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The Devil's Staircase Dimensions and Measure-Theoretical Entropy of Maps with Horizontal Gap*

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This work elucidates the measure-theoretical entropy and dimensions of a unimodal map with a horizontal gap. The measure-theoretical entropy and dimensions of the F_t (which is defined later) are shown to form a devil's staircase structure with respect to the gap size t. Pesin's formula for gap maps is also considered.

KEY WORDS: Upper (lower) box dimension; Hausdoff dimension; devil's staircase; Pesin's formula.

1. INTRODUCTION

This study addresses a special family of maps (gap maps), which arise in the communication with chaos in cellular neural networks. (Readers should refer to [2].) Such maps are constructed by cutting a gap into a given unimodal map f.

The most interesting invariants to be considered in determining whether a map is chaotic are topological entropy, measure-theoretical entropy, dimensions and Lyapunov exponents. In [9], the authors examined such maps and observed the devil's staircase structure of the entropy function with respect to the gap size. Then in [2], the authors applied kneading theory to prove rigorously this presence of such structure for a

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map f with strong transitivity. In [4], Misiurewicz proved that this structure can be applied to more general maps by considering a dynamic mean rather than kneading theory.

Nature questions arise regarding what happens to the measure-theoretical entropy for certain measure for these families of maps, and what about their dimensions (Hausdorff, upper box and lower box dimension are considered).

This investigation addresses whether for some function f there exist some measure μ (defined later), where the measure-theoretical entropy $h_\mu(F_t)$ (F_t is a family of gap maps induced by f) with respect to this measure μ the devil's staircase structure remains; that is, $h_\mu(t) \equiv h_\mu(F_t)$ is a monotonic and continuous function. Furthermore, the union of the constant part of $h_\mu(t)$ is open and dense in parameter space, i.e., $\{t \in [0,1] | h_\mu(t) \text{ is constant}\}$ is open and dense in $[0, 1]$.

Moreover, for a given f, the Hausdorff, lower box and upper box dimension can be determined. For these three dimensions, the devil's staircase structure also obtains. Additionally, these three dimensions are equal for some function f.

Similar results of the well-known Pesin's dimension theorem for these maps were obtained.

This paper is organized as follows. Section 2 provides some well-known definitions, results and the main theorem. Section 3 presents a geometric structure developed by Moran, Pesin and Weiss. Section 4 includes relevant proof of the main theorems.

2. PRELIMINARIES

This section presents some definitions and well-known theorems without proof. f is assumed to be a continuous unimodal map.

Definition 1. A continuous map f: $[0, 1] \to [0, 1]$ is unimodal if there exist $c \in [0, 1]$ such that

(i) $f(0) = f(1) = 0$,
(ii) $f(x)$ is monotonically increasing on $[0, c]$ and decreasing on $[c, 1]$.

c is called turning point herein.

Definition 2. Tent map f: $[0, 1] \to [0, 1]$ with slope $\lambda > 0$ is defined

$$T_\lambda(x) = \begin{cases} \lambda_x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ \lambda(1-x) & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Definition 3. Given a continuous map f, for $t \in [0,1]$, we define $U_t\{= x \mid f(x) > t\}$ and the gap map $F_t(x)$ with respect to $f(x)$ is

$$F_t(x) = \begin{cases} t & \text{if } 0 \in U_t, \\ f(x) & \text{if } x \in [0,1] \backslash U_t. \end{cases}$$

Furthermore,

$$\begin{aligned} C(t) &= C(F_t) \equiv \{x \in [0,1] \mid f^n(x) \notin U_t, \quad \text{for all } n \geqslant 1\}, \\ C_m(t) &= C_m(F_t) \equiv \{x \in [0,1] \mid f^k(x) \notin U_t, \quad \text{for } 1 \leqq k \leqq m\}, \\ D(t) &= D(F_t) \equiv \{x \in [0,1] \mid f^k(x) \in U_t, \quad \text{for some } k \geqslant 1\}. \end{aligned}$$

The box dimensions are defined as follows.

Definition 4. For a compact set $A \subset R^m$, the lower box dimension of A, $\underline{\dim}_B(A)$, is defined by

$$\underline{\dim}_B(A) = \liminf_{\varepsilon \to 0} \frac{\log N_A(\varepsilon)}{\log \varepsilon^{-1}},$$

where $N_A(\varepsilon)$ is the least number of balls of radius ε needed to cover A. The upper box dimension of A is similarly defined, by

$$\overline{\dim}_B(A) = \limsup_{\varepsilon \to 0} \frac{\log N_A(\varepsilon)}{\log \varepsilon^{-1}}.$$

In particular, the lower and upper box dimension of $C(t)$ for $t \in (0,1)$ are defined by

$$\underline{\dim}_B(t) \equiv \underline{\dim}_B(C(F_t))$$

and

$$\overline{\dim}_B(t) \equiv \overline{\dim}_B(C(F_t)),$$

respectively.

