interval of $[t_0, \infty)$, and

$$co\big[a_{kl}(x_{i,l}(t))\big] = \begin{cases} a'_{kl}, & |x_{i,l}(t)| < T_l, \\ a''_{kl}, & |x_{i,l}(t)| > T_l, \\ \big[\check{a}_{kl}, \hat{a}_{kl}\big], & |x_{i,l}(t)| = T_l, \end{cases} \quad co\big[b_{kl}(x_{i,l}(t-\tau_l))\big] = \begin{cases} b'_{kl}, & |x_{i,l}(t-\tau_l)| < T_l, \\ b''_{kl}, & |x_{i,l}(t-\tau_l)| > T_l, \\ \Big[\check{b}_{kl}, \hat{b}_{kl}\Big], & |x_{i,l}(t-\tau_l)| = T_l. \end{cases}$$

Notably,

$$co\big[a_{kl}(x_{i,l}(t))\big] \subseteq \big[\check{a}_{kl}, \hat{a}_{kl}\big] \subseteq [-\bar{a}_{kl}, \bar{a}_{kl}]; \;\; co\big[b_{kl}\big(x_{i,l}(t-\tau_l)\big)\big] \subseteq \Big[\check{b}_{kl}, \hat{b}_{kl}\Big] \subseteq \big[-\bar{b}_{kl}, \bar{b}_{kl}\big] \tag{10}$$

for all $t \geq t_0$, $(i, k) \in \mathcal{A}_x$, and $l \in \mathcal{K}$. We denote by $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ an arbitrary solution of Eq. (9), where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))^T$, $i \in \mathcal{N}$. System (4) is said to attain global (complete) synchronization if

$$z_{i,k}(t) := x_{i,k}(t) - x_{i+1,k}(t) \to 0 \text{ as } t \to \infty \text{ for } (i, k) \in \mathcal{A}_z$$

for every solution $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ of Eq. (9), where $\mathcal{A}_z := \{(j, l) : j \in \mathcal{N} - \{N\},\ l \in \mathcal{K}\}$ is the subscript set of $z_{i,k}$.

As discussed above, existing studies on synchronization of coupled MNNs commonly considered linear couplings or couplings consisting of a linear term and a sign function term. The coupled MNNs that were considered commonly have internal time delays within individual networks, and have no transmission delays across different networks. In addition, there exist far fewer studies on the synchronization of multiple MNNs than on master-slave MNNs. The studies that considered the synchronization of multiple MNNs required that all nonzero off-diagonal entries of the connection matrix have the same signs. To the best of our knowledge, the global synchronization of multiple MNNs (4) with a transmission delay or nonlinear h_k in coupling terms $U_{i,k}(t)$, or which admits the coexistence of excitatory and inhibitory connections is not yet established. In this paper, we develop a novel approach to establish the global synchronization of multiple MNNs (4). In the proposed approach, system (4) can involve both the internal time delays within individual networks and transmission delays across different networks. The function h_k in the coupling term $U_{i,k}(t)$ can be nonlinear. Moreover, the off-diagonal entries of connection matrix can have mixed signs; hence, the coupling scheme can admit the coexistence of excitatory and inhibitory connections. The paper is organized as follows. Section 2 provides the basis for investigating the global synchronization of system (4). In Section 2.1, analyses of the asymptotic behavior of a type of scalar differential inequality and a type of scalar differential equation are presented. We establish the global dissipativity of system (4) in Section 2.2. Herein, the global dissipativity is a property that indicates the existence of a globally attracting set for a system, cf. [10]. A system of differential equations associated with the synchronization of system (4) is derived in Section 2.3. We establish the global synchronization of system (4) in Section 3. Section 4 demonstrates the theories presented using three numerical examples. The paper is concluded in Section 5.

2. Preliminaries.

In this paper, for a real-valued function $y(t)$ and an interval $[a, b]$, we say that $y(t)\to[a, b]$ as $t\to\infty$ if $dist(y(t), [a, b])$ tends to zero as $t\to\infty$, or equivalently, $a\le\liminf_{t\to\infty}y(t)\le\limsup_{t\to\infty}y(t)\le b$, where $dist(y(t),[a,b]):=\inf\{|y(t)-\zeta|:\zeta\in[a,b]\}$ is referred to as the distance from $y(t)$ to $[a, b]$. For a function $f(t)$, to indicate the magnitude of $f(t)$ at time t subsequent to T, we set $|f|^{\max}(T):=\sup\{|f(t)|:t\ge T\}$ for $T\ge t_0$. Notably, $|f|^{\max}(T_1)\le|f|^{\max}(T_2)$ if $T_1>T_2$. We further set $|f|^{\max}(\infty):=\lim_{T\to\infty}|f|^{\max}(T)$.

2.1. A differential inequality and a differential equation.

Let us now introduce a type of differential inequality. Assume that $g^{\pm}$ are strictly decreasing real-valued functions, with $g^-(\xi)\le g^+(\xi)$ for all $\xi\in\mathbb{R}$, and have unique zeros at $q^{\pm}$. Let $u(t)$ be an absolutely continuous function on any compact interval of $[t_0, \infty)$ that satisfies

$$g^-(u(t))\le\dot{u}(t)\le g^+(u(t))\text{ for a.e. }t\ge\bar{t}\ge t_0. \tag{11}$$

Proposition 2.1. *If $u(t)$ satisfies (11), then $u(t)\to[q^-,q^+]$ as $t\to\infty$.*

Proof. Recall that g^+ is a strictly decreasing function, and has a unique zero at q^+. If $u(t)\in(q^+,\infty)$ for $t\in[t_1, t_2]$, where $t_2>t_1\ge\bar{t}$, then $g^+(u(t))<0$ for $t\in[t_1, t_2]$, and

$$u(\tilde{t}_2)-u(\tilde{t}_1)=\int_{\tilde{t}_1}^{\tilde{t}_2}\dot{u}(s)ds\le\int_{\tilde{t}_1}^{\tilde{t}_2}g^+(u(s))ds<0$$

for all $\tilde{t}_1,\tilde{t}_2\in[t_1,t_2]$ with $\tilde{t}_2>\tilde{t}_1$. Thus, $u(t)$ is strictly decreasing if $u(t)$ remains in (q^+,∞). This reveals that $(-\infty, q]$ is a positively invariant set for $u(t)$ for any $q\ge q^+$. Suppose that $u(t)\in(q^+,\infty)$ for $t=t_3\in(\bar{t},\infty)$, then for an arbitrary $\epsilon>0$ with $\epsilon<u(t_3)-q^+$, let us claim that $u(t)$ eventually enters, and then remains in $(-\infty,q^++\epsilon]$. Assume otherwise that $u(t)\in(q^++\epsilon,\infty)$ for all $t\ge t_3$. Then, it follows that

$$u(t)=u(t_3)+\int_{t_3}^{t}\dot{u}(s)ds\le u(t_3)+\int_{t_3}^{t}g^+(u(s))ds\le u(t_3)+\int_{t_3}^{t}g^+(q^++\epsilon)ds$$

for $t>t_3$, which contradicts $u(t)\in(q^++\epsilon,\infty)$ since $\int_{t_3}^{t}g^+(q^++\epsilon)ds\downarrow-\infty$ as $t\to\infty$. From the above arguments, it follows that $\limsup_{t\to\infty}u(t)\le q^+$. Applying similar arguments reveals that $\liminf_{t\to\infty}u(t)\ge q^-$. □

Next, let us introduce a type of scalar differential equation. Let $z(t)$ be an absolutely continuous function on any compact interval of $[t_0, \infty)$, and satisfy that there exist some $\bar{\tau},\tilde{q},\bar{q}\ge0$, with $0<\tilde{q}<\bar{q}$, and some $\tilde{t}_0\ge t_0+\bar{\tau}$, such that

$$z(t)\in\left[-2\bar{q},2\bar{q}\right]\text{ if }t\ge\tilde{t}_0-\bar{\tau},\quad\text{and }z(t)\to\left[-2\tilde{q},2\tilde{q}\right]\subset\left[-2\bar{q},2\bar{q}\right]\text{ as }t\to\infty. \tag{12}$$

Consider that $z(t)$ satisfies the scalar differential equation:

$$\dot{z}(t)=-\lambda\text{sign}(z(t))+\alpha(t)z(t)+\beta(t)z(t-\tau_\beta)+\gamma(t)z(t-\tau_\gamma)+w(t)+e(t) \tag{13}$$

for a. e. $t \geq t_0$, where $\tau_\beta, \tau_\gamma \in [0, \bar{\tau}]$, sign(·) is defined in Eq. (7), $w(t)$ is a bounded function, and

$$\alpha(t) \in [\check{\alpha}, \hat{\alpha}],\ \ \beta(t) \in [-\bar{\beta}, \bar{\beta}],\ \ \gamma(t) \in [-\bar{\gamma}, \bar{\gamma}],\ \ |e(t)| \leq \lambda \text{ for } t \geq \tilde{t}_0, \tag{14}$$

where $\bar{\beta}, \bar{\gamma}, \lambda \geq 0$, and $\check{\alpha}, \hat{\alpha} \in \mathbb{R}$.

Lemma 2.1. *If $z(t)$ satisfies Eq. (13), then*

$$\dot{z}(t) \leq \alpha(t)z(t) + \beta(t)z(t-\tau_\beta) + \gamma(t)z(t-\tau_\gamma) + w(t) + \begin{cases} 0, & z(t) > 0, \\ \lambda + |e|^{\max}(\tilde{t}_0), & z(t) \leq 0, \end{cases} \tag{15}$$

$$\dot{z}(t) \geq \alpha(t)z(t) + \beta(t)z(t-\tau_\beta) + \gamma(t)z(t-\tau_\gamma) + w(t) - \begin{cases} 0, & z(t) < 0, \\ \lambda + |e|^{\max}(\tilde{t}_0), & z(t) \geq 0, \end{cases} \tag{16}$$

for a. e. $t \geq \tilde{t}_0$.

Proof. Recall that $|e(t)| \leq \lambda$ for all $t \geq \tilde{t}_0$, which implies $-\lambda + |e|^{\max}(\tilde{t}_0) \leq 0$. Subsequently, as seen from Eq. (13), it can be directly observed that Eqs. (15) and (16) hold. □

Define

$$\hat{h}(\xi) := 2\bar{q}(\bar{\beta} + \bar{\gamma}) + |w|^{\max}(\tilde{t}_0) + \begin{cases} \hat{\alpha}\xi, & \xi > 0, \\ \check{\alpha}\xi + \lambda + |e|^{\max}(\tilde{t}_0), & \xi \leq 0, \end{cases}$$

$$\check{h}(\xi) := -\hat{h}(-\xi).$$

If $\hat{\alpha} < 0$ and $\bar{\beta} + \bar{\gamma} > 0$, then $\hat{h}(\xi) \geq \check{h}(\xi)$ for $\xi \in \mathbb{R}$; moreover, $\hat{h}$ and $\check{h}$ are strictly decreasing, and have unique zeros at $\hat{A}^h$ and $\check{A}^h$, respectively, where $\hat{A}^h = -\check{A}^h = [2\bar{q}(\bar{\beta} + \bar{\gamma}) + |w|^{\max}(\tilde{t}_0)]/(-\hat{\alpha}) > 0$. From $\hat{h}$ and $\check{h}$, the following lemma draws a preliminary estimate on the asymptotic dynamics for $z(t)$, which satisfies Eqs. (12)–(14).

Lemma 2.2. *Consider $z(t)$, which satisfies Eqs. (12)–(14). If $\hat{\alpha} < 0$ and $\bar{\beta} + \bar{\gamma} > 0$, then there exist some sufficiently small $\varepsilon_h > 0$ and some $T_\varepsilon > \tilde{t}_0$ such that*

$$\check{h}(z(t)) + \varepsilon_h \leq \dot{z}(t) \leq \hat{h}(z(t)) - \varepsilon_h \text{ for a. e. } t \geq T_\varepsilon. \tag{17}$$

Subsequently, there exists some $T'_\varepsilon \geq T_\varepsilon + \bar{\tau}$ such that $z(t) \in [-\hat{A}^h, \hat{A}^h]$ for all $t \geq T'_\varepsilon - \bar{\tau}$.

Proof. Recall from Eqs. (12)–(14) that $|\beta(t)| \leq \bar{\beta}$ and $|\gamma(t)| \leq \bar{\gamma}$ for all $t \geq \tilde{t}_0$, and $z(t) \to [-2\tilde{q}, 2\tilde{q}] \subset [-2\bar{q}, 2\bar{q}]$ as $t \to \infty$. In addition, $\bar{\beta} + \bar{\gamma} > 0$ implies $\bar{q}(\bar{\beta} + \bar{\gamma}) > 0$. Thus, there exist some sufficiently small $\varepsilon_h > 0$ and some $T_\varepsilon > \tilde{t}_0$ such that $|\beta(t)z(t-\tau_\beta) + \gamma(t)z(t-\tau_\gamma)| < 2\bar{q}(\bar{\beta} + \bar{\gamma}) - \varepsilon_h$ for $t \geq T_\varepsilon$. Accordingly, it follows from Lemma 2.1 that

$$\dot{z}(t) \leq 2\bar{q}(\bar{\beta} + \bar{\gamma}) + |w|^{\max}(\tilde{t}_0) - \varepsilon_h + \begin{cases} \hat{\alpha}z(t), & z(t) > 0, \\ \check{\alpha}z(t) + \lambda + |e|^{\max}(\tilde{t}_0), & z(t) \leq 0, \end{cases}$$

which yields that $\dot{z}(t) \leq \hat{h}(z(t)) - \varepsilon_h$ for a. e. $t \geq T_\varepsilon$. That $\dot{z}(t) \geq \check{h}(z(t)) + \varepsilon_h$ for a. e. $t \geq T_\varepsilon$ can be verified similarly. We hence verify Eq. (17). Applying Proposition 2.1 and Eq. (17) reveals that $z(t) \to [\check{A}^h_\varepsilon, \hat{A}^h_\varepsilon]$ as $t \to \infty$, where $\hat{A}^h_\varepsilon$ (resp., $\check{A}^h_\varepsilon$) is the unique solution of $\hat{h}(\cdot) - \varepsilon_h = 0$ (resp., $\check{h}(\cdot) + \varepsilon_h = 0$). Recall that $[\check{A}^h_\varepsilon, \hat{A}^h_\varepsilon] \subset [-\hat{A}^h, \hat{A}^h]$. Thus, there exists some $T'_\varepsilon \geq T_\varepsilon + \bar{\tau}$ such that $z(t) \in [-\hat{A}^h, \hat{A}^h]$ for all $t \geq T'_\varepsilon - \bar{\tau}$. □

Let us introduce the following condition:

Condition (S_0): $-\hat{\alpha} - \bar{\beta} - \bar{\gamma} - |w|^{\max}(\tilde{t}_0)/(2\bar{q}) > 0$.

