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The multifractal spectra for the recurrence rates of beta-transformations

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ABSTRACT

In this paper, we show a handy approximate approach to provide a lower bound of the Hausdorff dimension of a given subset in $[0, 1)$ related to β-transformation dynamical system. Here approximation means from special class with β-shift satisfying the specification property or being subshift of finite type to general $\beta > 1$. As an application, we obtain the multifractal spectra for the recurrence rate of the first return time of β-transformation, including the cases returning to the ball and cylinder.

1. Introduction

Let $(X, \mathcal{B}, \mu, T, d)$ be a metric measure-preserving system (m.m.p.s.), by which we mean that (X, d) is a metric space, $\mathcal{B}$ is a σ-field containing the Borel σ-field of X and $(X, \mathcal{B}, \mu, T)$ is a measure-preserving dynamical system. Under the assumption that (X, d) has a countable base, Poincaré recurrence theorem implies that μ-almost all $x \in X$ is recurrent in the sense

$$\liminf_{n\to\infty} d\big(T^n x, x\big) = 0 \tag{1.1}$$

(for example, see [11]). Later, Boshernitzan [4] has improved it by a quantitative result

$$\liminf_{n\to\infty} n^{1/\alpha} d\big(T^n x, x\big) < \infty, \quad \mu\text{-almost everywhere (a.e. for short)},$$

where α is the dimension of the space in some sense.

The above results describe whether or not a point is recurrent and how far the orbit will return to the initial point. Recurrence time is an important aspect used to characterize the behaviors of orbits in

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dynamical systems. Of the research conducted on recurrence time, the first return time of a point has been well studied in the last decade. The first return time of a point $x \in X$ into the set A is defined by

$$\tau_A(x) = \inf\{k \in \mathbb{N} : T^k x \in A\}.$$

Ornstein and Weiss [21] proved that for a finite partition ξ of X, if there exists a T-invariant ergodic Borel probability measure μ, then

$$\lim_{n\to\infty} \frac{\log \tau_{\xi_n(x)}(x)}{n} = h_\mu(\xi), \quad \mu\text{-a.e.}$$

where $\xi_n(x)$ is the intersection of $\xi, T^{-1}(\xi), \cdots, T^{-n+1}(\xi)$ which contains x, and $h_\mu(\xi)$ denotes the measure-theoretic entropy of T with respect to the partition ξ. Feng and Wu [10] considered the recurrence set of the one-sided shift space on m symbols $(\{0, 1, \ldots, m-1\}^{\mathbb{N}}, \sigma)$, where the partition ξ is the cylinders sets $\{[0], [1], \ldots, [m-1]\}$. They proved that the set

$$\left\{x \in \{0, 1, \ldots, m-1\}^{\mathbb{N}} : \liminf_{n\to\infty} \frac{\log \tau_{\xi_n(x)}(x)}{n} = \alpha,\ \limsup_{n\to\infty} \frac{\log \tau_{\xi_n(x)}(x)}{n} = \gamma\right\}$$

has Hausdorff dimension one for any $0 \le \alpha \le \gamma \le +\infty$ (see also [26]). Lau and Shu [15] extended this result to the dynamical systems with specification property by considering the topological entropy instead of Hausdorff dimension. Barreira and Saussol [2] replaced the cylinders $\xi_n(x)$ with the balls $B(x, r)$ according to quantity

$$\tau_r(x) = \inf\{n \ge 1 : T^n x \in B(x, r)\},$$

and defined the lower and upper recurrence rates of x by

$$\underline{R}(x) = \liminf_{r\to 0} R_r(x), \qquad \overline{R}(x) = \limsup_{r\to 0} R_r(x),$$

where $R_r(x) = \frac{\log \tau_r(x)}{-\log r}$. They proved that

$$\underline{R}(x) = \underline{d}_\mu(x), \qquad \overline{R}(x) = \bar{d}_\mu(x), \quad \mu\text{-a.e.} \tag{1.2}$$

with the conditions that μ has a so-called *long return time* (see [2]) and $\underline{d}_\mu(x) > 0$ for μ-a.e. x, where $\underline{d}_\mu(x)$, $\bar{d}_\mu(x)$ are the lower and upper pointwise dimensions of μ at a point $x \in X$ respectively. A simple consequence of this result is a reformulation of Boshernitzan's theory by noting that

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) = 0$$

holds for all $\alpha > \underline{d}_\mu(x)$. Many researchers have studied the problem when the formulation (1.2) holds from many different viewpoints. For example, Saussol [25, Theorem 3] proved that formulation (1.2) holds if the transformation T is piecewise Lipschitz with some condition and the decay of the correlation is super-polynomial.

Let $A(R_r(x))$ be the set of the accumulation points of $R_r(x)$ as $r \to 0$ and J a compact sub-interval of $(0, +\infty)$. Olsen [20] studied the following set

$$G \cap \{x \in K : A(R_r(x)) = J\}$$

for the self-conformal set (satisfying a certain separation condition) K with the natural self-map induced by the shift, where G is an open set with $G \cap K \neq \emptyset$. He proved that such a set shares the same Hausdorff dimension as K. This result can be applied to the case of N-adic transformation with $N \in \mathbb{N}$.

In this investigation we consider the similar problem for the β-transformation T_β with any $\beta > 1$, which includes the cases of full-shift ($\beta = N$), subshift of finite type, and cases with, but not limited to, specification condition. We use the notation $\tau_r^\beta(x)$, $R_r^\beta(x)$, $\underline{R}^\beta(x)$, $\overline{R}^\beta(x)$ to emphasize the dynamical system $([0,1), T_\beta)$. Denote by μ_β the T_β-invariant measure equivalent with the Lebesgue measure $\mathcal{L}$.

Firstly, we prove the following.

Proposition 1.1. *The set $A(R_r^\beta(x))$ is a closed interval for any $x \in [0,1)$.*

Proof. When $\lim_{r\to 0} R_r^\beta(x)$ exists, the accumulation set just contains one point and then the claim holds. Now we consider the case that such limit does not exist, say $a := \liminf_{r\to 0} R_r^\beta(x) < \limsup_{r\to 0} R_r^\beta(x) := b$.

For any $a < c < b$, fix arbitrary small $\delta > 0$, choose a decreasing sequence $\{r_k\}$ tending to zero as $k \to \infty$ such that $R_{r_k}^\beta(x) \le c \le R_{(1+\delta)r_k}^\beta(x)$. Such sequence can be found since $a < b$. Noting that $\tau_{(1+\delta)r_k}^\beta(x) \le \tau_{r_k}^\beta(x)$, we know that $R_{(1+\delta)r_k}^\beta(x) \le R_{r_k}^\beta(x)(1 + \frac{\log(1+\delta)}{\log r_k})$. Therefore,

$$R_{r_k}^\beta(x) \le c \le R_{(1+\delta)r_k}^\beta(x) \le R_{r_k}^\beta(x)\left(1 + \frac{\log(1+\delta)}{\log r_k}\right),$$

which implies $\lim_{k\to\infty} R_{r_k}^\beta(x) = c$. Thus we get the desired. $\square$

It is known that the dynamical system $([0,1), T_\beta)$ satisfies the conditions of metric theorem (Theorem 3) in [25] by noting that the measure μ_β is exponentially mixing (see [23]). Applying this metric result and noting that $\underline{d}_{\mu_\beta}(x) = \bar{d}_{\mu_\beta}(x) = 1$ for $\mathcal{L}$-almost every $x \in [0,1)$, we obtain

$$\lim_{r\to 0} R_r^\beta(x) = 1$$

for $\mathcal{L}$-almost every $x \in [0,1)$.

Theorem 1.1. *Let $\beta > 1$ be any real number and J a closed interval in $[0,+\infty]$. Denote*

$$G_J^\beta = \{x \in [0,1) : A(R_r^\beta(x)) = J\}.$$

Then

$$\dim_{\mathrm{H}} G_J^\beta = 1,$$

where $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension.*

Remark 1. Due to Proposition 1.1, Theorem 1.1 is equivalent to say $\dim_{\mathrm{H}} G_{\alpha,\gamma}^\beta = 1$, where

$$G_{\alpha,\gamma}^\beta = \left\{x \in [0,1) : \liminf_{r\to 0} R_r^\beta(x) = \alpha,\ \limsup_{r\to 0} R_r^\beta(x) = \gamma\right\} \tag{1.3}$$

with $0 \le \alpha \le \gamma \le +\infty$.