We note that conditions (S_0) and $\bar{\beta} + \bar{\gamma} > 0$ imply $\hat{\alpha} < 0$ and $(\bar{\beta} + \bar{\gamma})\hat{A}^h < 2\bar{q}(\bar{\beta} + \bar{\gamma})$; hence, there exists an $\varepsilon_0 > 0$ with $\varepsilon_0 < \varepsilon_h$ such that

$$(\bar{\beta} + \bar{\gamma})\hat{A}^h + \varepsilon_0 < 2\bar{q}(\bar{\beta} + \bar{\gamma}). \tag{18}$$

For each $T \geq t_0$, we introduce the following functions:

$$\hat{h}^{(0)}(\xi, T) := (\bar{\beta} + \bar{\gamma})\hat{A}^h + |w|^{\max}(T) + \varepsilon_0 + \begin{cases} \hat{\alpha}\xi, & \xi > 0, \\ \check{\alpha}\xi + \lambda + |e|^{\max}(\tilde{t}_0), & \xi \leq 0, \end{cases}$$

$$\check{h}^{(0)}(\xi, T) := -\hat{h}^{(0)}(-\xi, T).$$

From Eq. (18) and $|w|^{\max}(T) \leq |w|^{\max}(\tilde{t}_0)$ for $T \geq \tilde{t}_0$, it follows that

$$\check{h}(\xi) < \check{h}^{(0)}(\xi, T) < \hat{h}^{(0)}(\xi, T) < \hat{h}(\xi) \tag{19}$$

for all $\xi \in \mathbb{R}$ and $T \geq \tilde{t}_0$, under the conditions (S_0) and $\bar{\beta} + \bar{\gamma} > 0$. Let $\check{p}^{(0)}(T)$ (resp., $\hat{p}^{(0)}(T)$) be the unique solution of $\check{h}^{(0)}(\cdot, T) = 0$ (resp., $\hat{h}^{(0)}(\cdot, T) = 0$). Notably, $\check{p}^{(0)}(T) = -\hat{p}^{(0)}(T) < 0$ and $[-\hat{p}^{(0)}(T), \hat{p}^{(0)}(T)] \subset [-\hat{A}^h, \hat{A}^h]$ for all $T \geq \tilde{t}_0$.

Lemma 2.2 shows a preliminary estimate on the asymptotic dynamics for $z(t)$ based on Eq. (17). Therein, Eq. (17) refers to an estimation on $\dot{z}(t)$ via the lower bound $\check{h}(\cdot) + \varepsilon_h$ and upper bound $\hat{h}(\cdot) - \varepsilon_h$. Based on Eqs. (19) and (20), the following lemma reveals that $\check{h}^{(0)}(\cdot, T) + \varepsilon_0$ and $\hat{h}^{(0)}(\cdot, T) - \varepsilon_0$ provide lower and upper bounds finer than $\check{h}(\cdot) + \varepsilon_h$ and $\hat{h}(\cdot) - \varepsilon_h$, respectively, for the dynamics of $z(t)$, as time increases.

Lemma 2.3. *Consider $z(t)$, which satisfies Eqs. (12)–(14). If conditions (S_0) and $\bar{\beta} + \bar{\gamma} > 0$ hold, then for an arbitrary fixed $T \geq T'_\varepsilon$, there exists some $T_0 \geq T$ such that*

$$\check{h}^{(0)}(z(t), T) + \varepsilon_0 \leq \dot{z}(t) \leq \hat{h}^{(0)}(z(t), T) - \varepsilon_0 \tag{20}$$

for a. e. $t \geq T_0$. Hence, there exists some $T_1 \geq T_0 + \bar{\tau}$ such that $z(t) \in [-\hat{p}^{(0)}(T), \hat{p}^{(0)}(T)]$ for all $t \geq T_1 - \bar{\tau}$. Herein, T'_ε and ε_0 are introduced in Lemma 2.2 and Eq. (18), respectively.

Proof. Fix an arbitrary $T \geq T'_\varepsilon$. By Lemma 2.2, we obtain that $z(s) \in [-\hat{A}^h, \hat{A}^h]$ for all $s \geq T_0 - \bar{\tau}$ if $T_0 \geq T$. Consequently, applying Lemma 2.1 yields that

$$\dot{z}(t) \leq (\bar{\beta} + \bar{\gamma})\hat{A}^h + |w|^{\max}(T) + \begin{cases} \hat{\alpha}z(t), & z(t) > 0, \\ \check{\alpha}z(t) + \lambda + |e|^{\max}(\tilde{t}_0), & z(t) \leq 0, \end{cases}$$

which yields $\dot{z}(t) \leq \hat{h}^{(0)}(z(t), T) - \varepsilon_0$ for a. e. $t \geq T_0$. Similarly, $\dot{z}(t) \geq \check{h}^{(0)}(z(t), T) + \varepsilon_0$ for a. e. $t \geq T_0$ can be verified. Thus, there exists some $T_0 \geq T$ such that Eq. (20) holds. Applying Proposition 2.1 and Eq. (20) yields that $z(t) \to [\check{p}_\varepsilon^{(0)}(T), \hat{p}_\varepsilon^{(0)}(T)]$ as $t \to \infty$, where $\check{p}_\varepsilon^{(0)}(T)$ (resp., $\hat{p}_\varepsilon^{(0)}(T)$) is the unique solution of $\check{h}^{(0)}(\cdot, T) + \varepsilon_0 = 0$ (resp., $\hat{h}^{(0)}(\cdot, T) - \varepsilon_0 = 0$). Notably, $[\check{p}_\varepsilon^{(0)}(T), \hat{p}_\varepsilon^{(0)}(T)] \subset [-\hat{p}^{(0)}(T), \hat{p}^{(0)}(T)]$. Thus, there exists some $T_1 \geq T_0 + \bar{\tau}$ such that $z(t) \in [-\hat{p}^{(0)}(T), \hat{p}^{(0)}(T)]$ for $t \geq T_1 - \bar{\tau}$. □

Lemmas 2.2 and 2.3 demonstrate the formulation of lower and upper bounds for the dynamics of $z(t)$, which satisfies Eqs. (12)–(14) in succession. In the same spirit, we shall formulate finer lower and upper bounds iteratively to capture the asymptotic dynamics of $z(t)$. Now, let us set a decreasing sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_1 < \varepsilon_0$ and that $\varepsilon_k \to 0$ as $k \to \infty$. We then define the following functions iteratively. For $k \in \mathbb{N}$ and $T \geq t_0$,

$$\hat{h}^{(k)}(\xi, T) := (\bar{\beta} + \bar{\gamma})\hat{p}^{(k-1)}(T) + |w|^{\max}(T) + \varepsilon_k + \begin{cases} \hat{\alpha}\xi, & \xi > 0, \\ \check{\alpha}\xi + \lambda + |e|^{\max}(\tilde{t}_0), & \xi \leq 0, \end{cases}$$

$$\check{h}^{(k)}(\xi, T) := -\hat{h}^{(k)}(-\xi, T),$$

where $\hat{p}^{(k-1)}(T) > 0$ is the unique solution of $\hat{h}^{(k-1)}(\cdot, T) = 0$. Notably, $\check{p}^{(k)}(T) := -\hat{p}^{(k)}(T)$ is the unique solution of $\check{h}^{(k)}(\cdot, T) = 0$. Functions $\hat{h}^{(k)}(\cdot, T)$ and $\check{h}^{(k)}(\cdot, T)$ are formulated to provide more delicate control on the dynamics of $z(t)$ as k or T increases. More precisely, $\hat{h}^{(k)}(\cdot, T)$ (resp. $\check{h}^{(k)}(\cdot, T)$) decreases (resp. increases) with respect to k and T; accordingly, $[\check{p}^{(k)}(T), \hat{p}^{(k)}(T)] = [-\hat{p}^{(k)}(T), \hat{p}^{(k)}(T)]$ shrinks to some interval, say $[-\bar{p}, \bar{p}]$, as $k \to \infty$ and $T \to \infty$. Moreover, it can be shown that for each $T \geq T'_\varepsilon$ and $k \in \mathbb{N}$, $z(t) \to [-\hat{p}^{(k)}(T), \hat{p}^{(k)}(T)]$ as $t \to \infty$. Consequently, $z(t) \to [-\bar{p}, \bar{p}]$ as $t \to \infty$. We summarize these properties in the following lemma and proposition.

Lemma 2.4. *Assume that conditions (S_0) and $\bar{\beta} + \bar{\gamma} > 0$ hold. Then, for each $T \geq t_0$, the sequences $\{\hat{p}^{(k)}(T)\}_{k \geq 0}$ can be defined iteratively, where $\hat{p}^{(k)}(T) > 0$. Moreover, (i) for any fixed $k \in \mathbb{N} \cup \{0\}$, $\hat{p}^{(k)}(T)$ is decreasing with respect to $T \geq t_0$; (ii) for any $T \geq t_0$, there exists $p(T) \geq 0$ such that $\hat{p}^{(k)}(T) \to p(T)$ decreasingly as $k \to \infty$; (iii) there exists $\bar{p} \geq 0$ such that $p(T) \to \bar{p}$ decreasingly as $T \to \infty$; (iv) $0 \leq p(T) \leq |w|^{\max}(T)/(-\hat{\alpha} - \bar{\beta} - \bar{\gamma})$ for any $T \geq t_0$; (v) $\cap_{T \geq t_0}[-p(T), p(T)] = [-\bar{p}, \bar{p}]$, and $0 \leq \bar{p} \leq |w|^{\max}(\infty)/(-\hat{\alpha} - \bar{\beta} - \bar{\gamma})$.*

Proof. Assume that condition (S_0) holds. By mathematical induction and Eq. (19), we can show that

$$\hat{h}^{(k)}(\xi, T) \leq \hat{h}^{(k-1)}(\xi, T) < \hat{h}(\xi) \text{ for all } \xi \in \mathbb{R}, \ k \in \mathbb{N}, \text{ and } T \geq t_0. \tag{21}$$

As $\hat{h}^{(k)}(\cdot, T)$ is the vertical shift of $\hat{h}(\cdot)$ and $\lim_{\xi \to 0^+} \hat{h}^{(k)}(\xi, T) \geq \varepsilon_k > 0$, $\hat{p}^{(k)}(T)$ is well defined, with $0 < \hat{p}^{(k)}(T) < \hat{A}^h$, for all $k \in \mathbb{N} \cup \{0\}$ and $T \geq t_0$ recalling Eq. (21). Recall that the term $|w|^{\max}(T)$ in $\hat{h}^{(k)}(\cdot, T)$ satisfies

$$|w|^{\max}(T_1) \leq |w|^{\max}(T_2) \text{ if } T_1 \geq T_2. \tag{22}$$

By Eqs. (21) and (22), $\hat{p}^{(k)}(T)$ is decreasing with respect to both T and k, which yields assertions (i)–(iii). Obviously, assertion (v) follows from assertions (iii) and (iv).

Next, let us verify assertion (iv) to complete the proof. It is obvious that $0 \leq p(T) < \hat{A}^h$ for $T \geq t_0$. Below, let us assume that $p(T) > 0$ since assertion (iv) holds if $p(T) = 0$. For any fixed $T \geq t_0$, $\{\hat{h}^{(k)}(\cdot, T)|_{\mathcal{I}_n}\}_{k \geq 1}$ and $\{\hat{h}^{(k)}(\cdot, T)|_{\mathcal{I}_p}\}_{k \geq 1}$, where $\mathcal{I}_n := [-\hat{A}^h, -p(T)]$ and $\mathcal{I}_p := [p(T), \hat{A}^h]$, are uniformly bounded and equicontinuous since $|\hat{h}^{(k)}(\xi, T)| \leq \hat{h}(\check{A}^h)$ and $(\hat{h}^{(k)})'(\xi, T) \leq |\check{\alpha}|$ for $\xi \in \mathcal{I}_n \cup \mathcal{I}_p$ and $k \in \mathbb{N}$. In addition, $\hat{h}^{(k)}(\cdot, T)$ decreases with respect to k. Applying the Ascoli-Arzela Theorem, there exists a continuous function $\hat{h}^{(\infty)}(\cdot, T)$ defined on $\mathcal{I}_n \cup \mathcal{I}_p$ such that

$$\hat{h}^{(k)}(\cdot, T) \downarrow \hat{h}^{(\infty)}(\cdot, T) \text{ uniformly on } \mathcal{I}_n \cup \mathcal{I}_p \tag{23}$$

as $k \to \infty$. With Eq. (23), $\hat{p}^{(k)}(T) \to p(T)$, the continuity of $\hat{h}^{(k)}(\cdot, T)|_{\mathcal{I}_n \cup \mathcal{I}_p}$ and $\hat{h}^{(\infty)}(\cdot)$, and the fact that $\varepsilon_k \to 0$ as $k \to \infty$, we can derive the following properties for $\hat{h}^{(\infty)}(\cdot, T)$:

(P1): $\hat{h}^{(\infty)}(\xi, T) = (\bar{\beta} + \bar{\gamma})p(T) + |w|^{\max}(T) + \begin{cases} \hat{\alpha}\xi, & \xi \in \mathcal{I}_p, \\ \check{\alpha}\xi + \lambda + |e|^{\max}(\tilde{t}_0), & \xi \in \mathcal{I}_n, \end{cases}$

(P2): $\hat{h}^{(\infty)}(p(T), T) = 0.$

Applying (P1) and (P2) yields $p(T) = |w|^{\max}(T)/(-\hat{\alpha} - \bar{\beta} - \bar{\gamma})$. We have justified assertion (iv). □

Proposition 2.2. *Consider* $z(t)$, *which satisfies Eqs. (12)–(14). If condition* (S_0) *holds, then* $z(t) \to [-\bar{p}, \bar{p}]$ as $t \to \infty$, *where* $0 \le \bar{p} \le |w|^{\max}(\infty)/(-\hat{\alpha} - \bar{\beta} - \bar{\gamma})$.

Proof. Let us first consider the case $\bar{\beta} + \bar{\gamma} > 0$. By the spirit in concluding Lemma 2.3, we can verify by induction that for an arbitrary fixed $T \ge T'_\varepsilon$ and $n \in \mathbb{N}$, there exists an increasing sequence $\{T_k\}_{k=0}^n$, with $T_{k+1} \ge T_k + \bar{\tau}$ for $k = 0, 1, \ldots, n-1$ and $T_0 \ge T$, such that

$$\begin{cases} \check{h}^{(k)}(z(t), T) + \varepsilon_k \le \dot{z}(t) \le \hat{h}^{(k)}(z(t), T) - \varepsilon_k \text{ for a. e. } t \ge T_k, \ k = 0, 1, \ldots, n-1, \\ z(t) \in \left[-\hat{p}^{(k)}(T), \hat{p}^{(k)}(T)\right] \text{ for all } t \ge T_{k+1} - \bar{\tau}, \ k = 0, 1, \ldots, n-1. \end{cases} \tag{24}$$

Recall from Lemma 2.4 that $\hat{p}^{(k)}(T) \to p(T)$ decreasingly as $k \to \infty$, and $p(T) \to \bar{p}$ decreasingly as $T \to \infty$, where $0 \le \bar{p} \le |w|^{\max}(\infty)/(-\hat{\alpha} - \bar{\beta} - \bar{\gamma})$. Based on Eq. (24), we obtain that for $T \ge T'_\varepsilon$, $z(t) \to [-p(T), p(T)]$ as $t \to \infty$. Thus, $z(t) \to [-\bar{p}, \bar{p}]$ as $t \to \infty$.

If $\bar{\beta} + \bar{\gamma} = 0$, then $\beta(t) = \gamma(t) \equiv 0$, which yields that for $T \ge \tilde{t}_0$,

$$\check{g}(z(t), T) \le \dot{z}(t) \le \hat{g}(z(t), T) \tag{25}$$

for a. e. $t \ge T$, recalling Lemma 2.1, where

$$\hat{g}(\xi, T) := |w|^{\max}(T) + \begin{cases} \hat{\alpha}\xi, & \xi > 0, \\ \check{\alpha}\xi + \lambda + |e|^{\max}(\tilde{t}_0), & \xi \le 0, \end{cases} \qquad \check{g}(\xi, T) := -\hat{g}(-\xi, T).$$

Applying Proposition 2.1 and Eq. (25) reveals that $z(t) \to [-p(T), p(T)]$ as $t \to \infty$ for $T \ge \tilde{t}_0$, where $p(T) = |w|^{\max}(T)/(-\hat{\alpha}) \ge 0$. Notably, $p(T)$ is decreasing with respect to T. It follows that $z(t) \to [-\bar{p}, \bar{p}]$ as $t \to \infty$, where $0 \le \bar{p} = \lim_{T \to \infty} p(T) \le |w|^{\max}(\infty)/(-\hat{\alpha} - \bar{\beta} - \bar{\gamma})$. Thus, the proof is completed. □

2.2. Global dissipativity of system (4).

We first consider the following condition:

Condition (S1)*: For each $l \in \mathcal{K}$, there exist some $\bar{\rho}_l^f, \bar{\rho}_l^h \ge 0$ such that $|f_l(\xi)| \le \bar{\rho}_l^f$ and $|h_l(\xi)| \le \bar{\rho}_l^h$ for all $\xi \in \mathbb{R}$.

With $\bar{\rho}_l^f$ and $\bar{\rho}_l^h$ in condition (S1)*, we successively define

$$\bar{q}_k := \left[\bar{I}_k + \gamma_k \bar{\varpi}\left(2\bar{\rho}_k^h + \bar{s}\right) + \sum_{l \in \mathcal{K}} \left(\bar{a}_{kl} + \bar{b}_{kl}\right)\bar{\rho}_l^f\right] / d_k,$$
$$\tilde{q}_k := \left[\bar{I}_k + \gamma_k \bar{\varpi}\left(2\tilde{\rho}_k^h + \bar{s}\right) + \sum_{l \in \mathcal{K}} \left(\bar{a}_{kl} + \bar{b}_{kl}\right)\tilde{\rho}_l^f\right] / d_k, \tag{26}$$

$$\hat{H}_k := \sup\left\{h'_k(\xi) : \xi \in \left[-\bar{q}_k, \bar{q}_k\right]\right\} \ge 0, \quad \check{H}_k := \inf\left\{h'_k(\xi) : \xi \in \left[-\bar{q}_k, \bar{q}_k\right]\right\} \ge 0 \tag{27}$$

for $k \in \mathcal{K}$, where

$$\bar{\varpi} := \max\left\{\sum_{j \in \mathcal{N}} |w_{ij}| : i \in \mathcal{N}\right\}, \tag{28}$$

$$\tilde{\rho}_l^h := \max\left\{|h_l(\xi)| : \xi \in \left[-\tilde{q}_l, \tilde{q}_l\right]\right\} \le \bar{\rho}_l^h, \quad \tilde{\rho}_l^f := \max\left\{|f_l(\xi)| : \xi \in \left[-\tilde{q}_l, \tilde{q}_l\right]\right\} \le \bar{\rho}_l^f. \tag{29}$$

Obviously, $\tilde{q}_k \le \bar{q}_k$ for $k \in \mathcal{K}$. Let us consider the following condition, which is slightly stronger than (S1)*:

Condition (S1): Condition (S1)* holds, and $\tilde{q}_k < \bar{q}_k$ for all $k \in \mathcal{K}$.

The following proposition provides an estimated globally attracting region for system (4).

Proposition 2.3. *Assume that condition (S1) holds. If* $\mathbf{X}(t) = (\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ *is a solution of system (9), where* $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))^T$, $i \in \mathcal{N}$, *then*

$x_{i,k}(t) \to \left[-\tilde{q}_k, \tilde{q}_k\right] \subset \left[-\bar{q}_k, \bar{q}_k\right]$ as $t \to \infty$

for all $(i, k) \in \mathcal{A}_x$. *Thus, there exists some* $\tilde{t}_0 \ge t_0 + \tau_M$ *such that* $x_{i,k}(t) \in [-\bar{q}_k, \bar{q}_k]$ for all $t \ge \tilde{t}_0 - \tau_M$ and $(i, k) \in \mathcal{A}_x$, *where* $\tau_M := \max\{\max\{\tau_l : l \in \mathcal{K}\}, \tau_T\}$.

Proof. Under condition (S1), $\mathbf{X}(t)$ satisfies

$$\begin{cases} co\left[a_{kl}(x_{i,l}(t))\right]f_l\left(x_{i,l}(t)\right) \subseteq \left[-\bar{a}_{kl}\bar{\rho}_l^f, \bar{a}_{kl}\bar{\rho}_l^f\right], co\left[b_{kl}\left(x_{i,l}(t-\tau_l)\right)\right]f_l\left(x_{i,l}(t-\tau_l)\right) \\ \quad \subseteq \left[-\bar{b}_{kl}\bar{\rho}_l^f, \bar{b}_{kl}\bar{\rho}_l^f\right], \\ |U_{i,k}(t)| \le \gamma_k\varpi\left(2\bar{\rho}_k^h + \bar{s}\right) \end{cases} \tag{30}$$

for $t \ge t_0$, $(i, k) \in \mathcal{A}_x$, and $l \in \mathcal{K}$ based on Eqs. (6), (7), (10), and (28). Applying Eqs. (3) and (30) reveals that $x_{i,k}(t)$, where $(i, k) \in \mathcal{A}_x$, satisfies

$$g_k^-(x_{i,k}(t)) \le \dot{x}_{i,k}(t) \le g_k^+(x_{i,k}(t)) \text{ for a. e. } t \ge t_0, \tag{31}$$

where $g_k^{\pm}(\xi) := -d_k\xi \pm [\bar{I}_k + \gamma_k\varpi(2\bar{\rho}_k^h + \bar{s}) + \sum_{l\in\mathcal{K}}(\bar{a}_{kl} + \bar{b}_{kl})\bar{\rho}_l^f]$. The functions $g_k^{\pm}$ are strictly decreasing and have unique zeros at $\pm\bar{q}_k$. Applying Proposition 2.1 and Eq. (31) yields

$$x_{i,k}(t) \to \left[-\bar{q}_k, \bar{q}_k\right] \text{ as } t \to \infty \tag{32}$$

for all $(i, k) \in \mathcal{A}_x$. From Eqs. (29) and (32), it follows that

$$f_k(x_{i,k}(t)) \to \left[-\tilde{\rho}_k^f, \tilde{\rho}_k^f\right] \text{ and } h_k(x_{i,k}(t)) \to \left[-\tilde{\rho}_k^h, \tilde{\rho}_k^h\right] \text{ as } t \to \infty \tag{33}$$

for all $(i, k) \in \mathcal{A}_x$, which yields that for an arbitrary $\epsilon > 0$, there exists some $t_\epsilon \ge t_0$ such that

$$\tilde{g}_k^-(x_{i,k}(t)) - \epsilon \le \dot{x}_{i,k}(t) \le \tilde{g}_k^+(x_{i,k}(t)) + \epsilon \tag{34}$$

for a. e. $t \ge t_\epsilon$ and all $(i, k) \in \mathcal{A}_x$, where $\tilde{g}_k^{\pm}(\xi) := -d_k\xi \pm [\bar{I}_k + \gamma_k\varpi(2\tilde{\rho}_k^h + \bar{s}) + \sum_{l\in\mathcal{K}}(\bar{a}_{kl} + \bar{b}_{kl})\tilde{\rho}_l^f]$ based on Eqs. (3), (6)–(10), (28), and (33). The functions $\tilde{g}_k^{\pm}$ are strictly decreasing and have unique zeros at $\pm\tilde{q}_k$. Applying Proposition 2.1, Eq. (34), and the condition (S1), we get $x_{i,k}(t) \to [-\tilde{q}_k, \tilde{q}_k] \subset [-\bar{q}_k, \bar{q}_k]$ as $t \to \infty$ for all $(i, k) \in \mathcal{A}_x$. □

2.3. Differential equations associated with the synchronization of system (4).

Throughout this subsection, suppose that $\mathbf{X}(t) = (\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ is an arbitrary solution of system (9), where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))^T$, $i \in \mathcal{N}$; then set

$$\Delta_{i,k}^f(t) := f_k(x_{i,k}(t)) - f_k(x_{i+1,k}(t)), \quad \Delta_{i,k}^h(t) := h_k(x_{i,k}(t)) - h_k(x_{i+1,k}(t)), \tag{35}$$

and

$$\Delta^a_{i,kl}(t) := co\big[a_{kl}(x_{i,l}(t))\big] - co\big[a_{kl}(x_{i+1,l}(t))\big], \quad \Delta^b_{i,kl}(t) := co\big[b_{kl}(x_{i,l}(t))\big] - co\big[b_{kl}(x_{i+1,l}(t))\big] \tag{36}$$

for $t \geq t_0 - \tau_M$, $i \in \mathcal{N} - \{N\}$, and $k, l \in \mathcal{K}$, where τ_M is defined in Proposition 2.3. Consider $\mathbf{Z}(t) = (\mathbf{z}_1(t), \dots, \mathbf{z}_{N-1}(t))^T$, where $\mathbf{z}_i(t) = (z_{i,1}(t), \dots, z_{i,K}(t))^T := \mathbf{x}_i(t) - \mathbf{x}_{i+1}(t)$, $i = 1, \dots, N-1$. Then $\mathbf{Z}(t)$ satisfies the following differential inclusions:

$$\begin{aligned}\dot{z}_{i,k}(t) \in & -d_k z_{i,k}(t) + U_{i,k}(t) - U_{i+1,k}(t) + \sum_{l\in\mathcal{K}} co\big[a_{kl}(x_{i,l}(t))\big] f_l\big(x_{i,l}(t)\big) \\ & - \sum_{l\in\mathcal{K}} co\big[a_{kl}\big(x_{i+1,l}(t)\big)\big] f_l\big(x_{i+1,l}(t)\big) \\ + & \sum_{l\in\mathcal{K}} co\big[b_{kl}\big(x_{i,l}(t-\tau_l)\big)\big] f_l\big(x_{i,l}(t-\tau_l)\big) - \sum_{l\in\mathcal{K}} co\big[b_{kl}\big(x_{i+1,l}(t-\tau_l)\big)\big] f_l\big(x_{i+1,l}(t-\tau_l)\big)\end{aligned} \tag{37}$$

for $t \geq t_0$ and $(i, k) \in \mathcal{A}_z$. Notably, system (4) attains global synchronization if $z_{i,k}(t) \to 0$ as $t \to \infty$ for all $(i, k) \in \mathcal{A}_z$. Define a synchronous set corresponding to system (4) as

$$\mathcal{S} := \big\{(\Phi, \dots, \Phi)^T \in C\big([-\tau_M, 0], \mathbb{R}^{NK}\big) : \Phi \in C\big([-\tau_M, 0], \mathbb{R}^K\big)\big\}, \tag{38}$$

where τ_M is defined in Proposition 2.3. For a coupled system, the invariance of the synchronous set under the evolution generated by the system is a prerequisite to the synchronization of the system. We note that the invariance of $\mathcal{S}$ is guaranteed if system (4) is without a transmission delay ($\tau_T = 0$); however, the invariance may be lost if system (4) has a transmission delay, cf. Example 2 in Section 4. We ensure the invariance of $\mathcal{S}$ for system (4) with transmission delay by imposing the condition:

$$\chi_i := \sum_{j\in\mathcal{N}} w_{ij} = \chi \text{ for all } i \in \mathcal{N}. \tag{39}$$

Thus, this paper considers system (4), which satisfies the condition:

Condition (S2): Eq. (39) holds if $\tau_T \neq 0$.

Now, let us explain the main idea of our approach to establish the global synchronization of system (4). First, we show that $\mathbf{Z}(t)$ satisfies differential equations of the form:

$$\begin{aligned}\dot{z}_{i,k}(t) = & -\lambda_{i,k}\mathrm{sign}\big(z_{i,k}(t)\big) + \alpha_{i,k}(t) z_{i,k}(t) + \beta_{i,k}(t) z_{i,k}(t-\tau_k) \\ & + \gamma_{i,k}(t) z_{i,k}(t-\tau_T) + w_{i,k}(t) + e_{i,k}(t)\end{aligned} \tag{40}$$

for a. e. $t \geq t_0$ and $(i, k) \in \mathcal{A}_z$, where

$$\lambda_{i,k} := \bar{s}\gamma_k\big[w_{i(i+1)} + w_{(i+1)i}\big] \tag{41}$$

with γ_k, $w_{i(i+1)}$, and $w_{(i+1)i}$ defined in Eq. (6) and $\bar{s}$ in Eq. (7). The precise formulations and properties of $\alpha_{i,k}(t)$, $\beta_{i,k}(t)$, $\gamma_{i,k}(t)$, $w_{i,k}(t)$, and $e_{i,k}(t)$ are derived through Lemmas 2.5 and 2.6. Notably, each component in Eq. (40) takes the form Eq. (13). Then, applying Proposition 2.2 yields that for each $(i, k) \in \mathcal{A}_z$, there exists some $\bar{p}_{i,k} \geq 0$ such that $z_{i,k}(t) \to [-\bar{p}_{i,k}, \bar{p}_{i,k}]$ as $t \to \infty$. In Proposition 3.1 and Theorem 3.1, we perform an iterative argument to establish that $\bar{p}_{i,k} = 0$, which yields that $z_{i,k}(t) \to 0$ as $t \to \infty$ for all $(i, k) \in \mathcal{A}_z$; hence, system (4) attains global synchronization.