Choose $\alpha = \gamma$ in (1.3), then we have the following.

Corollary 1. *Let $\beta > 1$ be a real number and $0 \le \alpha \le +\infty$. Denote*

$$G_\alpha^\beta = \Big\{ x \in [0,1) : \lim_{r\to 0} R_r^\beta(x) = \alpha \Big\}.$$

Then $\dim_{\mathrm{H}} G_\alpha^\beta = 1$.

Now we turn to consider the first return time of the point to the cylinders containing itself. For any $n \in \mathbb{N}$ and $x \in [0,1)$, define

$$\tau_n^\beta(x) = \inf\{ m \ge 1 : T_\beta^m x \in I_n(x) \},$$

where $I_n(x)$ is the cylinder of n containing x. Let $R_n^\beta(x) = \frac{\log \tau_n^\beta(x)}{n}$. Denoted by $A(R_n^\beta(x))$ the set of all accumulation points of $R_n^\beta(x)$ as $n \to \infty$. Similarly with Proposition 1.1 and Theorem 1.1, we can prove the following proposition and theorem.

Proposition 1.2. *The set $A(R_n^\beta(x))$ is a closed interval for any $x \in [0,1)$.*

Theorem 1.2. *Let $\beta > 1$ be any real number and J a closed interval in $[0,+\infty]$. Denote*

$$E_J^\beta = \{ x \in [0,1) : A\big(R_n^\beta(x)\big) = J \}.$$

Then

$$\dim_{\mathrm{H}} E_J^\beta = 1.$$

The paper is organized as follows. Definitions and known results of β-transformations, as well as Hausdorff dimensions and measures, are given in Section 2. In Section 3, we provide a kind of approximation method from the specification case to the general case followed by a detailed proof for Theorem 1.2 and Theorem 1.1 in Section 4.

2. Preliminaries

2.1. Basic notions and notation for β-transformations

Rényi [24] introduced the β-expansions of real numbers in 1957, where $1 < \beta \in \mathbb{R}$. More specifically stated, the β-expansion of $x \in [0,1)$ is the following

$$x = \sum_{n=1}^\infty \frac{\varepsilon_n(x,\beta)}{\beta^n}, \tag{2.4}$$

where $\varepsilon_1(x,\beta) = [\beta x]$, $[x]$ is the integer part of x and $\varepsilon_n(x,\beta) = \varepsilon_1(T_\beta^{n-1}(x),\beta)$ for all $n \ge 2$. Here T_β is the *β-transformation* on the unit interval $[0,1)$ defined as

$$T_\beta(x) = \beta x - [\beta x].$$

The numbers $\varepsilon_1(x,\beta), \varepsilon_2(x,\beta), \ldots, \varepsilon_n(x,\beta), \ldots$ are the *β-digits* of the β-expansion of x and this sequence is denoted by $\varepsilon(x,\beta)$, that is,

$$\varepsilon(x,\beta) = \big(\varepsilon_1(x,\beta), \varepsilon_2(x,\beta), \ldots, \varepsilon_n(x,\beta), \ldots\big).$$

Sometimes we write $\varepsilon_n(x)$ instead of $\varepsilon_n(x,\beta)$ if there is no confusion. It is well known that the Lebesgue measure is T_β-invariant and ergodic when β is an integer. When $\beta \notin \mathbb{N}$, Rényi [24] proved that there exists a unique invariant measure μ_β which is equivalent to the Lebesgue measure (the density formula was given by Gel'fond [12] and Parry [22] independently). Furthermore, the β-transformation is ergodic and strong mixing with respect to μ_β (see Fan et al. [8], Philipp [23] and Rényi [24]).

From the definition of β-digit $\{\varepsilon_n(\cdot,\beta)\}$, we know that the set of possible values of β-digits is $\mathcal{A}_\beta = \{0,1,\dots,\beta-1\}$ when β is an integer, otherwise, $\mathcal{A}_\beta = \{0,1,\dots,[\beta]\}$. Let $(\mathcal{A}_\beta^{\mathbb{N}},\sigma)$ be the symbolic dynamics with σ the shift transformation on $\mathcal{A}_\beta^{\mathbb{N}}$. For any words u,v in the symbolic space, uv denotes the concatenation of u and v. Denote $w|_n$ as the prefix of the sequence $w \in \mathcal{A}_\beta^{\mathbb{N}}$ with length n. The finite word u^n $(n \in \mathbb{N})$ and sequence u^∞ mean $\underbrace{uu\cdots u}_{n}$ and $uu\cdots u\cdots$ respectively. We denote by Σ_β the set of the admissible sequences in $\mathcal{A}_\beta^{\mathbb{N}}$, that is,

$$\Sigma_\beta = \{w \in \mathcal{A}_\beta^{\mathbb{N}} : \text{there exists some } x \in [0,1) \text{ such that } \varepsilon(x,\beta) = w\}.$$

Let Σ_β^n be the set of admissible words of length n, that is,

$$\Sigma_\beta^n = \{w \in \mathcal{A}_\beta^n : \text{there exists some } x \in [0,1) \text{ such that } \varepsilon(x,\beta)|_n = w\}.$$

When β is an integer, Σ_β is simply $\mathcal{A}_\beta^{\mathbb{N}}$ (or more precisely $\mathcal{A}_\beta^{\mathbb{N}} = S_\beta$ defined below); when β is not an integer, Σ_β was characterized by Parry [22] (see Theorem 2.1 below) by the β-expansion of the number 1, denoted by $\varepsilon(1,\beta)$, which can be obtained in a similar manner as the β-expansion of numbers in $[0,1)$. We say that $\varepsilon(1,\beta)$ is infinite if there are infinitely many non-zero elements in the sequence $\varepsilon(1,\beta)$, otherwise, it is said to be finite. For finite case, i.e., $\varepsilon(1,\beta) = (\varepsilon_1(1),\cdots,\varepsilon_n(1),0^\infty)$ with $\varepsilon_n(1) \neq 0$ for some $n \geq 1$, we take $\varepsilon^*(1,\beta) = (\varepsilon_1(1),\varepsilon_2(1),\cdots,\varepsilon_{n-1}(1),(\varepsilon_n(1)-1))^\infty$ as the infinite expansion of 1. We will still write $\varepsilon^*(1,\beta)$ instead of $\varepsilon(1,\beta)$ for infinite cases for the sake of simplicity so that there is no ambiguity in the rest of this paper. To state the following theorem, we give two notations $\prec$ and $\preceq$, the lexicographical orders on $\mathcal{A}_\beta^{\mathbb{N}}$. That is, let $w,w' \in \mathcal{A}_\beta^{\mathbb{N}}$, then $w \prec w'$ means that there exists $n \geq 1$ such that $w_n < w'_n$ and $w_j = w'_j$ for all $j < n$, and $w \preceq w'$ means that $w \prec w'$ or $w = w'$.

Theorem 2.1. *(See [22].) Let $\beta > 1$ be a real number and $\varepsilon^*(1,\beta)$ the infinite expansion of the number 1. Then $w \in \Sigma_\beta$ if and only if*

$$\sigma^k(w) \prec \varepsilon^*(1,\beta) \quad \textit{for all } k \geq 0.$$

Let S_β be the closure of the set Σ_β. It is well known that $S_\beta = \mathcal{A}_\beta^{\mathbb{N}}$ when β is an integer and otherwise, $(S_\beta, \sigma|_{S_\beta})$ is a subshift of $(\mathcal{A}_\beta^{\mathbb{N}},\sigma)$, where $\sigma|_{S_\beta}$ is the restriction of σ to S_β. Theorem 2.1 implies the following characterization of S_β.