Using the connection matrix $W = [w_{ij}]_{1\le i,j\le N}$ and χ_i defined in Eq. (39), we define $\tilde{W} = [\tilde{w}_{ij}]_{1\le i,j\le N}$, where

$$\tilde{w}_{ij} = \begin{cases} w_{ii} - \chi_i, & i = j \in \mathcal{N}, \\ w_{ij}, & i, j \in \mathcal{N} \text{ and } i \neq j. \end{cases} \tag{42}$$

We then define the matrix $\bar{W}$ derived from $\tilde{W}$ as follows:

$$\bar{W} = [\bar{w}_{ij}]_{1\le i,j\le N-1} := \mathbf{C}\tilde{W}\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}, \tag{43}$$

where

$$\mathbf{C} := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times N}.$$

Referring to [44], $\bar{W}$ is well defined, and satisfies $\mathbf{C}\tilde{W} = \bar{W}\mathbf{C}$, which yields

$$\sum_{j\in\mathcal{N}} [\tilde{w}_{ij} - \tilde{w}_{(i+1)j}] h_k(x_{j,k}(t)) = \sum_{j\in\mathcal{N}-\{N\}} \bar{w}_{ij}\Delta^h_{j,k}(t) \tag{44}$$

for all $(i, k) \in \mathcal{A}_z$ and $t \geq t_0 - \tau_M$, based on Eq. (35). From Eq. (44), we derive the following lemma.

Lemma 2.5. *If condition (S2) holds, then* $\mathbf{Z}(t)$ *satisfies*

$$\dot{z}_{i,k}(t) \in -\lambda_{i,k}\text{sign}(z_{i,k}(t)) + I^\alpha_{i,k}(t) + I^\beta_{i,k}(t) + I^\gamma_{i,k}(t) + I^w_{i,k}(t) + I^e_{i,k}(t) \tag{45}$$

for a. e. $t \geq t_0$ *and* $(i, k) \in \mathcal{A}_z$, *where* $\lambda_{i,k}$ *is defined in Eq. (41),*

$$I^\alpha_{i,k}(t) = -d_k z_{i,k}(t) + co[a_{kk}(x_{i,k}(t))]\Delta^f_{i,k}(t) + \begin{cases} \bar{w}_{ii}\gamma_k\Delta^h_{i,k}(t), & \tau_T = 0, \\ -\chi\gamma_k\Delta^h_{i,k}(t), & \tau_T \neq 0, \end{cases}$$

$$I^\beta_{i,k}(t) = co[b_{kk}(x_{i,k}(t-\tau_k))]\Delta^f_{i,k}(t-\tau_k),$$

$$I^\gamma_{i,k}(t) = \begin{cases} 0, & \tau_T = 0, \\ \gamma_k(\bar{w}_{ii} + \chi)\Delta^h_{i,k}(t-\tau_T), & \tau_T \neq 0, \end{cases}$$

$$I^w_{i,k}(t) = \sum_{l\in\mathcal{K}-\{k\}} co[a_{kl}(x_{i,l}(t))]\Delta^f_{i,l}(t) + \sum_{l\in\mathcal{K}-\{k\}} co[b_{kl}(x_{i,l}(t-\tau_l))]\Delta^f_{i,l}(t-\tau_l)$$
$$+ \gamma_k \sum_{j\in\mathcal{N}-\{i,N\}} \bar{w}_{ij}\Delta^h_{j,k}(t-\tau_T),$$

$$I^e_{i,k}(t) = \sum_{l\in\mathcal{K}} \Delta^a_{i,kl}(t) f_l(x_{i+1,l}(t)) + \sum_{l\in\mathcal{K}} \Delta^b_{i,kl}(t-\tau_l) f_l(x_{i+1,l}(t-\tau_l))$$
$$+ \gamma_k \sum_{j\in\mathcal{N}-\{i,i+1\}} [w_{ij}s(x_{j,k}(t) - x_{i,k}(t)) - w_{(i+1)j}s(x_{j,k}(t) - x_{i+1,k}(t))].$$

Proof. As seen from Eq. (6) and $\chi_i = \sum_{j\in\mathcal{N}} w_{ij}$ in Eq. (39), we observe that

$$U_{i,k}(t) = \gamma_k \sum_{j\in\mathcal{N}} w_{ij}[h_k(x_{j,k}(t-\tau_T)) + s(x_{j,k}(t) - x_{i,k}(t))] - \gamma_k\chi_i h_k(x_{i,k}(t))$$

$$= \gamma_k(w_{ii}-\chi_i)h_k\big(x_{i,k}(t-\tau_T)\big)+\gamma_k\sum_{j\in\mathcal{N}-\{i\}}w_{ij}h_k\big(x_{j,k}(t-\tau_T)\big)$$
$$+\gamma_k\sum_{j\in\mathcal{N}}w_{ij}s\big(x_{j,k}(t)-x_{i,k}(t)\big)-\gamma_k\chi_i\big[h_k\big(x_{i,k}(t)\big)-h_k\big(x_{i,k}(t-\tau_T)\big)\big]$$

for all $(i,k)\in\mathcal{A}_x$ and $t\geq t_0$. This yields that

$$U_{i,k}(t)=\gamma_k\sum_{j\in\mathcal{N}}\tilde{w}_{ij}h_k\big(x_{j,k}(t-\tau_T)\big)+\gamma_k\sum_{j\in\mathcal{N}}w_{ij}s\big(x_{j,k}(t)-x_{i,k}(t)\big)$$
$$-\gamma_k\chi_i\big[h_k\big(x_{i,k}(t)\big)-h_k\big(x_{i,k}(t-\tau_T)\big)\big] \tag{46}$$

for all $(i,k)\in\mathcal{A}_x$ and $t\geq t_0$, recalling that $\tilde{w}_{ij}=w_{ij}$ if $i\neq j$ and $\tilde{w}_{ii}=w_{ii}-\chi_i$, cf. (42).

Assume that condition (S2) holds. If $\tau_T=0$, it follows from Eq. (46) that

$$\begin{aligned}
&U_{i,k}(t)-U_{i+1,k}(t)\\
&\quad=\gamma_k\sum_{j\in\mathcal{N}}\big[\tilde{w}_{ij}-\tilde{w}_{(i+1)j}\big]h_k\big(x_{j,k}(t)\big)\\
&\quad+\gamma_k\sum_{j\in\mathcal{N}}\big[w_{ij}s\big(x_{j,k}(t)-x_{i,k}(t)\big)-w_{(i+1)j}s\big(x_{j,k}(t)-x_{i+1,k}(t)\big)\big]\\
&\quad=\gamma_k\sum_{j\in\mathcal{N}-\{N\}}\bar{w}_{ij}\Delta^h_{j,k}(t)+\gamma_k\sum_{j\in\mathcal{N}}\big[w_{ij}s\big(x_{j,k}(t)-x_{i,k}(t)\big)-w_{(i+1)j}s\big(x_{j,k}(t)-x_{i+1,k}(t)\big)\big],
\end{aligned} \tag{47}$$

where the latter equality in Eq. (47) holds due to Eq. (44).

If $\tau_T\neq 0$ and Eq. (39) holds, using Eqs. (35) and (46) reveals

$$U_{i,k}(t)-U_{i+1,k}(t)=-\chi\gamma_k\big[\Delta^h_{i,k}(t)-\Delta^h_{i,k}(t-\tau_T)\big]+\gamma_k\sum_{j\in\mathcal{N}}\big[\tilde{w}_{ij}-\tilde{w}_{(i+1)j}\big]h_k\big(x_{j,k}(t-\tau_T)\big)$$
$$+\gamma_k\sum_{j\in\mathcal{N}}\big[w_{ij}s\big(x_{j,k}(t)-x_{i,k}(t)\big)-w_{(i+1)j}s\big(x_{j,k}(t)-x_{i+1,k}(t)\big)\big]. \tag{48}$$

Applying Eqs. (44) and (48) then yields

$$U_{i,k}(t)-U_{i+1,k}(t)=-\chi\gamma_k\big[\Delta^h_{i,k}(t)-\Delta^h_{i,k}(t-\tau_T)\big]+\gamma_k\sum_{j\in\mathcal{N}-\{N\}}\bar{w}_{ij}\Delta^h_{j,k}(t-\tau_T)$$
$$+\gamma_k\sum_{j\in\mathcal{N}}\big[w_{ij}s\big(x_{j,k}(t)-x_{i,k}(t)\big)-w_{(i+1)j}s\big(x_{j,k}(t)-x_{i+1,k}(t)\big)\big]. \tag{49}$$

We note that if $I,J\subseteq\mathbb{R}$ and $x,y\in\mathbb{R}$, then

$$Ix-Jy\subseteq I(x-y)+(I-J)y. \tag{50}$$

The property in Eq. (50) holds because if $u\in Ix-Jy$, then $u=sx-ty=s(x-y)+(s-t)y\in I(x-y)+(I-J)y$ for some $s\in I$, $t\in J$. From Eq. (50), we can derive

$$\begin{aligned}
&\sum_{l\in\mathcal{K}}co\big[a_{kl}(x_{i,l}(t))\big]f_l(x_{i,l}(t))-\sum_{l\in\mathcal{K}}co\big[a_{kl}(x_{i+1,l}(t))\big]f_l\big(x_{i+1,l}(t)\big)\\
&\quad\subseteq\sum_{l\in\mathcal{K}}co\big[a_{kl}(x_{i,l}(t))\big]\Delta^f_{i,l}(t)+\sum_{l\in\mathcal{K}}\Delta^a_{i,kl}(t)f_l(x_{i+1,l}(t)),
\end{aligned} \tag{51}$$

and

$$\sum_{l\in\mathcal{K}} co\big[b_{kl}\big(x_{i,l}(t-\tau_l)\big)\big]f_l\big(x_{i,l}(t-\tau_l)\big)-\sum_{l\in\mathcal{K}} co\big[b_{kl}\big(x_{i+1,l}(t-\tau_l)\big)\big]f_l\big(x_{i+1,l}(t-\tau_l)\big)$$
$$\subseteq \sum_{l\in\mathcal{K}} co\big[b_{kl}\big(x_{i,l}(t-\tau_l)\big)\big]\Delta^f_{i,l}(t-\tau_l)+\sum_{l\in\mathcal{K}}\Delta^b_{i,kl}(t-\tau_l)f_l\big(x_{i+1,l}(t-\tau_l)\big). \tag{52}$$

Applying Eqs. (7), (47), and (49)–(52), it can be verified that $\mathbf{Z}(t)$ satisfies Eq. (45). □

Lemma 2.5 revealed that differential inclusions (37) can be recast into differential inclusions (45). In the following lemma, we further show that $\mathbf{Z}(t)$, which satisfies differential inclusions (45), can satisfy certain differential equations. To prepare for such a lemma, by using $\hat{H}_k$ and $\check{H}_k$ in Eq. (27), χ in Eq. (39), and $\bar{w}_{ii}$ in Eq. (43), we first define

$$\check{H}^w_{i,k}=\begin{cases}\check{H}_k, & \bar{w}_{ii}\geq 0,\\ \hat{H}_k, & \bar{w}_{ii}<0,\end{cases}\quad \hat{H}^w_{i,k}=\begin{cases}\hat{H}_k, & \bar{w}_{ii}\geq 0,\\ \check{H}_k, & \bar{w}_{ii}<0,\end{cases}\quad \check{H}^\chi_k=\begin{cases}\check{H}_k, & \chi\geq 0,\\ \hat{H}_k, & \chi<0,\end{cases}\quad \hat{H}^\chi_k=\begin{cases}\hat{H}_k, & \chi\geq 0,\\ \check{H}_k, & \chi<0.\end{cases} \tag{53}$$

Then, we further set

$$\check{\alpha}_{i,k}=-d_k-\bar{a}_{kk}\bar{v}^f_k+\begin{cases}\gamma_k\bar{w}_{ii}\check{H}^w_{i,k}, & \tau_T=0,\\ -\chi\gamma_k\hat{H}^\chi_k, & \tau_T\neq 0,\end{cases} \tag{54}$$

$$\hat{\alpha}_{i,k}=-d_k+\bar{a}_{kk}\bar{v}^f_k+\begin{cases}\gamma_k\bar{w}_{ii}\hat{H}^w_{i,k}, & \tau_T=0,\\ -\chi\gamma_k\check{H}^\chi_k, & \tau_T\neq 0,\end{cases} \tag{55}$$

$$\bar{\beta}_{i,k}=\bar{b}_{kk}\bar{v}^f_k,\quad \bar{\gamma}_{i,k}=\begin{cases}0, & \tau_T=0,\\ \gamma_k|\bar{w}_{ii}+\chi|\hat{H}_k, & \tau_T\neq 0,\end{cases} \tag{56}$$

$$\rho^w_{i,k}=2\Bigg[\sum_{l\in\mathcal{K}-\{k\}}(\bar{a}_{kl}+\bar{b}_{kl})\tilde{\rho}^f_l+\gamma_k\tilde{\rho}^h_k\sum_{j\in\mathcal{N}-\{i,N\}}|\bar{w}_{ij}|\Bigg], \tag{57}$$

$$\rho^e_{i,k}=\sum_{l\in\mathcal{K}}\big(d^a_{kl}+d^b_{kl}\big)\tilde{\rho}^f_l+\bar{s}\gamma_k\sum_{j\in\mathcal{N}-\{i,i+1\}}(|w_{ij}|+|w_{(i+1)j}|), \tag{58}$$

where w_{ij} and γ_k are introduced in Eq. (6), $\bar{v}^f_k$ in Eq. (2), $\bar{s}$ in Eq. (7), $\bar{a}_{kl}$, $\bar{b}_{kl}$ d^a_{kl}, and d^b_{kl} in Eq. (8), $\hat{H}_k$ in Eq. (27), $\tilde{\rho}^h_l$ and $\tilde{\rho}^f_l$ in Eq. (29), χ in Eq. (39), and $\bar{w}_{ij}$ in Eq. (43).