Corollary 2. *(See [3,16,22].) Let $\beta > 1$ be a real number and $\varepsilon^*(1,\beta)$ the infinite expansion of the number 1. Then*

$$S_\beta = \{w \in \mathcal{A}_\beta^{\mathbb{N}} : \sigma^k w \preceq \varepsilon^*(1,\beta) \textit{ for all } k \geq 0\}.$$

Proposition 2.1. *(See [22].) The function $\beta \mapsto \varepsilon^*(1,\beta)$ is increasing with respect to the variable $\beta > 1$. Therefore, if $1 < \beta_1 < \beta_2$, then*

$$\Sigma_{\beta_1} \subset \Sigma_{\beta_2}, \qquad \Sigma_{\beta_1}^n \subset \Sigma_{\beta_2}^n \quad (\textit{for all } n \geq 1) \quad \textit{and} \quad S_{\beta_1} \subset S_{\beta_2}.$$

Topological entropy of T_β and the measure-theoretical entropy of μ_β share the same value $\log\beta$, and μ_β is the unique measure of maximal entropy (see Dajani and Kraaikamp [6], Hofbauer [13], Ito and Takahashi [14]). In 1989, Blanchard [3] outlined a classification for all numbers $\beta > 1$ according to the topological properties of S_β, furthermore, the Lebesgue measures and Hausdorff dimensions of all classes were calculated by Schmeling [27]. Recently, Li and Wu [17] provided another classification by the quantity $\ell_n(\beta)$, which is defined as

$$\ell_n(\beta) = \sup\{k \geq 0 : \varepsilon^*_{n+j}(1,\beta) = 0 \text{ for all } 1 \leq j \leq k\} \tag{2.5}$$

for all $n \geq 0$. Let

$$A_0 = \Big\{\beta \in (1,+\infty) : \limsup_{n\to\infty} \ell_n(\beta) < \infty,\ \text{i.e.,}\ \{\ell_n(\beta)\} \text{ is bounded}\Big\}$$

and $A_1 = (1,+\infty)\backslash A_0$. The key function $\ell_n(\beta)$ states the maximal length of the string of 0's following $\varepsilon_n(1,\beta)$ in $\varepsilon(1,\beta)$. All β's such that S_β is a subshift of finite type are contained in A_0, and moreover, $\beta \in A_0$ if and only if S_β satisfies the specification property. Buzzi [5] proved that the set of $\beta > 1$ such that the map T_β has the specification property is of zero Lebesgue measure. It is known that the set A_0 has full Hausdorff dimension (see [27]) and is dense in $(1,\infty)$ (see [22]).

Definition 2.1. For any $w \in \Sigma^n_\beta$, we call

$$I_n(w) = \{x \in [0,1) : \varepsilon_1(x) = w_1,\ \varepsilon_2(x) = w_2,\ \ldots,\ \varepsilon_n(x) = w_n\},$$

a cylinder of order n. It is a left-closed and right-open interval. Furthermore, if $|I_n(w)| = \beta^{-n}$, we say $I_n(w)$ is full or w is full.

The full cylinder $I_n(w)$ means that any admissible word can be concatenated following w (see also [9]). The following lemma from [17] describing a way to get full cylinders, will be used to prove Lemma 4.1 and Lemma 4.2 below.

Lemma 2.1. *(See [17].) Let $\beta > 1$ be a real number and $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ an admissible word. Denote $M_n(\beta) = \max_{1\leq k\leq n}\{\ell_k(\beta)\}$, then for any $m > M_n(\beta)$, the cylinder*

$$I_{n+m}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \underbrace{0, \cdots, 0}_{m})$$

is a full cylinder of order $n + m$ and its length equals $\beta^{-(n+m)}$.

It is simple to deduce that $|I_n(w)| \leq \beta^{-n}$ for any $w \in \Sigma^n_\beta$, where $|\cdot|$ denotes the length of an interval. The following proposition characterizes the sizes of cylinders by the classification in [17].

Proposition 2.2. *(See [17].) $\beta \in A_0$ if and only if there exists a constant C such that for all $x \in [0,1)$ and $n \geq 1$,*

$$C\beta^{-n} \leq |I_n(x)| \leq \beta^{-n}.$$

Define a projection function π_β from S_β to $[0,1]$ as the following:

$$\pi_\beta(w) = \sum_{i=1}^{\infty} \frac{w_i}{\beta^i} \quad \text{where } w = (w_1, w_2, \ldots, w_i, \ldots) \in S_\beta. \tag{2.6}$$

Then π_β is one-to-one except at the countable many points for which the β-expansions are finite and the restriction of π_β to which is two-to-one. It is easy to know that π_β is continuous and $\pi_\beta \circ \sigma = T_\beta \circ \pi_\beta$.

2.2. Hausdorff dimensions and measures

Let us recall the definitions of both the Hausdorff measures and dimensions, as well as a useful *mass distribution principle* which will be used later. A finite or countable collection of subsets $\{U_i\}$ of $\mathbb{R}$ is called a δ-cover of a set $E \subset \mathbb{R}$ if $|U_i| < \delta$ for all i and $E \subset \bigcup_{i=1}^{\infty} U_i$. Let E be a subset of $\mathbb{R}$ and $s \geq 0$. For all $\delta > 0$, we define

$$\mathcal{H}^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}.$$

The s-dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

We know that there exists a critical point s_0 such that $\mathcal{H}^s(E) = \infty$ if $s < s_0$ and $\mathcal{H}^s(E) = 0$ if $s > s_0$. This point is called the Hausdorff dimension of E, denoted by $\dim_{\mathrm{H}} E$, that is,

$$\dim_{\mathrm{H}} E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

The following mass distribution principle is usually used to estimate a lower bound for the Hausdorff dimension of a set. We refer to Falconer [7] and Mattila [18] for further properties of Hausdorff dimension.

Theorem 2.2 *(Mass distribution principle). Let $E \subset \mathbb{R}$ and μ be a finite measure with $\mu(E) > 0$. Suppose that there exist $s \geq 0$, $C > 0$ and $\delta > 0$ such that*

$$\mu(U) \leq C|U|^s \tag{2.7}$$

for all sets U with $|U| \leq \delta$, where $|U|$ denotes the diameter of the set U. Then

$$\dim_{\mathrm{H}} E \geq s.$$

Remark 2. In (2.7), we can replace the set U by any ball $B(x, r)$ of radius r centered at x with r which is sufficiently small.

3. Approximation method for the β-shift

Let $1 < \beta' < \beta$. Since $\Sigma_{\beta'} \subset \Sigma_\beta$, we know that $H^{\beta'}_\beta := \pi_\beta(\Sigma_{\beta'})$ is a Cantor set of $\pi_\beta(\Sigma_\beta) = [0, 1)$. Let $m \geq 1$ and denote

$$F^\beta_m = \{x \in [0, 1) : 0^m \notin \varepsilon(x, \beta)\},$$

where $0^m \notin \varepsilon(x, \beta)$ means that the word 0^m does not appear in $\varepsilon(x, \beta)$. Sometimes we use the notations $I^\beta_n(x)$ and $I^{\beta'}_n(x)$ to distinguish the cylinders of n containing x w.r.t. β-expansion and β'-expansion respectively.

Remark 3.

$$x \in H^{\beta'}_\beta \iff \varepsilon(x, \beta) \in \Sigma_{\beta'}.$$

Define the function $h : H_\beta^{\beta'} \to [0,1)$ as

$$h(x) = \pi_{\beta'}\big(\varepsilon(x,\beta)\big).$$

Theorem 3.1. *(1) For any $x \in H_\beta^{\beta'}$, we have*

$$\varepsilon\big(h(x),\beta'\big) = \varepsilon(x,\beta).$$

(2) The function h is bijective and strictly increasing on $H_\beta^{\beta'}$.

(3) The function h is continuous on $H_\beta^{\beta'}$.

(4) If additionally assume $\beta' \in A_0$ with $M = \max\{\ell_n(\beta') : n \geq 1\}$, then h is Hölder continuous on $H_\beta^{\beta'}$, moreover,

$$\big|h(x) - h(y)\big| \leq 2\beta'^{\,M+1}|x-y|^{\frac{\log\beta'}{\log\beta}} \tag{3.8}$$

for any $x, y \in H_\beta^{\beta'}$.

(5) If additionally assume $\beta' \in A_0$ with $M = \max\{\ell_n(\beta') : n \geq 1\}$ and $m > M$, then

$$\big|h(x) - h(y)\big| \geq \beta'^{\,-(m+1)}|x-y|^{\frac{\log\beta'}{\log\beta}} \tag{3.9}$$

for any $x, y \in H_\beta^{\beta'} \cap F_m^\beta$.

Proof. (1) It is clear from the definitions of $H_\beta^{\beta'}$ and h.

(2) Suppose $h(x) = h(y)$; by (1), we have $\varepsilon(x,\beta) = \varepsilon(h(x),\beta') = \varepsilon(h(y),\beta') = \varepsilon(y,\beta)$ which implies $x = y$. That is, h is injective. For any $z \in [0,1)$, take $x = \pi_\beta(\varepsilon(z,\beta')) \in H_\beta^{\beta'}$. It is easy to check that $h(x) = z$, that is, h is surjective.

For any $x < y$, we have $\varepsilon(x,\beta) \prec \varepsilon(y,\beta)$. Thus $h(x) < h(y)$ since $\pi_{\beta'}$ is strictly increasing on $\Sigma_{\beta'}$.

(3) Let $x \in H_\beta^{\beta'}$. We will prove that h is continuous at x, that is,

$$\lim_{\substack{y\to x,\\ y\in H_\beta^{\beta'}}} h(y) = h(x). \tag{3.10}$$

If $\varepsilon(x,\beta)$ is infinite, then there exist infinitely many $n \in \mathbb{N}$ such that $y \in I_n^\beta(x)$, that is, $\varepsilon(y,\beta)|_n = \varepsilon(x,\beta)|_n$. By (1), we have $\varepsilon(h(y),\beta')|_n = \varepsilon(h(x),\beta')|_n$, that is, $h(y) \in I_n^{\beta'}(h(x))$, which implies (3.10) holds. If $\varepsilon(x,\beta)$ is finite, that is, x is the endpoint of some cylinders, note that $H_\beta^{\beta'}$ is a Cantor set, we know that y cannot approach x from left. Then $y \to x$ means that y tends to x from right. Since the cylinder $I_n(x)$ is left-closed and right-open, we have that $y \in I_n^\beta(x)$ for infinitely many n. Thus (3.10) holds similarly with the infinite case.

(4) Without loss of generality, we assume $x > y$ since it is similar for the case $x < y$ and (3.8) holds trivially if $x = y$. Let $n \geq 1$ be the smallest integer such that $\varepsilon_n(x,\beta) > \varepsilon_n(y,\beta)$. We divide the left proof to two cases according to $\varepsilon_n(x,\beta) = \varepsilon_n(y,\beta) + 1$ or not.

Case I: $\varepsilon_n(x,\beta) > \varepsilon_n(y,\beta) + 1$. By (1) and $2 \leq \varepsilon_n(x,\beta) - \varepsilon_n(y,\beta) \leq \beta'$, we have

$$\big|h(x) - h(y)\big| = \big(\varepsilon_n(x,\beta) - \varepsilon_n(y,\beta)\big)\beta'^{\,-n} + \big(T_{\beta'}^n h(x) - T_{\beta'}^n h(y)\big)\beta'^{\,-n} \leq \big(\beta'+1\big)\beta'^{\,-n}$$

and

$$|x-y| = \big(\varepsilon_n(x,\beta) - \varepsilon_n(y,\beta)\big)\beta^{-n} + \big(T_\beta^n x - T_\beta^n y\big)\beta^{-n} \geq \beta^{-n}.$$

Therefore,

$$\left|h(x)-h(y)\right| \le \left(\beta'+1\right)|x-y|^{\frac{\log\beta'}{\log\beta}}.$$

Case II: $\varepsilon_n(x,\beta)=\varepsilon_n(y,\beta)+1$. Denote

$$j=\min\left\{k\ge 1: \varepsilon_{n+1}(x,\beta)\cdots\varepsilon_{n+k}(x,\beta)\ne 0^k \text{ or } \varepsilon_{n+1}(y,\beta)\cdots\varepsilon_{n+k}(y,\beta)\ne \varepsilon\left(1,\beta'\right)\big|_k\right\}.$$

By the definition of j, there is at least one other cylinder, denoted by $I^{\beta}_{n+j}(w)$, between $I^{\beta}_{n+j}(x)$ and $I^{\beta}_{n+j}(y)$. Since $I^{\beta'}_{n+j+M+1}(w0^{M+1})$ is full for $\Sigma_{\beta'}$, we know that the cylinder $I^{\beta}_{n+j+M+1}(w0^{M+1})$ is full for Σ_{β}. It implies

$$|x-y|\ge \left|I^{\beta}_{n+j+M+1}\left(w0^{M+1}\right)\right|=\beta^{-(n+j+M+1)}.$$

By (1) and the definition of j, the cylinders $I^{\beta'}_{n+j-1}(h(y))$ and $I^{\beta'}_{n+j-1}(h(x))$ are consecutive in $\Sigma^{n+j-1}_{\beta'}$. Then

$$\left|h(x)-h(y)\right|\le 2\beta'^{\,-(n+j-1)}\le 2\beta'^{\,M+2}|x-y|^{\frac{\log\beta'}{\log\beta}}.$$

(5) Let n be the smallest integer such that $\varepsilon_n(x,\beta)\ne\varepsilon_n(y,\beta)$, also that for $\varepsilon_n(h(x),\beta')\ne\varepsilon_n(h(y),\beta')$ by (1). Then $x,y\in I^{\beta}_{n-1}(x)=I^{\beta}_{n-1}(y)$, and thus

$$|x-y|\le\beta^{-(n-1)}.$$

We assume, without loss of generality, $x>y$, which indicates $h(x)>h(y)$ by (2). Since $x\in F^{\beta}_m$, we know that $h(x)\in F^{\beta'}_m$. So $h(x)$ and $h(y)$ lie on the two sides of the cylinder $I^{\beta'}_{n+m}(\varepsilon(h(x),\beta')|n,0^m)$, which is full since $m>M$. Thus

$$\left|h(x)-h(y)\right|\ge\beta'^{\,-(n+m)}\ge\beta'^{\,-(m+1)}|x-y|^{\frac{\log\beta'}{\log\beta}}. \qquad\square$$

Corollary 3.

$$\dim_{\mathrm H} H^{\beta'}_{\beta}=\frac{\log\beta'}{\log\beta}.$$

Proof. On the one hand, by Theorem 3.1(4), we have

$$1=\dim_{\mathrm H} h\left(H^{\beta'}_{\beta}\right)\le\frac{\log\beta}{\log\beta'}\dim_{\mathrm H} H^{\beta'}_{\beta}.$$

Then $\dim_{\mathrm H} H^{\beta'}_{\beta}\ge\frac{\log\beta'}{\log\beta}$.

On the other hand, by the relationship between Hausdorff dimension and topological entropy in symbolic space S_β, we know that $\dim_{\mathrm H}\Sigma_{\beta'}=\frac{\log\beta'}{\log\beta}$. Since the projection $\pi_\beta: S_\beta\to[0,1)$ is Lipschitz, that is, $|\pi_\beta(w)-\pi_\beta(w')|\le d(w,w')$ for any $w,w'\in S_\beta$, where $d(w,w')=\beta^{-\inf\{k\ge 0: w_{k+1}\ne w'_{k+1}\}}$, we have

$$\dim_{\mathrm H} H^{\beta'}_{\beta}=\dim_{\mathrm H}\pi_\beta(\Sigma_{\beta'})\le\dim_{\mathrm H}\Sigma_{\beta'}=\frac{\log\beta'}{\log\beta}.$$

That is, $\dim_{\mathrm H} H^{\beta'}_{\beta}\le\frac{\log\beta'}{\log\beta}$. $\square$

Remark 4. (1) From Corollary 3, we know $\lim_{\beta'\to\beta} \dim_{\mathrm{H}} H_\beta^{\beta'} = 1$.