Lemma 2.6. *Assume that conditions (S1) and (S2) hold. There exist real-valued functions $\alpha_{i,k}(t)$, $\beta_{i,k}(t)$, $\gamma_{i,k}(t)$, $w_{i,k}(t)$, and $e_{i,k}(t)$ for all $(i,k)\in\mathcal{A}_z$ and $t\geq t_0$ such that $\mathbf{Z}(t)$ satisfies*

$$\dot{z}_{i,k}(t)=-\lambda_{i,k}\mathrm{sign}(z_{i,k}(t))+\alpha_{i,k}(t)z_{i,k}(t)+\beta_{i,k}(t)z_{i,k}(t-\tau_k)+\gamma_{i,k}(t)z_{i,k}(t-\tau_T)$$
$$+w_{i,k}(t)+e_{i,k}(t) \tag{59}$$

for a. e. $t\geq t_0$ and $(i,k)\in\mathcal{A}_z$, where $\alpha_{i,k}(t)$, $\beta_{i,k}(t)$, $\gamma_{i,k}(t)$, and $w_{i,k}(t)$ satisfy

$$\alpha_{i,k}(t)\in\big[\check{\alpha}_{i,k},\hat{\alpha}_{i,k}\big],\quad \beta_{i,k}(t)\in\big[-\bar{\beta}_{i,k},\bar{\beta}_{i,k}\big],\quad \gamma_{i,k}(t)\in\big[-\bar{\gamma}_{i,k},\bar{\gamma}_{i,k}\big], \tag{60}$$

and

$$|w_{i,k}(t)| \leq \sum_{l\in\mathcal{K}-\{k\}} \left[\bar{a}_{kl}\bar{\nu}_l^f|z_{i,l}(t)| + \bar{b}_{kl}\bar{\nu}_l^f|z_{i,l}(t-\tau_l)|\right] + \gamma_k\hat{H}_k \sum_{j\in\mathcal{N}-\{i,N\}} |\bar{w}_{ij}||z_{j,k}(t-\tau_T) \quad (61)$$

for all $t \geq \tilde{t}_0$ *and* $(i,k) \in \mathcal{A}_z$. *In addition,*

$$|w_{i,k}|^{\max}(\tilde{t}_0) \leq \rho_{i,k}^w,\ |e_{i,k}|^{\max}(\tilde{t}_0) \leq \rho_{i,k}^e, \quad (62)$$

where $\tilde{t}_0$ *is defined in Proposition 2.3.*

Proof. Recall condition (S2). We merely prove the case that $\tau_T \neq 0$ and Eq. (39) holds, and the case that $\tau_T = 0$ can be treated similarly. Applying Proposition 2.3, Eqs. (29) and (35) reveals that $x_{j,l}(s) \in [-\bar{q}_l, \bar{q}_l]$; hence,

$$|f_l(x_{j,l}(s))| \leq \tilde{\rho}_l^f,\ \ |h_l(x_{j,l}(s))| \leq \tilde{\rho}_l^h \quad (63)$$

for all $s \geq \tilde{t}_0 - \tau_M$ and $(j,l) \in \mathcal{A}_x$, and

$$|\Delta_{j,l}^f(s)| \leq 2\tilde{\rho}_l^f,\ \ |\Delta_{j,l}^h(s)| \leq 2\tilde{\rho}_l^h \quad (64)$$

for all $s \geq \tilde{t}_0 - \tau_M$ and $(j,l) \in \mathcal{A}_z$ under condition (S1). Recall Eq. (35) and $z_{j,l}(\cdot) = x_{j,l}(\cdot) - x_{j+1,l}(\cdot)$. From Eq. (2), for each $(j,l) \in \mathcal{A}_z$ and $s \geq t_0$, there exists some $\tilde{\nu}_{j,l}(s)$ with

$$|\tilde{\nu}_{j,l}(s)| \leq \bar{\nu}_l^f \quad (65)$$

such that

$$\Delta_{j,l}^f(s) = \tilde{\nu}_{j,l}(s) z_{j,l}(s). \quad (66)$$

In addition, applying the mean-value theorem yields

$$\Delta_{j,l}^h(s) = h_l'(\xi_{j,l}(s)) z_{j,l}(s) \quad (67)$$

for all $(j,l) \in \mathcal{A}_z$ and $s \geq t_0$, where $\xi_{j,\,l}(s)$ is some value between $x_{j,\,l}(s)$ and $x_{j+1,l}(s)$. Recall Eq. (27). Notably,

$$h_l'(\xi_{j,l}(s)) \in \left[\check{H}_l, \hat{H}_l\right] \subseteq \left[-\hat{H}_l, \hat{H}_l\right] \quad (68)$$

for all $(j,l) \in \mathcal{A}_z$ and $s \geq \tilde{t}_0$, as $\xi_{j,l}(s) \in [-\bar{q}_l, \bar{q}_l]$ for all $s \geq \tilde{t}_0$. Recall from Lemma 2.5 that $\mathbf{Z}(t)$ satisfies Eq. (45) under condition (S2). As seen from Eqs. (8), (10), and (36),

$$\Delta_{i,kl}^a(t) \subseteq \left[-d_{kl}^a, d_{kl}^a\right],\ \ \Delta_{i,kl}^b(t-\tau_l) \subseteq \left[-d_{kl}^b, d_{kl}^b\right] \quad (69)$$

for $i \in \mathcal{N} - \{N\}$, $k, l \in \mathcal{K}$, and $t \geq t_0$. From Eqs. (10), (66), (67), and (69), there exist some $\tilde{a}_{i,kl}(t)$ and $\tilde{b}_{i,kl}(t)$ for $i \in \mathcal{N}$, $k, l \in \mathcal{K}$, and $t \geq t_0$, with

$$\tilde{a}_{i,kl}(t) \in co\left[a_{kl}(x_{i,l}(t))\right] \subseteq [-\bar{a}_{kl}, \bar{a}_{kl}],\ \tilde{b}_{i,kl}(t) \in co\left[b_{kl}(x_{i,l}(t-\tau_l))\right] \subseteq \left[-\bar{b}_{kl}, \bar{b}_{kl}\right], \quad (70)$$

and some $\tilde{\delta}_{i,kl}^a(t)$ and $\tilde{\delta}_{i,kl}^b(t)$ for $i \in \mathcal{N} - \{N\}$, $k, l \in \mathcal{K}$, and $t \geq t_0$, with

$$\tilde{\delta}_{i,kl}^a(t) \in \Delta_{i,kl}^a(t) \subseteq \left[-d_{kl}^a, d_{kl}^a\right],\ \tilde{\delta}_{i,kl}^b(t) \in \Delta_{i,kl}^b(t-\tau_l) \subseteq \left[-d_{kl}^b, d_{kl}^b\right], \quad (71)$$

such that $\mathbf{Z}(t)$, which satisfies Eq. (45), now satisfies

$$\dot{z}_{i,k}(t) = -\lambda_{i,k}\mathrm{sign}(z_{i,k}(t)) + \alpha_{i,k}(t) z_{i,k}(t) + \beta_{i,k}(t) z_{i,k}(t-\tau_k) + \gamma_{i,k}(t) z_{i,k}(t-\tau_T)$$
$$+ w_{i,k}(t) + e_{i,k}(t) \quad (72)$$

for a. e. $t \geq t_0$ and $(i, k) \in \mathcal{A}_z$, where

$$\alpha_{i,k}(t) = -d_k + \tilde{a}_{i,kk}(t)\tilde{v}_{i,k}(t) - \chi\gamma_k h'_k(\xi_{i,k}(t)), \tag{73}$$

$$\beta_{i,k}(t) = \tilde{b}_{i,kk}(t)\tilde{v}_{i,k}(t-\tau_k), \tag{74}$$

$$\gamma_{i,k}(t) = \gamma_k(\bar{w}_{ii} + \chi)h'_k(\xi_{i,k}(t-\tau_T)), \tag{75}$$

$$\begin{aligned} w_{i,k}(t) = &\sum_{l\in\mathcal{K}-\{k\}} \left[\tilde{a}_{i,kl}(t)\tilde{v}_{i,l}(t)z_{i,l}(t) + \tilde{b}_{i,kl}(t)\tilde{v}_{i,l}(t-\tau_l)z_{i,l}(t-\tau_l)\right] \\ &+ \gamma_k \sum_{j\in\mathcal{N}-\{i,N\}} \bar{w}_{ij}h'_k\big(\xi_{j,k}(t-\tau_T)\big)z_{j,k}(t-\tau_T), \end{aligned} \tag{76}$$

$$\begin{aligned} e_{i,k}(t) = &\sum_{l\in\mathcal{K}} \left[\tilde{\delta}^a_{i,kl}(t)f_l(x_{i+1,l}(t)) + \tilde{\delta}^b_{i,kl}(t)f_l\big(x_{i+1,l}(t-\tau_l)\big)\right] \\ &+ \gamma_k \sum_{j\in\mathcal{N}-\{i,i+1\}} \left[w_{ij}s\big(x_{j,k}(t) - x_{i,k}(t)\big) - w_{(i+1)j}s\big(x_{j,k}(t) - x_{i+1,k}(t)\big)\right]. \end{aligned} \tag{77}$$

Accordingly, $\mathbf{Z}(t)$ satisfies Eq. (59), as seen from Eqs. (72)–(77). Applying Eqs. (53)–(56), (65), (68), and (70) yields that $\alpha_{i,k}(t)$, $\beta_{i,k}(t)$, $\gamma_{i,k}(t)$, and $w_{i,k}(t)$ in Eqs. (73)–(76) satisfy Eqs. (60) and (61). As seen from Eqs. (64), (66), and (67),

$$|\tilde{v}_{j,l}(s)z_{j,l}(s)| \leq 2\tilde{\rho}^f_l, \quad |h'_l(\xi_{j,l}(s))z_{j,l}(s)| \leq 2\tilde{\rho}^h_l, \tag{78}$$

for all $s \geq \tilde{t}_0 - \tau_M$ and $(j, l) \in \mathcal{A}_z$. As seen from Eq. (7),

$$|s(x_{j,l}(s) - x_{i,l}(s))| \leq \bar{s} \tag{79}$$

for $s \geq t_0$, $i, j \in \mathcal{N}$, and $l \in \mathcal{K}$. By Eqs. (57), (58), (63), (70), (71), (78), and (79), $w_{i,k}(t)$ in Eq. (76) and $e_{i,k}(t)$ in Eq. (77) satisfy that $w_{i,k}(t) \leq \rho^w_{i,k}$ and $e_{i,k}(t) \leq \rho^e_{i,k}$ for $t \geq \tilde{t}_0$, which yields Eq. (62). Thus, we complete the proof. □

3. Main results.

This section establishes the global synchronization of system (4). Recall that $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ is an arbitrary solution of system (9), where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))^T$, $i \in \mathcal{N}$. Applying Lemma 2.6 shows that $(\mathbf{z}_1(t), \ldots, \mathbf{z}_{N-1}(t))^T$ satisfies system (59), where $\mathbf{z}_i(t) = (z_{i,1}(t), \ldots, z_{i,K}(t))^T = \mathbf{x}_i(t) - \mathbf{x}_{i+1}(t)$, $i = 1, \ldots, N-1$. In the following discussion, let us preview the main process for showing $z_{i,k}(t) = x_{i,k}(t) - x_{i,k+1}(t) \to 0$ as $t \to \infty$ for all $(i, k) \in \mathcal{A}_z$. This then establishes the global synchronization of system (4). Let us first relabel the two-dimensional indices in system (59) to one-dimensional indices through the bijective mapping ℓ: $\mathcal{A}_z \to \{1, \ldots, K(N-1)\}$ defined by

$$\ell(i, k) = (i-1) \times K + k. \tag{80}$$

The assignment of the label $\ell(i,k)$ corresponds to the sequence (i,k) generated by considering the order of i and k in succession; more precisely,

$$\ell(i,k) < \ell(j,l) \quad \text{if } i < j; \quad \ell(i,k) < \ell(i,l) \quad \text{if } k < l. \tag{81}$$

By labeling $z_{\ell(i,k)} := z_{i,k}$, system (59) becomes

$$\begin{aligned}\dot{z}_{\ell(i,k)}(t) = & -\lambda_{i,k}\text{sign}(z_{\ell(i,k)}(t)) + \alpha_{i,k}(t)z_{\ell(i,k)}(t) + \beta_{i,k}(t)z_{\ell(i,k)}(t-\tau_k) \\ & + \gamma_{i,k}(t)z_{\ell(i,k)}(t-\tau_T) + w_{i,k}(t) + e_{i,k}(t)\end{aligned} \tag{82}$$

for a. e. $t \geq t_0$ and $(i,k) \in \mathcal{A}_z$, where each $z_{\ell(i,k)}$ satisfies

$$\begin{aligned} & z_{\ell(i,k)}(t) \in \left[-2\bar{q}_k, 2\bar{q}_k\right] \text{ for all } t \geq \tilde{t}_0 - \tau_M, \text{ and } z_{\ell(i,k)}(t) \to \left[-2\tilde{q}_k, 2\tilde{q}_k\right] \\ & \quad \subset \left[-2\bar{q}_k, 2\bar{q}_k\right] \text{ as } t \to \infty \end{aligned}$$

based on Proposition 2.3.

With the quantities in Eqs. (26), (41), and (54)–(58), we introduce the following conditions:

Condition (S3): $\lambda_{i,k} > \rho^e_{i,k}$ for all $(i,k) \in \mathcal{A}_z$.

Condition (S4): $L_{i,k} := -\hat{\alpha}_{i,k} - \bar{\beta}_{i,k} - \bar{\gamma}_{i,k} - \rho^w_{i,k}/(2\bar{q}_k) > 0$ for all $(i,k) \in \mathcal{A}_z$.

The reason why conditions (S3) and (S4) are needed in the synchronization theory can be deduced from in the following. From Lemma 2.6 and condition (S3), $\alpha_{i,k}(t)$, $\beta_{i,k}(t)$, $\gamma_{i,k}(t)$, $w_{i,k}(t)$, and $e_{i,k}(t)$ in Eq. (82) satisfy Eqs. (60), (61), and $|e_{i,k}(t)| \leq \rho^e_{i,k} < \lambda_{i,k}$ for all $t \geq \tilde{t}_0$ and $(i,k) \in \mathcal{A}_z$. Hence, each component equation of $z_{\ell(i,k)}$ in Eq. (82) satisfies Eqs. (12)–(14) with $z(t) = z_{\ell(i,k)}(t)$, $\tilde{q} = \tilde{q}_k$, $\bar{q} = \bar{q}_k$, $\lambda = \lambda_{i,k}$, $\alpha(t) = \alpha_{i,k}(t)$, $\beta(t) = \beta_{i,k}(t)$, $\gamma(t) = \gamma_{i,k}(t)$, $w(t) = w_{i,k}(t)$, $e(t) = e_{i,k}(t)$, $\tau_\beta = \tau_k$, $\tau_\gamma = \tau_T$, $[\check{\alpha}, \hat{\alpha}] = [\check{\alpha}_{i,k}, \hat{\alpha}_{i,k}]$, $[-\bar{\beta}, \bar{\beta}] = [-\bar{\beta}_{i,k}, \bar{\beta}_{i,k}]$, $[-\bar{\gamma}, \bar{\gamma}] = [-\bar{\gamma}_{i,k}, \bar{\gamma}_{i,k}]$; moreover, condition (S_0) is satisfied under condition (S4). Applying Proposition 2.2 then yields that $z_{\ell(i,k)}(t) \to [-\bar{p}_{\ell(i,k)}, \bar{p}_{\ell(i,k)}]$ for each $(i,k) \in \mathcal{A}_z$; moreover,

$$0 \leq \bar{p}_{\ell(i,k)} \leq |w_{i,k}|^{\max}(\infty)/\eta_{i,k}, \tag{83}$$

where

$$\eta_{i,k} := -\hat{\alpha}_{i,k} - \bar{\beta}_{i,k} - \bar{\gamma}_{i,k}. \tag{84}$$

The following proposition shows that $\bar{p}_{\ell(i,k)}$ in Eq. (83) can further be estimated iteratively.