(2) Note that

$$B := \{x \in [0,1) : \text{the orbit of } x \text{ under } T_\beta \text{ is not dense in } [0,1]\} = \bigcup_{m=0}^{\infty} F_m^\beta,$$

in [16], where the authors proved $\dim_{\mathrm{H}} B = 1$, then $\sup_{m\ge 0} \dim_{\mathrm{H}} F_m^\beta = 1$, that is,

$$\lim_{m\to\infty} \dim_{\mathrm{H}} F_m^\beta = 1.$$

The function h induces a method to provide a lower bound of the Hausdorff dimension of a given set $E \subset [0,1)$. Firstly, consider a subset $E \cap H_\beta^{\beta'} \subset E$ and use the Hölder function h in Theorem 3.1 to transfer it to $h(E \cap H_\beta^{\beta'})$, whose dimension may be easier to be calculated by choosing $\beta' \in A_0$ or β' satisfying that $S_{\beta'}$ is subshift of finite type. Secondly, give a lower bound of $\dim_{\mathrm{H}} h(E \cap H_\beta^{\beta'})$ and then by the Hölder exponent of h (Theorem 3.1(4)) obtain a lower bound of $\dim_{\mathrm{H}} E \cap H_\beta^{\beta'}$, also that of $\dim_{\mathrm{H}} E$. That is,

$$\dim_{\mathrm{H}} E \ge \dim_{\mathrm{H}} E \cap H_\beta^{\beta'} \ge \frac{\log \beta'}{\log \beta} \dim_{\mathrm{H}} h\big(H \cap H_\beta^{\beta'}\big).$$

Finally, let β' approximate to β.

In the following section, we will apply this approximate method to prove Theorem 1.1 and Theorem 1.2.

4. Proof of Theorem 1.2 and Theorem 1.1

In this section we give a detailed proof for Theorem 1.2 and Theorem 1.1. First, we obtain several lemmas for $\beta \in A_0$ and then go on to prove Theorem 1.2 and Theorem 1.1 using the approximation method given in last section.

4.1. The case of bases in A_0

Let $\beta \in A_0$ and $M_\beta \ge \max\{\ell_n(\beta) : n \ge 1\}$. Denote

$$W_N = \{0^{M_\beta} w 0^{M_\beta} : w \in \Sigma_\beta^{N-2M_\beta}\},$$

where $2M_\beta \le N \in \mathbb{N}$ and by $W_N^{\mathbb{N}}$ the set of sequences $u_1 u_2 \cdots u_n \cdots$ with $u_n \in W_N$. Let $m \in \mathbb{N}$ and put

$$F_{N,m}^\beta = \{x \in [0,1) : \varepsilon(x,\beta) = \big(0^{M_\beta} w_n 0^{M_\beta}\big)_{n\ge 1} \in W_N^{\mathbb{N}},\ 0^m \notin w_n \text{ for all } n \in \mathbb{N}\}.$$

Lemma 4.1. *Let $\beta \in A_0$. For any $N > 2M_\beta$ and $M_\beta < m \le N - 2M_\beta$, we have*

$$\dim_{\mathrm{H}}\big(F_{N,m}^\beta\big) \ge s_{N,m}^\beta := 1 - \frac{2M_\beta}{N} - \frac{N-2M_\beta}{mN}\left(1 - \frac{\log(\beta-1)}{\log\beta}\right). \tag{4.11}$$

Proof. We will show a mass distribution μ supported on $F_{N,m}^\beta$ and then apply the mass distribution principle.

Firstly, we define the measure μ as a weak limit of a sequence of measures $\{\mu_k\}_{k\ge1}$ given step by step.

Step I. Define $\mu_1(I_1(\varepsilon_1)) = 1$ if $\varepsilon_1 = 0$ and otherwise, $\mu_1(I_1(\varepsilon_1)) = 0$.

Step II. Assuming μ_{k-1} is well defined, now we define the measure μ_k by the following three cases according to the position k. Denote the set

$$P = \{iN + j, (i+1)N - M_\beta + j : i = 0, 1, 2, \dots,\ j = 1, 2, \dots, M_\beta\}.$$

Case (i). $k \in P$. Define $\mu_k(I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k)) = \mu_{k-1}(I_{k-1}(\varepsilon_1, \dots, \varepsilon_{k-1}))$ if $\varepsilon_k = 0$ and otherwise, $\mu_k(I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k)) = 0$.

Case (ii). $k = iN + M_\beta + j$ with $i = 0, 1, 2, \dots$ and $1 \le j < m$. Let

$$\mu_k\big(I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k)\big) = \frac{|I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k)|}{|I_{k-1}(\varepsilon_1, \dots, \varepsilon_{k-1})|}\mu_{k-1}\big(I_{k-1}(\varepsilon_1, \dots, \varepsilon_{k-1})\big). \tag{4.12}$$

Case (iii). $k = iN + M_\beta + j$ with $i = 0, 1, 2, \dots$ and $m \le j \le N - 2M_\beta$. If $\varepsilon_1 \cdots \varepsilon_{k-1}$ does not end up with 0^{m-1}, then $\mu_k(I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k))$ is defined by the formula (4.12) and otherwise,

$$\mu_k\big(I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k)\big) = \begin{cases} 0 & \text{if } \varepsilon_k = 0, \\ \frac{|I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_k)|}{|I_{k-1}(\varepsilon_1, \dots, \varepsilon_{k-1}) \setminus I_k(\varepsilon_1, \dots, \varepsilon_{k-1}, 0)|}\mu_{k-1}(I_{k-1}(\varepsilon_1, \dots, \varepsilon_{k-1})) & \text{if } \varepsilon_k \ne 0. \end{cases}$$

Step III. Continuing the procedures in Step II as $k \to \infty$, we obtain a sequence of measures $\{\mu_k\}_{k \ge 1}$ satisfying the condition

$$\mu_k\big(I_k(\varepsilon_1, \dots, \varepsilon_k)\big) = \sum_{(\varepsilon_1, \dots, \varepsilon_k, \varepsilon_{k+1}) \in \Sigma_\beta^{k+1}} \mu_{k+1}\big(I_{k+1}(\varepsilon_1, \dots, \varepsilon_k, \varepsilon_{k+1})\big)$$

for any $(\varepsilon_1, \dots, \varepsilon_k) \in \Sigma_\beta^k$ and $k \ge 1$.

Step IV. Denote by μ a weak limit of the sequence of measures $\{\mu_k\}_{k \ge 1}$.

From the construction of μ, we know that it is supported on $F_{N,m}^\beta$.

Secondly, we will prove that the measure μ satisfies the condition (2.7) for any cylinder and any ball.

Step (a). For any $0^{M_\beta} w 0^{M_\beta} \in W_N$, we claim that

$$\mu\big(I_N\big(0^{M_\beta} w 0^{M_\beta}\big)\big) \le \left(\frac{\beta}{\beta - 1}\right)^{\frac{N - 2M_\beta}{m}} \frac{1}{\beta^{N - 2M_\beta}}. \tag{4.13}$$

In fact, by Step II Case (i), we know $\mu(I_N(0^{M_\beta} w 0^{M_\beta})) = \mu(I_{N - M_\beta}(0^{M_\beta} w))$. Let j $(0 \le j \le \frac{N - 2M_\beta}{m})$ be the times that the word 0^{m-1} appears in w and

$$w = \varepsilon_1 \cdots \varepsilon_{i_1} 0^{m-1} \varepsilon_{i_1 + m} \cdots \varepsilon_{i_j} 0^{m-1} \varepsilon_{i_j + m} \cdots \varepsilon_{N - 2M_\beta}.$$

That is, i_l $(1 \le l \le j)$ are just the positions which the word 0^{m-1} follows. Therefore, combining Step II Case (ii) and Case (iii) in the construction, $\mu(I_{N - M_\beta}(0^{M_\beta} w))$ can be written as, in the following paragraph of calculations, we omit the subindex of the orders of the cylinders for simplicity,

$$\begin{aligned} & \frac{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{N - 2M_\beta})|}{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{N - 2M_\beta - 1})|} \cdots \frac{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j + m + 1})|}{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1} \varepsilon_{i_j + m})|} \mu\big(I\big(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1} \varepsilon_{i_j + m}\big)\big) \\ &= \frac{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{N - 2M_\beta})|}{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1} \varepsilon_{i_j + m})|} \frac{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1} \varepsilon_{i_j + m})| \cdot \mu(I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1}))}{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1})| \setminus |I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1} 0)|} \\ &= \big|I\big(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{N - 2M_\beta}\big)\big| \frac{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-1})|}{\beta^{-(M_\beta + i_j + m - 1)} - \beta^{-(M_\beta + i_j + m)}} \frac{\mu(I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-2}))}{|I(0^{M_\beta} \varepsilon_1 \cdots \varepsilon_{i_j} 0^{m-2})|}, \end{aligned}$$

where the last equality is from Lemma 2.1 and the definition of the measure μ. Continuing the computation, the quantity above is equal to

$$\begin{aligned}
&\frac{\beta}{\beta-1}\frac{|I_{N-M_\beta}(0^{M_\beta}\varepsilon_1\cdots\varepsilon_{N-2M_\beta})|}{|I_{M_\beta+i_j+m-2}(0^{M_\beta}\varepsilon_1\cdots\varepsilon_{i_j}0^{m-2})|}\mu\big(I_{M_\beta+i_j+m-2}\big(0^{M_\beta}\varepsilon_1\cdots\varepsilon_{i_j}0^{m-2}\big)\big)\\
&=\cdots\\
&=\left(\frac{\beta}{\beta-1}\right)^j\frac{|I_{N-M_\beta}(0^{M_\beta}\varepsilon_1\cdots\varepsilon_{N-2M_\beta})|}{|I_{M_\beta}(0^{M_\beta})|}\mu\big(I_{M_\beta}\big(0^{M_\beta}\big)\big)\\
&\le\left(\frac{\beta}{\beta-1}\right)^{\frac{N-2M_\beta}{m}}\frac{1}{\beta^{N-2M_\beta}},
\end{aligned}$$

where the last inequality holds because $j\le\frac{N-2M_\beta}{m}$, $|I_{N-M_\beta}(0^{M_\beta}\varepsilon_1\cdots\varepsilon_{N-2M_\beta})|\le\beta^{-(N-M_\beta)}$, $|I_{M_\beta}(0^{M_\beta})|=\beta^{-M_\beta}$ and $\mu(I_{M_\beta}(0^{M_\beta}))=1$. Thus (4.13) holds.

Step (b). We will prove that (2.7) holds for any cylinder $I_k(\varepsilon_1,\cdots,\varepsilon_k)$. Without loss of generality, assume $\varepsilon_1\cdots\varepsilon_k$ is the prefix of some word in $F^\beta_{N,m}$, otherwise, (2.7) will naturally hold since $\mu(I_k(\varepsilon_1,\cdots,\varepsilon_k))=0$. Noting that $I_k(\varepsilon_1,\cdots,\varepsilon_k)\subset I_{t_k}(0^{M_\beta}w_1 0^{M_\beta}\cdots 0^{M_\beta}w_{[\frac{k}{N}]}0^{M_\beta})$, where $t_k=[\frac{k}{N}]N$, we know

$$\begin{aligned}
\mu\big(I_k(\varepsilon_1,\cdots,\varepsilon_k)\big)&\le\mu\big(I_{t_k}\big(0^{M_\beta}w_1 0^{M_\beta}\cdots 0^{M_\beta}w_{[\frac{k}{N}]}0^{M_\beta}\big)\big)\\
&=\mu\big(I_N\big(0^{M_\beta}w_1 0^{M_\beta}\big)\big)\cdots\mu\big(I_N\big(0^{M_\beta}w_{[\frac{k}{N}]}0^{M_\beta}\big)\big)\\
&\le\left(\left(\frac{\beta}{\beta-1}\right)^{\frac{N-2M_\beta}{m}}\frac{1}{\beta^{N-2M_\beta}}\right)^{[\frac{k}{N}]}\\
&\le\left(\frac{1}{\beta^k}\right)^{s^\beta_{N,m}}\le C^{-s^\beta_{N,m}}\big|I_k(\varepsilon_1,\cdots,\varepsilon_k)\big|^{s^\beta_{N,m}},
\end{aligned}$$

where the equality is from the construction of the measure μ, the second inequality is derived from (4.13) and C is an absolute constant in Proposition 2.2.

Step (c). For any ball $B(x,r)$, there exists $k\in\mathbb{N}$ such that $\beta^{-k-1}<r\le\beta^{-k}$, then $B(x,r)$ can be covered by at most $2C^{-1}$ adjoint cylinders at level k by Proposition 2.2. Combining this and Step (b), we know that (2.7) holds for any ball.

Finally, the application of the mass distribution principle (Theorem 2.2) implies (4.11). □

Recall

$$F^\beta_m=\big\{x\in[0,1):0^m\notin\varepsilon(x,\beta)\big\}.$$

We remark that the set F^β_m is related to the dynamical systems with holes (for example, see [1]) and also can be written as the following type of badly approximable points

$$F^\beta_m=\big\{x\in[0,1):T^n_\beta x\ge\beta^{-m}\ \text{for all}\ n\in\mathbb{N}\big\}.$$

The general badly approximable set for $\beta=2$ was studied in [19]. Note that $F^\beta_{N,m}\subset F^\beta_{2m+2M_\beta}$; letting $N\to\infty$ in Lemma 4.1, we obtain the following.

Remark 5. Let $\beta\in A_0$. For any $m>2M_\beta$, we have

$$\dim_{\mathrm{H}}\big(F^\beta_m\big)\ge1-\frac{2}{m-2M_\beta}\left(1-\frac{\log(\beta-1)}{\log\beta}\right).$$

Lemma 4.2. *Let $\beta \in A_0$ and J be a closed interval. Then*

$$\dim_{\mathrm{H}}\left(E_J^\beta \cap F_{3m}^\beta\right) \geq s_m^\beta := 1 - \frac{1}{m}\left(1 - \frac{\log(\beta-1)}{\log\beta}\right)$$

for any $m \geq 2M_\beta$.

Proof. The idea of the proof will eventually be to construct a function $f : F_{N,m}^\beta \to E_J^\beta \cap F_{3m}^\beta$ such that f^{-1} is nearly Lipschitz (in particular Hölder for every exponent < 1).

Choose a sequence $\{a_n\}$ in J such that $\{a_n\}$ is dense in J and $|a_{n+1} - a_n| \leq \frac{1}{n+1}$ and let $b_n = [e^{n(a_n + n^{-\frac{1}{2}})}]$, where $[\cdot]$ represents the integer part of a real number. For any given $N \in \mathbb{N}$ with $N > m - 2M_\beta$, we can obtain recursively a sequence $\{c_n\}$ of natural numbers such that

$$b_n \leq N + N\sum_{i=1}^{n} c_i + \sum_{i=1}^{n}(i + M_\beta + 1) < b_n + N. \tag{4.14}$$

It is simple to check that such $\{c_n\}$ can be uniquely determined. Denote

$$d_n = N + N\sum_{i=1}^{n} c_i + \sum_{i=1}^{n-1}(i + M_\beta + 1).$$

We will now define the function f on $F_{N,m}^\beta$. For any $x \in F_{N,m}^\beta$ with its β-expansion $\varepsilon(x,\beta) = (0^{M_\beta} w_n 0^{M_\beta})_{n\geq 1}$, we firstly construct a sequence $\{\xi^*\}$ from $\varepsilon(x,\beta)$. Write

$$\xi^{(0)} = \left(\xi_i^{(0)}\right) = \varepsilon_1\varepsilon_2\cdots\varepsilon_N 0^{M_\beta} w_1 0^{M_\beta} 0^{M_\beta} w_2 0^{M_\beta} \cdots 0^{M_\beta} w_n 0^{M_\beta} \cdots,$$

where $\varepsilon_1\varepsilon_2\cdots\varepsilon_N$ is the prefix of $\varepsilon(1,\beta)$ with length N, that is, $\xi^{(0)}$ is obtained by adding the word $\varepsilon_1\varepsilon_2\cdots\varepsilon_N$ before $\varepsilon(x,\beta)$. We have $\xi^{(0)} \in S_\beta$ using Lemma 2.1 since $\varepsilon(x,\beta)$ begins with the string of 0's with length M_β. Denote $u_1 := \xi^{(0)}|_1 0^{M_\beta} v_1$ with $v_1 \neq \xi_{1+M_\beta+1}^{(0)}$. Let

$$\xi^{(1)} = \left(\xi_i^{(1)}\right) = \xi^{(0)}\big|_{d_1} u_1 0^{M_\beta} w_{c_1+1} 0^{M_\beta} 0^{M_\beta} w_{c_1+2} 0^{M_\beta} \cdots,$$

that is, insert the word u_1 between the positions d_1 and $d_1 + 1$ of $\xi^{(0)}$. Assuming $\xi^{(k-1)}$ is well defined, we obtain $\xi^{(k)}$ according to inserting $u_k := \xi^{(k-1)}|_k 0^{M_\beta} v_k$ with $v_k \neq \xi_{k+M_\beta+1}^{(k-1)}$ between the positions d_k and $d_k + 1$ of $\xi^{(k-1)}$, that is,

$$\xi^{(k)} = \left(\xi_i^{(k)}\right) = \xi^{(k-1)}\big|_{d_k} u_k 0^{M_\beta} w_{c_k+1} 0^{M_\beta} 0^{M_\beta} w_{c_k+2} 0^{M_\beta} \cdots.$$