Proposition 3.1. *Assume that conditions (S1)–(S4) hold. Then, for each $(i,k) \in \mathcal{A}_z$, there exists a sequence $\{p^{(n)}_{\ell(i,k)}\}_{n=0}^{\infty}$, which satisfies*

$$0 \leq \bar{p}_{\ell(i,k)} \leq p^{(n)}_{\ell(i,k)} \tag{85}$$

for $n \geq 0$, where

$$p^{(n)}_{\ell(i,k)} := \begin{cases} \rho^w_{i,k}/\eta_{i,k}, & n = 0, \\ \Big[\sum_{l=1}^{k-1} \left(\bar{a}_{kl} + \bar{b}_{kl}\right)\bar{v}^f_l p^{(n)}_{\ell(i,l)} + \sum_{l=k+1}^{K} \left(\bar{a}_{kl} + \bar{b}_{kl}\right)\bar{v}^f_l p^{(n-1)}_{\ell(i,l)} & \\ +\gamma_k \hat{H}_k \Big(\sum_{j=1}^{i-1} |\bar{w}_{ij}| p^{(n)}_{\ell(j,k)} + \sum_{j=i+1}^{N-1} |\bar{w}_{ij}| p^{(n-1)}_{\ell(j,k)} \Big) \Big] \Big/ \eta_{i,k}, & n \geq 1. \end{cases} \tag{86}$$

Herein, γ_k is defined in Eq. (6), $\bar{v}^f_l$ in Eq. (2), $\bar{a}_{kl}$ and $\bar{b}_{kl}$ in Eq. (8), $\hat{H}_k$ in Eq. (27), $\bar{w}_{ij}$ in Eq. (43), $\rho^w_{i,k}$ in Eq. (57), and $\eta_{i,k}$ in Eq. (84).

Proof. Let us verify Eq. (85) by induction. First, we consider $n = 0$ and $(i,k) \in \mathcal{A}_z$. Recall from Eq. (62) that $|w_{i,k}|^{\max}(\infty) \leq |w_{i,k}|^{\max}(\tilde{t}_0) \leq \rho^w_{i,k}$ for all $(i,k) \in \mathcal{A}_z$. Thus,

$0 \leq \bar{p}_{\ell(i,k)} \leq \rho^w_{i,k}/\eta_{i,k} = p^{(0)}_{\ell(i,k)}$ for all $(i,k) \in \mathcal{A}_z$ based on Eq. (83). Next, we assume that for some $(i,k) \in \mathcal{A}_z$ and $n_0 \geq 1$, $0 \leq \bar{p}_{\ell(j,l)} \leq p^{(n)}_{\ell(j,l)}$ holds; hence, $z_{\ell(j,l)}(t) \to [-p^{(n)}_{\ell(j,l)}, p^{(n)}_{\ell(j,l)}]$ as $t \to \infty$, for $n \in \{0, 1, \ldots, n_0 - 1\}$, $(j,l) \in \mathcal{A}_z$ and $n = n_0$, $(j,l) \in \{(\tilde{j}, \tilde{l}) : \ell(\tilde{j}, \tilde{l}) < \ell(i,k)\}$. Applying Eq. (61) and the order properties of mapping ℓ, cf. (81), yields

$$\begin{aligned}
&|w_{i,k}|^{\max}(\infty) \\
&\leq \sum_{l=1}^{k-1} (\bar{a}_{kl} + \bar{b}_{kl})\bar{v}^f_l p^{(n_0)}_{\ell(i,l)} + \sum_{l=k+1}^{K} (\bar{a}_{kl} + \bar{b}_{kl})\bar{v}^f_l p^{(n_0-1)}_{\ell(i,l)} \\
&+ \gamma_k \hat{H}_k \left(\sum_{j=1}^{i-1} |\bar{w}_{ij}| p^{(n_0)}_{\ell(j,k)} + \sum_{j=i+1}^{N-1} |\bar{w}_{ij}| p^{(n_0-1)}_{\ell(j,k)} \right) \\
&= p^{(n_0)}_{\ell(i,k)} \eta_{i,k},
\end{aligned}$$

which implies $0 \leq \bar{p}_{\ell(i,k)} \leq |w_{i,k}|^{\max}(\infty)/\eta_{i,k} \leq p^{(n_0)}_{i,k}$ based on Eq. (83). We thus verify Eq. (85). □

For later use, we define the following matrix by mapping ℓ.

$$\mathbf{M} = D_{\mathbf{M}} - L_{\mathbf{M}} - U_{\mathbf{M}} := [m_{\imath\jmath}]_{1 \leq \imath, \jmath \leq K(N-1)} \tag{87}$$

with

$$m_{\imath\jmath} := \begin{cases} \eta_{i,k}, & \imath = \jmath = \ell(i,k), \\ -(\bar{a}_{kl} + \bar{b}_{kl})\bar{v}^f_l, & \imath = \ell(i,k),\ \jmath = \ell(i,l),\ l \neq k, \\ -\gamma_k \hat{H}_k |\bar{w}_{ij}|, & \imath = \ell(i,k),\ \jmath = \ell(j,k),\ i \neq j, \\ 0, & \text{otherwise}, \end{cases} \tag{88}$$

where $D_{\mathbf{M}}$, $-L_{\mathbf{M}}$, and $-U_{\mathbf{M}}$ represent the diagonal, strictly lower-triangular, and strictly upper-triangular parts of $\mathbf{M}$, respectively. From Eq. (88) and the order properties of mapping ℓ, cf. (81), and labeling $p^{(n)}_{\ell(i,k)} = p^{(n)}_\imath$ if $\ell(i,k) = \imath$ for all $n \in \mathbb{N} \cup \{0\}$ and $(i,k) \in \mathcal{A}_z$, Eq. (86) can be rewritten as follows:

$$p^{(0)}_\imath := \rho^w_{i,k}/\eta_{i,k}; \quad p^{(n)}_\imath := \left[-\sum_{\jmath=1}^{\imath-1} m_{\imath\jmath} p^{(n)}_\jmath - \sum_{\jmath=\imath+1}^{K(N-1)} m_{\imath\jmath} p^{(n-1)}_\jmath \right] / m_{\imath\imath}, \quad n \geq 1. \tag{89}$$

With $p^{(n)}_\imath$ defined in Eq. (89), $1 \leq \imath \leq K(N-1)$ and $n \in \mathbb{N} \cup \{0\}$, $\{(p^{(n)}_1, \ldots, p^{(n)}_{K(N-1)})^T\}_{n=0}^\infty$ is exactly the Gauss–Seidel iteration for solving the linear system

$$\mathbf{Mv} = \mathbf{0}. \tag{90}$$

The following theorem transforms the problem of global synchronization of system (4) into solving linear system (90).

Theorem 3.1. *Consider system (4) under conditions (S1)–(S4). Then, the system attains global synchronization if the Gauss–Seidel iteration for solving the linear system (90) converges to zero, the unique solution of (90), or equivalently,*

$$\lambda_{\text{syn}} := \max_{1 \leq \imath \leq K(N-1)} \left\{ |\lambda_\imath| : \ \lambda_\imath : \text{ eigenvalue of } (D_{\mathbf{M}} - L_{\mathbf{M}})^{-1} U_{\mathbf{M}} \right\} < 1. \tag{91}$$

Proof. Assume that conditions (S1)–(S4) hold. Recall that $(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))^T$ is an arbitrary solution of system (9), where $\mathbf{x}_i(t) = (x_{i,1}(t), \ldots, x_{i,K}(t))^T$, $i \in \mathcal{N}$. Recall that $z_{\ell(i,k)}(t) = x_{i,k}(t) - x_{i,k+1}(t) \to [-\bar{p}_{\ell(i,k)}, \bar{p}_{\ell(i,k)}]$ as $t \to \infty$, where $0 \le \bar{p}_{\ell(i,k)} \le p^{(n)}_{\ell(i,k)} = p^{(n)}_\imath$ if $\ell(i,k) = \imath$ for all $n \in \mathbb{N}$. Consequently, if $\{(p^{(n)}_1, \ldots, p^{(n)}_{K(N-1)})^T\}^\infty_{n=0}$, the Gauss–Seidel iteration for solving system (90), converges to zero, then $\bar{p}_{\ell(i,k)} = 0$ for all $(i,k) \in \mathcal{A}_z$. This then establishes the global synchronization of system (4). Recall that condition (S4) implies that all diagonal entries of $\mathbf{M}$ (i.e., $\eta_{i,k}$ for $(i,k) \in \mathcal{A}_z$) are positive. Accordingly, $(D_\mathbf{M} - L_\mathbf{M})^{-1}$ and λ_{syn} exist; moreover, the Gauss–Seidel iteration for solving system (90) converges to zero if and only if $\lambda_{\text{syn}} < 1$. Thus, we complete the proof. □

Remark 3.1. (i) As seen from Eqs. (84), (88), and the definitions of $\hat{\alpha}_{i,k}$, $\bar{\beta}_{i,k}$, and $\bar{\gamma}_{i,k}$ in Eqs. (55) and (56), the entries $m_{\imath\jmath}$ in matrix $\mathbf{M}$ are determined by d_k, γ_k, and $\bar{\upsilon}^f_k$ defined in Eqs. (1), (2), and (6), $\bar{a}_{kl}$ and $\bar{b}_{kl}$ in Eq. (8), $\hat{H}_k$ in Eq. (27), χ in Eq. (39), $\bar{w}_{ij}$ in Eq. (43), $\hat{H}^w_{i,k}$ and $\check{H}^\chi_k$ in Eq. (53), where d_k, γ_k, $\bar{\upsilon}^f_k$, $\hat{H}_k$, $\hat{H}^w_{i,k}$, $\check{H}^\chi_k$, $\bar{a}_{kl}$, and $\bar{b}_{kl}$ are nonnegative. In general, positive and sufficiently large diagonal entries of M (i.e., $\eta_{i,k} = -\hat{\alpha}_{i,k} - \bar{\beta}_{i,k} - \bar{\gamma}_{i,k}$, cf. (84)) promote the convergence of the Gauss-Seidel iteration for system (90). Accordingly, in general, the criterion in Theorem 3.1 prefers: positive $d_k + \chi\gamma_k\check{H}^\chi_k$ with large magnitude, small $\bar{a}_{kk}\bar{\upsilon}^f_k$, small $\bar{b}_{kk}\bar{\upsilon}^f_k$, and small $\gamma_k|\bar{w}_{ii} + \chi|\hat{H}_k$ if $\tau_T \neq 0$; and positive $d_k - \gamma_k\bar{w}_{ii}\hat{H}^w_{i,k}$ with large magnitude, small $\bar{a}_{kk}\bar{\upsilon}^f_k$, and small $\bar{b}_{kk}\bar{\upsilon}^f_k$ if $\tau_T = 0$, such that in each case, $\eta_{i,k}$ is positive and large. (ii) The criterion in Theorem 3.1 is independent of the network scale (N) and the magnitudes of the delays τ_k and τ_T (if $\tau_T \neq 0$). (iii) As seen from (6), the coupling terms $U_{i,k}(t)$ in system (4) are formulated with a sign function sign(·). The sign function was also utilized in the design of the discontinuous controllers to investigate the finite-time synchronization problem of coupled systems; for instance, see [45]. (iv) System (4) has discrete-time delays τ_l ($l \in K$) and τ_T. The present approach in this paper may also be extended to treat memristor-based neural networks with distributed delays, cf. [46].

4. Numerical examples

This section illustrates Proposition 2.3 and Theorem 3.1 using the following three examples. Example 1 demonstrates the application of Proposition 2.3 to provide an estimated globally attracting region for system (4). As seen from condition (S2), the synchronization criterion in Theorem 3.1 depends on whether the transmission delay (τ_T) exists, and requires that Eq. (39) holds if $\tau_T \neq 0$. In the following, Example 2 considers system (4) which is with $\tau_T = 0$ and does not satisfy Eq. (39); Example 3 considers system (4) which is with $\tau_T \neq 0$ and satisfies (39). The numerical simulation in Example 2 shows that the invariance of synchronous set S of system (4), defined in Eq. (38), is lost if considering $\tau_T \neq 0$ instead of $\tau_T = 0$. Examples 2 and 3 also show that the size of matrix M in the synchronization theory (cf. (87), (90), and Theorem 3.1) is larger if the scale of the considered networks is larger. In Examples 2 and 3, for a solution of system (9), we define the corresponding synchronization error:

$$Err(t) := \sqrt{\sum_{i \in \mathcal{N}-\{N\}} \sum_{k \in \mathcal{K}} [x_{i,k}(t) - x_{i+1,k}(t)]^2}.$$

Notably, the solution synchronizes if $Err(t) \to 0$ as $t \to \infty$; the solution is synchronous at time t if $Err(t) = 0$.

Example 1. Consider three coupled networks (4) in which each individual network (1) has two neurons, and for which $d_1 = d_2 = 0.5$, $\tau_1 = \tau_2 = 20$, $f_1(\xi) = f_2(\xi) = (|\xi + 0.5| - |\xi - 0.5|)/2$, $I_1(t) = I_2(t) \equiv 0$, $T_1 = 0.3$, $T_2 = 0.1$, and

$$a_{kl}(x_{i,l}(t)) = \begin{cases} a'_{kl} & |x_{i,l}(t)| < T_l, \\ a''_{kl} & |x_{i,l}(t)| \geq T_l, \end{cases} \quad b_{kl}(x_{i,l}(t - \tau_l)) = \begin{cases} b'_{kl} & |x_{i,l}(t - \tau_l)| < T_l, \\ b''_{kl} & |x_{i,l}(t - \tau_l)| \geq T_l, \end{cases}$$

for $i = 1, 2, 3$ and $k, l = 1, 2$, where

$$\begin{cases} \left(a'_{11}, a'_{12}, a'_{21}, a'_{22}\right) = (0.03, 0.01, -0.1, 0.2), \left(a''_{11}, a''_{12}, a''_{21}, a''_{22}\right) = (-0.02, -0.02, 0.1, 0.1), \\ \left(b'_{11}, b'_{12}, b'_{21}, b'_{22}\right) = (-0.6, -0.1, 0.1, -0.1), \left(b''_{11}, b''_{12}, b''_{21}, b''_{22}\right) = (-0.65, 0.2, -0.1, 0.1). \end{cases}$$

The coupling terms $U_{i,k}(t)$ in Eq. (6), $i = 1, 2, 3$, $k = 1, 2$, have $h_1(\xi) = h_2(\xi) = 0.5\tanh(2\xi)$, $\gamma_1 = \gamma_2 = 0.1$, $\bar{s} = 0.01$ in Eq. (7), $\tau_T = 20$, and w_{ij}, $1 \leq i, j \leq 3$, satisfying

$$W = [w_{ij}]_{1 \leq i,j \leq 3} := \begin{pmatrix} 0 & -0.4 & -0.1 \\ 0.05 & 0 & -0.35 \\ -0.1 & -0.35 & 0 \end{pmatrix}.$$

Notably, $\bar{\rho}_l^f$ and $\bar{\rho}_l^h$ in condition (S1)* and $\bar{I}_k$ in Eq. (3) can be chosen as

$$\bar{\rho}_1^f = \bar{\rho}_2^f = \bar{\rho}_1^h = \bar{\rho}_2^h = 0.5, \bar{I}_1 = \bar{I}_2 = 0. \tag{92}$$

From Eq. (8), we obtain

$$(\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}) = (0.03, 0.02, 0.1, 0.2), \left(\bar{b}_{11}, \bar{b}_{12}, \bar{b}_{21}, \bar{b}_{22}\right) = (0.65, 0.2, 0.1, 0.1). \tag{93}$$

We compute Eq. (28) to obtain

$$\bar{\varpi} = 0.5. \tag{94}$$

From Eqs. (92)–(94), we compute Eqs. (26) and (29) to obtain

$$\bar{q}_1 = 1.001, \bar{q}_2 = 0.601, \tilde{q}_1 \approx 0.9974, \tilde{q}_2 \approx 0.5844. \tag{95}$$

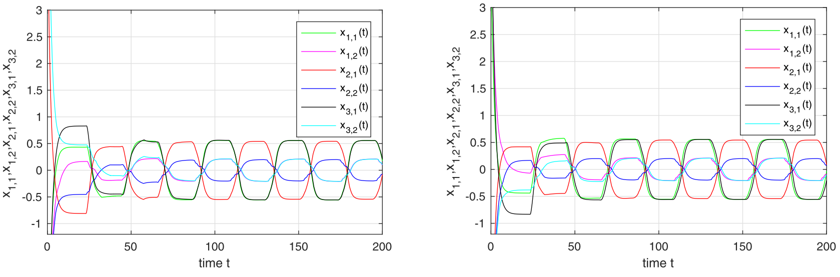
As seen from Eq. (95), condition (S1) holds. By Proposition 2.3, for any solution $(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))^T$ of the system,

$$x_{i,k}(t) \to \left[-\tilde{q}_k, \tilde{q}_k\right] \subset \left[-\bar{q}_k, \bar{q}_k\right] \text{ as } t \to \infty, \tag{96}$$

for $i = 1, 2, 3$ and $k = 1, 2$, where $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t))^T$, $\tilde{q}_1 \approx 0.9974$, $\tilde{q}_2 \approx 0.5844$, $\bar{q}_1 = 1.001$, and $\bar{q}_2 = 0.601$. Accordingly, $(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))^T$ eventually enters, and then remains in set $\mathcal{D} := \{(x_1, y_1, x_2, y_2, x_3, y_3)^T : -1.001 \leq x_i \leq 1.001 \text{ and } -0.601 \leq y_i \leq 0.601, i = 1, 2, 3\}$. Fig. 1(a) and (b) demonstrate the evolution of two solutions, which evolve from the constant initial conditions $(-5, -5, 5, -6, -6, 8)^T$ and $(4, 5, -5, -5, 6, -6)^T$ at $t_0 = 0$, respectively. They show that these two solutions eventually enter, and then remain in, some set contained in $\mathcal{D}$.

Example 2. Consider four coupled networks (4) in which each individual network (1) has four neurons, and for which $d_1 = 1$, $d_2 = 2$, $d_3 = d_4 = 1.5$, $\tau_l = 15$, $f_l(\xi) = (|\xi + 0.1| - |\xi - 0.1|)/2$, $I_1(t) \equiv 0$, $I_2(t) \equiv 0.15$, $I_3(t) = 0.3\sin(t/10)$, $I_4(t) = 0.2\cos(t/5)$, $T_l = 0.1$, and

$$a_{kl}(x_{i,l}(t)) = \begin{cases} a'_{kl} & |x_{i,l}(t)| < T_l, \\ a''_{kl} & |x_{i,l}(t)| \geq T_l, \end{cases} \quad b_{kl}(x_{i,l}(t - \tau_l)) = \begin{cases} b'_{kl} & |x_{i,l}(t - \tau_l)| < T_l, \\ b''_{kl} & |x_{i,l}(t - \tau_l)| \geq T_l, \end{cases}$$



(a) (b)

Fig. 1. Numerical simulation for Example 1.

for $i = 1, 2, 3, 4$ and $k, l = 1, 2, 3, 4,$ with

$$[a'_{kl}]_{1\le k,l\le 4} = \begin{pmatrix} 0.05 & 0.5 & 0.2 & 0.5 \\ 0.05 & 0.05 & 0.3 & 0.05 \\ 0 & 0.05 & 0.01 & 0 \\ 0.02 & 0 & 0.02 & -0.05 \end{pmatrix},$$

$$[a''_{kl}]_{1\le k,l\le 4} = \begin{pmatrix} -0.05 & 0.4 & 0.15 & 0.45 \\ -0.05 & 0 & 0.2 & -0.05 \\ 0.05 & 0 & -0.04 & 0.05 \\ -0.03 & 0.05 & -0.03 & 0 \end{pmatrix},$$

$$[b'_{kl}]_{1\le k,l\le 4} = \begin{pmatrix} -1.02 & 0 & 0.15 & 0.05 \\ -0.04 & -1.07 & 0.2 & 0 \\ 0 & 0.05 & -0.11 & -0.05 \\ -0.03 & 0.05 & -0.03 & 0.05 \end{pmatrix},$$

$$[b''_{kl}]_{1\le k,l\le 4} = \begin{pmatrix} -1.03 & -0.05 & 0.2 & -0.05 \\ 0.01 & -1.02 & 0.25 & 0.05 \\ 0.05 & 0.02 & -0.12 & 0.05 \\ 0.02 & 0 & 0.02 & 0 \end{pmatrix}.$$

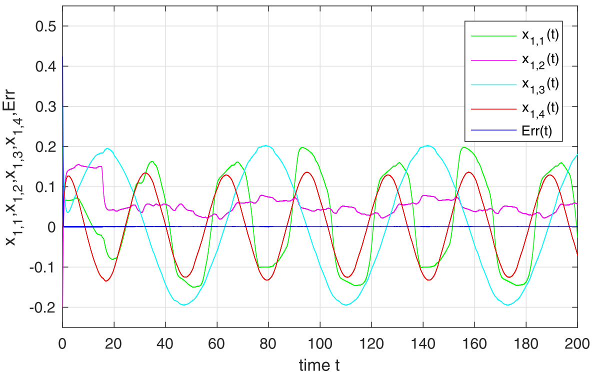
The coupling terms $U_{i,k}(t)$ in Eq. (6), $i = 1, 2, 3, 4,$ $k = 1, 2,$ are with $h_k(\xi) = 50\tanh(\xi/50)$ and $\gamma_k = 0.2$ for $k = 1, 2, 3, 4,$ $\bar{s} = 0.8$ in Eq. (7), $\tau_T = 0,$ and $w_{ij},$ $1 \le i, j \le 4,$ satisfying

$$W = [w_{ij}]_{1\le i,j\le 4} := \begin{pmatrix} 0 & 1.2 & 0 & -0.01 \\ 0.8 & 0 & 0.4 & 0 \\ 0 & 0.8 & 0 & 0.05 \\ 0 & 0 & 1.2 & 0 \end{pmatrix}. \quad (97)$$

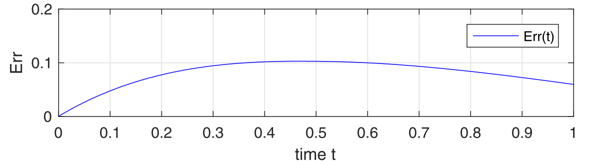
Notably, the considered system satisfies condition (S2) because $\tau_T = 0$. In addition, $\bar{\rho}_l^f$ and $\bar{\rho}_l^h$ in condition (S1)*, $\bar{\nu}_l^f$ in Eq. (2), and $\bar{I}_k$ in Eq. (3) can be chosen as

$$\bar{\rho}_l^f = 0.1, \;\; \bar{\rho}_l^h = 50, \bar{\nu}_l^f = 1, \bar{I}_1 = 0, \bar{I}_2 = 0.15, \bar{I}_3 = 0.3, \bar{I}_4 = 0.2, \quad (98)$$

for $l = 1, 2, 3, 4$. The quantities in Eq. (8) now satisfy



(a) $\tau_T = 0$



(b) $\tau_T = 2$

Fig. 2. Numerical simulation for Example 2.

$$[\bar{a}_{kl}]_{1\le k,l\le 4} = \begin{pmatrix} 0.05 & 0.5 & 0.2 & 0.5 \\ 0.05 & 0.05 & 0.3 & 0.05 \\ 0.05 & 0.05 & 0.04 & 0.05 \\ 0.03 & 0.05 & 0.03 & 0.05 \end{pmatrix}, \left[\bar{b}_{kl}\right]_{1\le k,l\le 4} = \begin{pmatrix} 1.03 & 0.05 & 0.2 & 0.05 \\ 0.04 & 1.07 & 0.25 & 0.05 \\ 0.05 & 0.05 & 0.12 & 0.05 \\ 0.03 & 0.05 & 0.03 & 0.05 \end{pmatrix}, \tag{99}$$

$$\left[d^a_{kl}\right]_{1\le k,l\le 4} = \begin{pmatrix} 0.1 & 0.1 & 0.05 & 0.05 \\ 0.1 & 0.05 & 0.1 & 0.1 \\ 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 \end{pmatrix}, \left[d^b_{kl}\right]_{1\le k,l\le 4} = \begin{pmatrix} 0.01 & 0.05 & 0.05 & 0.1 \\ 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.03 & 0.01 & 0.1 \\ 0.05 & 0.05 & 0.05 & 0.05 \end{pmatrix}. \tag{100}$$

From Eq. (97), $\bar{W}$ in Eq. (43) is now

$$\bar{W} = \left[\bar{w}_{ij}\right]_{1\le i,j\le 3} = \begin{pmatrix} -1.99 & 0.41 & 0.01 \\ 0.8 & -1,2 & 0.05 \\ 0 & 0.8 & -1.25 \end{pmatrix}, \tag{101}$$

and computing Eqs. (28) and (41) yields

$$\bar{\varpi} = 1.21, \left(\lambda_{1,k}, \lambda_{2,k}, \lambda_{3,k}\right) = (0.32, 0.192, 0.2), \tag{102}$$

where $k = 1, 2, 3, 4$. From Eqs. (98)–(102), we compute Eqs. (26)–(29) and (53) to obtain

$$\begin{cases} \bar{q}_1 \approx 24.6516, \bar{q}_2 \approx 12.3648, \bar{q}_3 \approx 16.4931, \bar{q}_4 \approx 16.4171, \\ \tilde{q}_1 \approx 11.5018, \tilde{q}_2 \approx 3.1975, \tilde{q}_3 \approx 5.4965, \tilde{q}_4 \approx 5.3985, \end{cases} \tag{103}$$

$$\tilde{\rho}_1^f = \tilde{\rho}_2^f = \tilde{\rho}_3^f = \tilde{\rho}_4^f = 0.1, \ \ \tilde{\rho}_1^h \approx 22.8310, \tilde{\rho}_2^h \approx 12.1188, \tilde{\rho}_3^h \approx 15.9198, \tilde{\rho}_4^h \approx 15.8515, \tag{104}$$

$$\begin{cases} \hat{H}_{i,1}^w = \check{H}_1 \approx 0.7915, \hat{H}_{i,2}^w = \check{H}_2 \approx 0.9413, \hat{H}_{i,3}^w = \check{H}_3 \approx 0.8986, \hat{H}_{i,4}^w = \check{H}_4 \approx 0.8995, \\ \check{H}_{i,1}^w = \check{H}_{i,2}^w = \check{H}_{i,3}^w = \check{H}_{i,4}^w = \hat{H}_1 = \hat{H}_2 = \hat{H}_3 = \hat{H}_4 = 1, \end{cases} \tag{105}$$

where $i = 1, 2, 3$. From Eqs. (98) and (103), it follows that condition (S1) holds. From Eqs. (97)–(101), (104), and (105), we compute Eqs. (54)–(58) to obtain

$$\begin{cases} \hat{\alpha}_{1,1} \approx -1.3650, \hat{\alpha}_{1,2} \approx -2.4246, \hat{\alpha}_{1,3} \approx -1.8977, \hat{\alpha}_{1,4} \approx -1.9080, \\ \hat{\alpha}_{2,1} \approx -1.2400, \hat{\alpha}_{2,2} \approx -2.2759, \hat{\alpha}_{2,3} \approx -1.7557, \hat{\alpha}_{2,4} \approx -1.7659, \\ \hat{\alpha}_{3,1} \approx -1.2479, \hat{\alpha}_{3,2} \approx -2.2853, \hat{\alpha}_{3,3} \approx -1.7647, \hat{\alpha}_{3,4} \approx -1.7749, \end{cases} \tag{106}$$

$$\bar{\beta}_{i,1} = 1.03, \bar{\beta}_{i,2} = 1.07, \bar{\beta}_{i,3} = 0.12, \bar{\beta}_{i,4} = 0.05, \bar{\gamma}_{i,k} = 0, \tag{107}$$

$$\begin{cases} \rho_{1,1}^w \approx 4.1356, \rho_{1,2}^w \approx 2.1840, \rho_{1,3}^w \approx 2.7345, \rho_{1,4}^w \approx 2.7070, \\ \rho_{2,1}^w \approx 8.0625, \rho_{2,2}^w \approx 4.2684, \rho_{2,3}^w \approx 5.4727, \rho_{2,4}^w \approx 5.4335, \\ \rho_{3,1}^w \approx 7.6059, \rho_{3,2}^w \approx 4.0260, \rho_{3,3}^w \approx 5.1543, \rho_{3,4}^w \approx 5.1165, \end{cases} \tag{108}$$

$$\begin{cases} \rho_{1,1}^e = 0.1166, \rho_{1,2}^e = 0.1206, \rho_{1,3}^e = 0.1046, \rho_{1,4}^e = 0.1056, \\ \rho_{2,1}^e = 0.1870, \rho_{2,2}^e = 0.1910, \rho_{2,3}^e = 0.1750, \rho_{2,4}^e = 0.1760, \\ \rho_{3,1}^e = 0.1790, \rho_{3,2}^e = 0.1830, \rho_{3,3}^e = 0.1670, \rho_{3,4}^e = 0.1680, \end{cases} \tag{109}$$

where $i = 1, 2, 3$ and $k = 1, 2, 3, 4$. The application of Eqs. (102) and (109) reveals that condition (S3) holds. By Eqs. (103) and (106)–(108), we compute $L_{i,k}$ in condition (S4) and $\eta_{i,k}$ in Eq. (84) to obtain

$$\begin{cases} L_{1,1} \approx 0.2511, L_{1,2} \approx 1.2663, L_{1,3} \approx 1.6948, L_{1,4} \approx 1.7756, \\ L_{2,1} \approx 0.0464, L_{2,2} \approx 1.0333, L_{2,3} \approx 1.4698, L_{2,4} \approx 1.5504, \\ L_{3,1} \approx 0.0636, L_{3,2} \approx 1.0525, L_{3,3} \approx 1.4884, L_{3,4} \approx 1.5690, \end{cases} \tag{110}$$

$$\begin{cases} \eta_{1,1} \approx 0.3350, \eta_{1,2} \approx 1.3546, \eta_{1,3} \approx 1.7777, \eta_{1,4} \approx 1.8580, \\ \eta_{2,1} \approx 0.2100, \eta_{2,2} \approx 1.2059, \eta_{2,3} \approx 1.6357, \eta_{2,4} \approx 1.7159, \\ \eta_{3,1} \approx 0.2179, \eta_{3,2} \approx 1.2153, \eta_{3,3} \approx 1.6447, \eta_{3,4} \approx 1.7249, \end{cases} \tag{111}$$

where Eq. (110) reveals that condition (S4) holds. Notably, ℓ defined in Eq. (80) now satisfies

$$\begin{cases} \ell(1,1) = 1, \ell(1,2) = 2, \ell(1,3) = 3, \ell(1,4) = 4, \ell(2,1) = 5, \ell(2,2) = 6, \\ \ell(2,3) = 7, \ell(2,4) = 8, \ell(3,1) = 9, \ell(3,2) = 10, \ell(3,3) = 11, \ell(3,4) = 12. \end{cases} \tag{112}$$