As this procedure continues, we get a sequence $\{\xi^{(k)}\}_{k\geq 1}$ with $\xi^{(k)}|_{d_k} = \xi^{(k-1)}|_{d_k}$ for all $k \geq 2$ and denote $\xi^* = (\xi_i^*)$ as the limit point of the sequence $\{\xi^{(k)}\}$. That is,

$$\xi^* = \varepsilon_1\cdots\varepsilon_N 0^{M_\beta} w_1 0^{M_\beta} \cdots 0^{M_\beta} w_{c_1} 0^{M_\beta} u_1 0^{M_\beta} w_{c_1+1} 0^{M_\beta} \cdots 0^{M_\beta} w_{c_n} 0^{M_\beta} u_n 0^{M_\beta} w_{c_n+1} 0^{M_\beta} \cdots.$$

According to Theorem 2.1, we know $\xi^{(k)} \in \Sigma_\beta$ and $\xi^* \in S_\beta$. Denote

$$x^* = \pi_\beta\left(\xi^*\right) = \frac{\xi_1^*}{\beta} + \frac{\xi_2^*}{\beta^2} + \cdots + \frac{\xi_n^*}{\beta^n} + \cdots.$$

Then $\varepsilon(x^*,\beta) = \xi^*$.

We claim that

$$d_{n-M_\beta} \le \tau_n^\beta(x^*) \le d_n \quad \text{for all } n > N. \tag{4.15}$$

Indeed, we have $\tau_n^\beta(x^*) \le d_n$ since $\varepsilon(x^*, \beta) = \xi^*$ and

$$\sigma^{d_n}(\xi^*) = u_n 0^{M_\beta} w_{c_n+1} 0^{M_\beta} \cdots = \xi^*\big|_n 0^{M_\beta} v_n 0^{M_\beta} w_{c_n+1} 0^{M_\beta} \cdots$$

from the construction of ξ^* and x^*. All that remains to be proven is $\tau_n^\beta(x^*) \ge d_{n-M_\beta}$, since the word $\xi^*|_n$ does not appear in any first d_{n-M_β} positions of ξ^* except the initial position. In fact, from the structure of x and the construction of ξ^*, we know that $\xi^*|_n$ does not appear in the positions lying in any $0^{M_\beta} w_i 0^{M_\beta}$ $(i \ge 1)$ since $\xi^*|_n$ begins with the first N digits of the β-expansion of the number 1 and the maximal length of the string of 0's in $\xi^*|_N$ is less than M_β. Combining $|u_i| = i + M_\beta + 1 < n$ for all $1 \le i < n - M_\beta - 1$ and because the last letter of u_i is not the same with $\xi^*_{i+M_\beta+1}$, we know that $\xi^*|_n$ does not appear in u_i $(1 \le i < n - M_\beta)$. So (4.15) holds.

We claim that

$$\lim_{n\to\infty}\left(\frac{\log \tau_n^\beta(x^*)}{n} - a_n\right) = 0, \tag{4.16}$$

which implies

$$A\left(\frac{\log \tau_n^\beta(x^*)}{n}\right) = J \tag{4.17}$$

since $\{a_n\}$ is dense in J. Now we verify the equality (4.16). Indeed, by (4.14), we have

$$b_n \le d_n + (n + M_\beta + 1) < b_n + N. \tag{4.18}$$

Combine (4.18) and (4.15), to obtain

$$b_{n-M_\beta} - (n+1) \le \tau_n^\beta(x^*) \le b_n + N - (n + M_\beta + 1) \le b_n \tag{4.19}$$

whenever $n > N$. Note that because $b_n = [e^{n(a_n + n^{-\frac{1}{2}})}]$, we know

$$\lim_{n\to\infty}\left(\frac{\log b_n}{n} - a_n\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{\log(b_{n-M_\beta} - (n+1))}{n} - a_n\right) = \lim_{n\to\infty}(a_{n-M_\beta} - a_n).$$

By $|a_{n+1} - a_n| \le \frac{1}{n+1}$, we have obtained $\lim_{n\to\infty}(a_{n-M_\beta} - a_n) = 0$. Therefore, by (4.19), we find that (4.16) holds.

Define the function f as $f(x) = x^*$ for any $x \in F_{N,m}^\beta$. Combining (4.17) and the structure of ξ^*, we know

$$f(F_{N,m}^\beta) \subset E_J^\beta \cap F_{3m}^\beta \tag{4.20}$$

whenever $m \ge 2M_\beta$.

We consider f^{-1} on $f(F_{N,m}^\beta)$ as $f^{-1}(x^*) = x$, that is, delete the first N digits and the digits between $d_i + 1$ and $d_i + i + M_\beta + 1$ positions for all $i \ge 1$. That is, the words $\varepsilon_1 \cdots \varepsilon_N$ and u_n $(n \ge 1)$ are removed from ξ^*. We claim that f^{-1} is $(1-\eta)$-Hölder for any $\eta > 0$. In fact, for any $x^*, y^* \in I_n(x^*)$, where n is the largest integer such that $y^* \in I_n(x^*)$ (assume $\varepsilon_{n+1}(x^*, \beta) > \varepsilon_{n+1}(y^*, \beta)$ without loss of generality), then $x, y \in I_{n'}(x)$ for some n' from the definition of f^{-1}. By (4.14), we know that d_n is of exponential rate and that the total number of deleted digits $|u_n|$ have a polynomial growth rate, therefore $n' \geq n(1-\eta)$ can be assured when n is large enough. Since $x^* \in F^\beta_{3m}$, we have

$$\begin{aligned}
&\left|x^* - y^*\right| \\
&= \frac{\varepsilon_{n+1}(x^*,\beta) - \varepsilon_{n+1}(y^*,\beta)}{\beta^{n+1}} + \frac{1}{\beta^{n+1}}\left(\frac{\varepsilon_{n+2}(x^*,\beta)}{\beta} + \cdots\right) - \frac{1}{\beta^{n+1}}\left(\frac{\varepsilon_{n+2}(y^*,\beta)}{\beta} + \cdots\right) \\
&\geq \frac{1}{\beta^{n+1}} + \frac{1}{\beta^{n+1+3m}} - \frac{1}{\beta^{n+1}} = \frac{1}{\beta^{n+1+3m}}.
\end{aligned}$$

Note that $|x-y| \leq \beta^{-n'} \leq \beta^{-n(1-\eta)}$, so

$$\left|f^{-1}\left(x^*\right) - f^{-1}\left(y^*\right)\right| \leq \left(\beta^{1+3m}\right)^{1-\eta}\left|x^* - y^*\right|^{1-\eta}.$$

Therefore, $\dim_{\mathrm{H}}(F^\beta_{N,m}) \leq \frac{1}{1-\eta}\dim_{\mathrm{H}} f(F^\beta_{N,m})$. Letting $\eta \to 0$, by (4.20) and Lemma 4.1, we have

$$\dim_{\mathrm{H}}\left(E^\beta_J \cap F^\beta_{3m}\right) \geq s^\beta_{N,m}.$$

Letting $N \to \infty$, we obtain $\dim_{\mathrm{H}}(E^\beta_J \cap F^\beta_{3m}) \geq s^\beta_m$. $\square$

Corollary 4. *If $\beta \in A_0$, then* $\dim_{\mathrm{H}} E^\beta_J = 1$.