Applying Eqs. (88), (98)–(101), (105), (111), and (112) yields that **M** defined in Eq. (87) is approximately $[M_{1j}]_{1\leq j\leq 2}$, which yields $\lambda_{\text{syn}}\approx 0.9784$, cf. (91), where

$$M_{11}=\begin{pmatrix} 0.3350 & -0.55 & -0.4 & -0.55 & -0.082 & 0\\ -0.09 & 1.3546 & -0.55 & -0.1 & 0 & -0.082\\ -0.1 & -0.1 & 1.7777 & -0.1 & 0 & 0\\ -0.06 & -0.1 & -0.06 & 1.8580 & 0 & 0\\ -0.16 & 0 & 0 & 0 & 0.2100 & -0.55\\ 0 & -0.16 & 0 & 0 & -0.09 & 1.2059\\ 0 & 0 & -0.16 & 0 & -0.1 & -0.1\\ 0 & 0 & 0 & -0.16 & -0.06 & -0.1\\ 0 & 0 & 0 & 0 & -0.16 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.16\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{12}=\begin{pmatrix} 0 & 0 & -0.002 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.002 & 0 & 0\\ -0.082 & 0 & 0 & 0 & -0.002 & 0\\ 0 & -0.082 & 0 & 0 & 0 & -0.002\\ -0.4 & -0.55 & -0.01 & 0 & 0 & 0\\ -0.55 & -0.1 & 0 & -0.01 & 0 & 0\\ 1.6357 & -0.1 & 0 & 0 & -0.01 & 0\\ -0.06 & 1.7159 & 0 & 0 & 0 & -0.01\\ 0 & 0 & 0.2179 & -0.55 & -0.4 & -0.55\\ 0 & 0 & -0.09 & 1.2153 & -0.55 & -0.1\\ -0.16 & 0 & -0.1 & -0.1 & 1.6447 & -0.1\\ 0 & -0.16 & -0.06 & -0.1 & -0.06 & 1.7249 \end{pmatrix}.$$

Thus, the coupled networks achieve global synchronization by Theorem 3.1. Fig. 2(a) shows the evolution of the components of the first network (i.e., $x_{1,\,k}(t)$, $k=1,2,3,4$) and the corresponding synchronization error $Err(t)$ for the solution, which evolves from the constant initial condition $(0.2,-0.2,0.3,0,0,-0.1,0.2,0,0.2,-0.2,0.3,0,0,-0.1,0.2,0)^T$ at $t_0=0$. This shows that the solution synchronizes.

In this example, W does not satisfy Eq. (39); hence, condition (S2) does not hold if it is considered that $\tau_T\neq 0$. Fig. 2(b) shows that by considering $\tau_T=2$ instead of $\tau_T=0$, then, for the solution that evolves from the synchronous initial condition $(\Phi(t),\ \Phi(t),\ \Phi(t),\ \Phi(t))^T$, $\Phi(t)=(e^{-t},-e^{-t},e^{-t},-e^{-t})^T$ at $t_0=0$, $Err(t)$ fails to remain zero. This indicates that the synchronous set $\mathcal{S}$, cf. (38), is not positively invariant.

Example 3. Consider three coupled networks (4) in which each individual network (1) has two neurons, and for which $d_1=2$, $d_2=1.2$, $\tau_1=\tau_2=20$, $f_1(\xi)=f_2(\xi)=(|\xi+0.1|-|\xi-0.1|)/2$, $I_1(t)\equiv 0.25$, $I_2(t)=0.1\sin(t/5)$, $T_1=T_2=0.05$, and

$$a_{kl}(x_{i,l}(t))=\begin{cases}a'_{kl} & |x_{i,l}(t)|<T_l,\\ a''_{kl} & |x_{i,l}(t)|\geq T_l,\end{cases}\quad b_{kl}(x_{i,l}(t-\tau_l))=\begin{cases}b'_{kl} & |x_{i,l}(t-\tau_l)|<T_l,\\ b''_{kl} & |x_{i,l}(t-\tau_l)|\geq T_l,\end{cases}$$

for $i=1,2,3$ and $k,l=1,2,$ where

$$\begin{cases} (a'_{11}, a'_{12}, a'_{21}, a'_{22}) = (0.01, 0.02, -0.02, -0.03), \ (a''_{11}, a''_{12}, a''_{21}, a''_{22}) \\ \quad = (-0.01, -0.02, 0.02, 0.03), \\ (b'_{11}, b'_{12}, b'_{21}, b'_{22}) = (-2.028, 0.1, 0.1, -0.2), \ (b''_{11}, b''_{12}, b''_{21}, b''_{22}) \\ \quad = (-2.026, -0.1, -0.1, 0.2). \end{cases}$$

The coupling terms $U_{i,k}(t)$ in Eq. (6), $i = 1, 2, 3$, $k = 1, 2$, have $h_1(\xi) = h_2(\xi) = 200\tanh(\xi/200)$, $\gamma_1 = 0.28$, $\gamma_2 = 0.2$, $\bar{s} = 2$ in Eq. (7), $\tau_T = 2$, and w_{ij}, $1 \le i, j \le 3$, satisfying

$$W = [w_{ij}]_{1\le i,j\le 3} := \begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.3 & 0.7 & 0 \end{pmatrix}. \tag{113}$$

From Eq. (113), the considered system uses $\tau_T = 2$, and satisfies Eq. (39) with $\chi = 1$; hence, condition (S2) is satisfied. Notably, $\bar{\rho}_l^f$ and $\bar{\rho}_l^h$ in condition (S1)*, $\bar{\upsilon}_l^f$ in Eq. (2), and $\bar{I}_k$ in Eq. (3) can be chosen as

$$\bar{\rho}_1^f = \bar{\rho}_2^f = 0.1, \bar{\rho}_1^h = \bar{\rho}_2^h = 200, \bar{\upsilon}_1^f = \bar{\upsilon}_2^f = 1, \bar{I}_1 = 0.25, \bar{I}_2 = 0.1. \tag{114}$$

From Eq. (8), we obtain

$$\begin{cases} (\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}) = (0.01, 0.02, 0.02, 0.03), \left(\bar{b}_{11}, \bar{b}_{12}, \bar{b}_{21}, \bar{b}_{22}\right) = (2.028, 0.1, 0.1, 0.2), \\ \left(d^a_{11}, d^a_{12}, d^a_{21}, d^a_{22}\right) = (0.02, 0.04, 0.04, 0.06), \left(d^b_{11}, d^b_{12}, d^b_{21}, d^b_{22}\right) = (0.002, 0.2, 0.2, 0.4). \end{cases} \tag{115}$$

From Eq. (113), $\bar{W}$ in Eq. (43) becomes

$$\bar{W} = [\bar{w}_{ij}]_{1\le i,j\le 2} = \begin{pmatrix} -1.3 & 0.5 \\ 0 & -1.7 \end{pmatrix}, \tag{116}$$

and we compute Eqs. (28) and (41) to obtain

$$\varpi = 1, \ \lambda_{1,1} = 0.616, \lambda_{1,2} = 0.44, \lambda_{2,1} = 0.784, \lambda_{2,2} = 0.56. \tag{117}$$

From Eqs. (114)–(117), we compute Eqs. (26)–(29) and (53) to obtain

$$\bar{q}_1 \approx 56.5129, \bar{q}_2 \approx 67.1125, \tilde{q}_1 \approx 15.9284, \tilde{q}_2 \approx 22.0132, \tag{118}$$

$$\tilde{\rho}_1^f = \tilde{\rho}_2^f = 0.1, \tilde{\rho}_1^h \approx 55.0554, \tilde{\rho}_2^h \approx 64.7020, \tag{119}$$

$$\check{H}_1^\chi = \check{H}_1 \approx 0.9242, \check{H}_2^\chi = \check{H}_2 \approx 0.8953, \hat{H}_1^\chi = \hat{H}_1 = \hat{H}_2^\chi = \hat{H}_2 = 1. \tag{120}$$

As seen from Eqs. (114) and (118), condition (S1) holds. From Eqs. (113)–(116), (119), (120), and $\chi = 1$, we compute Eqs. (54)–(58) to obtain

$$\hat{\alpha}_{i,1} \approx -2.2488, \hat{\alpha}_{i,2} \approx -1.3491, \bar{\beta}_{i,1} = 2.028, \bar{\beta}_{i,2} = 0.2, \bar{\gamma}_{1,1} = 0.084, \bar{\gamma}_{1,2} = 0.06, \tag{121}$$

$$\bar{\gamma}_{2,1} = 0.196, \bar{\gamma}_{2,2} = 0.14, \rho^w_{1,1} \approx 15.4395, \rho^w_{1,2} \approx 12.9644, \rho^w_{2,k} \approx 0.0240, \tag{122}$$

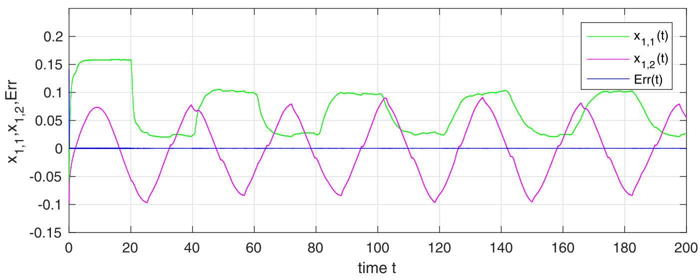


Fig. 3. Numerical simulation for Example 3.

$$\rho^e_{1,1} = 0.5302, \rho^e_{1,2} = 0.4300, \rho^e_{2,1} = 0.3622, \rho^e_{2,2} = 0.3100, \tag{123}$$

where $i = 1, 2$ and $k = 1, 2$. From Eqs. (117) and (123), it follows that condition (S3) holds. Using Eqs. (118), (121), and (122), we compute $L_{i,k}$ in condition (S4) and $\eta_{i,k}$ in Eq. (84) to obtain

$$L_{1,1} \approx 0.0002, L_{2,1} \approx 0.0246, L_{1,2} \approx 0.9925, L_{2,2} \approx 1.0089, \tag{124}$$

$$\eta_{1,1} \approx 0.1368, \eta_{2,1} \approx 0.0248, \eta_{1,2} \approx 1.0891, \eta_{2,2} \approx 1.0091. \tag{125}$$

Notably, Eq. (124) reveals that condition (S4) holds. The mapping ℓ defined in Eq. (80) now satisfies

$$\ell(1, 1) = 1, \ell(1, 2) = 2, \ell(2, 1) = 3, \ell(2, 2) = 4. \tag{126}$$

From Eqs. (88), (114)–(116), (120), (125), and (126), we obtain that $\mathbf{M}$ is approximately

$$\begin{pmatrix} 0.1368 & -0.12 & -0.14 & 0 \\ -0.12 & 1.0891 & 0 & -0.1 \\ 0 & 0 & 0.0248 & -0.12 \\ 0 & 0 & -0.12 & 1.0091 \end{pmatrix},$$

which yields $\lambda_{\text{syn}} \approx 0.5758$, cf. (91). Subsequently, the coupled networks achieve global synchronization by Theorem 3.1. Fig. 3 demonstrates the evolution of the components of the first network (i.e., $x_{1,k}(t)$, $k = 1, 2$) and the corresponding synchronization error $Err(t)$ for the solution, which evolves from the constant initial condition $(-0.1, -0.1, 0, -0.1, 0, 0)^T$ at $t_0 = 0$. This shows that the solution synchronizes.

Remark 4.1. As discussed in the Introduction, the existing work on synchronization of coupled MNNs mostly considered linear couplings or couplings that consist of a linear term and a sign function term. In addition, the work which considered multiple MNNs required that all nonzero off-diagonal entries of the connection matrix have the same signs. Notably, the couplings considered in Examples 2 and 3 consist of a nonlinear term and a sign function term; moreover, the off-diagonal entries of the connection matrix considered in Example 2 have mixed signs. To the best of our knowledge, synchronization of the systems in Examples 2 and 3 can not be concluded by existing techniques, even if the system in Example 2 is without a transmission delay (i.e., $\tau_T = 0$).

5. Conclusion

This study establishes the global synchronization of multiple MNNs. Previous studies on synchronization of coupled MNNs commonly considered linear couplings or couplings comprising a linear term and a sign function term; moreover, most studies did not consider transmission delays across different networks. In addition, the work that considered multiple MNNs required that all nonzero off-diagonal entries of the connection matrix have the same signs. The model under consideration in this paper can have both internal delays and transmission delay. The coupling functions consist of a nonlinear term and a sign term, and the off-diagonal entries of the connection matrix can have mixed signs. The derived synchronization criteria depend on whether the transmission delay exists, and it can be examined by straightforward computations. We implemented our theories to study the global synchronization of nonlinearly coupled MNNs in two examples, which cannot be treated by the existing approaches, cf. Remark 4.1. In our future work, we are interested in establishing the delay-magnitude-dependent synchronization criteria of multiple MNNs.

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