Proof. Since $E^\beta_J \cap F^\beta_{3m} \subset E^\beta_J$, Lemma 4.2 implies that Corollary 4 holds by letting $m \to \infty$. $\square$

Remark 6. For the recurrence rate $\tau^\beta_r(x)$ to the ball, we can similarly with Lemma 4.2 prove that

$$\dim_{\mathrm{H}}\left(G^\beta_J \cap F^\beta_{3m}\right) \geq s^\beta_m \tag{4.21}$$

for any $\beta \in A_0$. In fact, we construct the same x^* as Lemma 4.2; we claim that

$$I_{n+1}\left(x^*\right) \subset B\left(x^*, r\right) \subset I_{n-3N}\left(x^*\right) \tag{4.22}$$

for any $r > 0$ and $|I_{n+1}(x^*)| < r \leq |I_n(x^*)|$. So $\tau^\beta_{n-3N}(x^*) \leq \tau^\beta_r(x^*) \leq \tau^\beta_{n+1}(x^*)$, which implies $A(\frac{\log \tau^\beta_r(x^*)}{-\log r}) = J$ since $|I_n(x^*)| \approx \beta^{-n}$ (change the base e to β in $\{b_n\}$ that is, $b_n = [\beta^{n(a_n + n^{-\frac{1}{2}})}]$), thus (4.21) holds following the same argument with the proof of Lemma 4.2. Now we prove (4.22), indeed, $I_{n+1}(x^*) \subset B(x^*, r)$ is from $|I_{n+1}(x^*)| < r$. From the β-expansion of x^*, we know that 0^{m+2M_β} does not appear and $0^{M_\beta}0^{M_\beta}$ does appear in $\varepsilon_{n-3N+1}(x^*,\beta)\cdots\varepsilon_n(x^*,\beta)$. Then the full cylinder $I_n(\varepsilon_1(x^*,\beta),\cdots,\varepsilon_{n-3N}(x^*,\beta),0^{3N})$ lies on the left side of $I_n(x^*)$ and its length equals to β^{-n} ($\geq |I_n(x^*)|$). Similarly, we can find a full cylinder of order n inside $I_{n-3N}(x^*)$ lying on the right side of $I_n(x^*)$. Therefore, (4.22) holds.

4.2. General case for any β

Denote

$$E^{\beta,\beta'}_J = \left\{x \in H^{\beta'}_\beta : A\left(R^\beta_n(x)\right) = J\right\}$$

and

$$G^{\beta,\beta'}_J = \left\{x \in H^{\beta'}_\beta \cap F^\beta_m : A\left(R^\beta_r(x)\right) = J\right\}.$$

Recall $h : H^{\beta'}_\beta \to [0,1)$ defined as $h(x) = \pi_{\beta'}(\varepsilon(x,\beta))$.

Lemma 4.3. *For any given closed interval J, we obtain*

$$h\left(E_J^{\beta,\beta'}\right) = E_J^{\beta'}.$$

Proof. By Theorem 3.1(1), we know $\tau_n^{\beta'}(h(x)) = \tau_n^{\beta}(x)$ for any $x \in H_\beta^{\beta'}$ and $n \in \mathbb{N}$. Thus $h(E_J^{\beta,\beta'}) \subset E_J^{\beta'}$. Meanwhile, for any $y \in E_J^{\beta'}$, note that h is bijective, take $z = h^{-1}(y) \in H_\beta^{\beta'}$. We obtain $\varepsilon(z,\beta) = \varepsilon(h(z),\beta') = \varepsilon(y,\beta')$ by Theorem 3.1(1), thus $z \in E_J^{\beta,\beta'}$, which implies $E_J^{\beta'} \subset h(E_J^{\beta,\beta'})$. □

Finally, we will summarize the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\beta' \in A_0$ and $\beta' \le \beta$. According to Lemma 4.3 and Theorem 3.1(4), note that $E_J^{\beta,\beta'} \subset E_J^{\beta}$, then we have

$$\dim_{\mathrm{H}}\left(E_J^{\beta'}\right) = \dim_{\mathrm{H}}\left(h\left(E_J^{\beta,\beta'}\right)\right) \le \frac{\log\beta}{\log\beta'}\dim_{\mathrm{H}}\left(E_J^{\beta,\beta'}\right) \le \frac{\log\beta}{\log\beta'}\dim_{\mathrm{H}}\left(E_J^{\beta}\right).$$

That is, $\dim_{\mathrm{H}}(E_J^{\beta}) \ge \frac{\log\beta'}{\log\beta}\dim_{\mathrm{H}}(E_J^{\beta'})$. By applying Lemma 4.2 to β', we obtain

$$\dim_{\mathrm{H}}\left(E_J^{\beta}\right) \ge \frac{\log\beta'}{\log\beta}s_m^{\beta'}.$$

Since A_0 is dense in $(1,\infty)$, let $\beta' \to \beta$, and we obtain $\dim_{\mathrm{H}} E_J^{\beta} = 1$. □

Lemma 4.4. *Let $\beta' \in A_0$. For any given closed interval J and $m \in \mathbb{N}$. We have*

$$h\left(G_J^{\beta,\beta'}\right) = G_{J'}^{\beta'} \cap F_m^{\beta'},$$

where $J' = \frac{\log\beta}{\log\beta'}J$.

Proof. Applying Theorem 3.1(4) and (5) to any $x \in H_\beta^{\beta'}$ and $T_\beta^k x$, and noting that $h(T_\beta^k x) = T_{\beta'}^k h(x)$, we have

$$\beta'^{\,-(m+1)}\left|x - T_\beta^k x\right|^{\frac{\log\beta'}{\log\beta}} \le \left|h(x) - T_{\beta'}^k h(x)\right| \le 2\beta'^{\,M+1}\left|x - T_\beta^k x\right|^{\frac{\log\beta'}{\log\beta}}.$$

Thus

$$\tau_{c_1(r)}^{\beta'}\left(h(x)\right) \le \tau_r^{\beta}(x) \quad \text{and} \quad \tau_{c_2(r)}^{\beta}(x) \le \tau_r^{\beta'}\left(h(x)\right),$$

where $c_1(r) = 2\beta'^{\,M+1}r^{\frac{\log\beta'}{\log\beta}}$ and $c_2(r) = \beta'^{\,(m+1)}r^{\frac{\log\beta}{\log\beta'}}$. Therefore,

$$\underline{R}^{\beta'}\left(h(x)\right) = \frac{\log\beta}{\log\beta'}\underline{R}^{\beta}(x) \quad \text{and} \quad \overline{R}^{\beta'}\left(h(x)\right) = \frac{\log\beta}{\log\beta'}\overline{R}^{\beta}(x). \tag{4.23}$$

Noting that $h(F_m^{\beta}) = F_m^{\beta'}$ by Theorem 3.1(1), together with (4.23), we obtain $h(G_J^{\beta,\beta'}) = G_{J'}^{\beta'} \cap F_m^{\beta'}$. □

Proof of Theorem 1.1. By Lemma 4.4, we have

$$\dim_{\mathrm{H}} G_{J'}^{\beta'} \cap F_m^{\beta'} = \dim_{\mathrm{H}} h\left(G_J^{\beta,\beta'}\right) \le \frac{\log\beta}{\log\beta'}\dim_{\mathrm{H}} G_J^{\beta,\beta'} \le \frac{\log\beta}{\log\beta'}\dim_{\mathrm{H}} G_J^{\beta},$$

where the first inequality is from Theorem 3.1(4) and the second inequality is because $G_J^{\beta,\beta'} \subset G_J^{\beta}$. Applying (4.21) to β', we get

$$\dim_{\mathrm{H}} G_{J'}^{\beta'} \cap F_{3m}^{\beta'} \geq s_m^{\beta'}.$$

Thus

$$\dim_{\mathrm{H}} G_J^{\beta} \geq \frac{\log \beta'}{\log \beta} s_m^{\beta'}.$$

By letting $m \to \infty$, we obtain

$$\dim_{\mathrm{H}} G_J^{\beta} \geq \frac{\log \beta'}{\log \beta}.$$

Let $\beta' \to \beta$, and we obtain $\dim_{\mathrm{H}} G_J^{\beta} = 1$. □

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