Definition 5. Let X be a metric space with metric d. For $x \in X$ and $\rho > 0$, let

$$B_\rho(x) = \{y : d(x,y) \leqslant \rho\}.$$

The diameter of a cover κ of X is $\sup\{\text{diam } A : A \in \kappa\}$. Then the Hausdorff dimension of X is defined by

$$\dim_H(X) = \inf \left\{ \alpha : \liminf_{\varepsilon \to 0} \sum_{\substack{A \in \kappa \\ \kappa : \text{cover of } X \\ \text{with diam } \kappa \leqslant \varepsilon.}} (\text{diam } A)^\alpha = 0 \right\}$$

In particular, $\dim_H(t) \equiv \dim_H(C(F_t))$.

The definition of topological entropy is defined as follows.

Definition 6. Let $f: X \to X$ be uniformly continuous on the metric space X. For E, $F \subset X$ we say that $E(n, \delta)$–spans F (with respect to f), if for each $y \in F$ there is an $x \in E$ so that $d(f^k(x), f^k(y)) \leqslant \delta$ for all $0 \leqslant k \leqslant n$. We let $r_n(F, \delta) = r_n(F, \delta, f)$ denote the minimum cardinality of a set, which (n, δ)–spans F. If K is compact, then the continuity of f guarantees $r_n(F, \delta) < \infty$. For compact K we define

$$\bar{r}_f(K, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(K, \delta)$$

and

$$h(f, K) = \lim_{\delta \to 0} \bar{r}_f(K, \delta).$$

Finally let $h_{\text{top}}(f) = \sup_K h_{\text{top}}(f, K)$, where K varies over all compact subsets of X.

Here, some well-known theorems concerning topological entropy are presented without proof.

Theorem 1. *If f is a continuous map, then*

$$h_{\text{top}}(f) = \lim_{n \to \infty} \frac{\log \text{card}(P_n(f))}{n} = \lim_{n \to \infty} \frac{\log \text{lap}(f^n)}{n},$$

where $P_n(f)$ is the set of periodic points of period $\leqslant n$, card*(A) presents cardinality of A, and $lap(f^n)$ is the number laps of f^n, that is, the minimal number of intervals on which f^n is monotonic.*

Proof. Refer [1]. □

The measure-theoretical entropy is defined as follows.

Definition 7. Suppose f: $X \to X$ is a map on a metric space (X, d), where d is a metric. Let μ be a Borel measure on X with $\mu(X) = 1$, and μ is f-invariant (i.e., $\mu(f^{-1}(A)) = \mu(A)$ for every Borel set A). One can then define a measure theoretic entropy $h_\mu(f)$ as follows: Call $\alpha=\{A_1,\ldots,A_r\}$ a (finite) measurable partition of X if the A_i are disjoint measurable subsets of X covering X. Now set

$$H_m(\alpha)=\sum_{1\leqslant i_0,\ldots,i_{m-1}\leqslant r} -\mu\Big(\bigcap_{k=0}^{m-1} f^{-k}A_{i_k}\Big)\log \mu\Big(\bigcap_{k=0}^{m-1} f^{-k}A_{i_k}\Big).$$

Then the limit $h_\mu(f,\alpha)=\lim_{m\to\infty}(1/m)H_m(\alpha)$ exists and one defines $h_\mu(f)=\sup\{h_\mu(f,\alpha): \alpha \text{ is a finite measurable partition of } X\}$.

The well-known theorem that relates the topological and measure-theoretical entropy is stated as follows.

Theorem 2. (*Goodwyn*). *Let (X,d) be a compact metric space, let f: $X\to X$ be continuous map and μ be a f–invariant Borel measure on X with $\mu(X)=1$. Then $h_\mu(f)\leqslant h_{\text{top}}(f)$.*

The main results in the work of [2] and [4] are included.

Theorem 3. (*BHL & Misiurewicz*). *If f is a unimodal map with $h_{\text{top}}(f)>0$, the topological entropy function $h_{\text{top}}(t)=h_{\text{top}}(F_t)$ of the gap map F_t forms a devil's staircase function with respect to t; that is, $h_{\text{top}}(t)$ a monotonic, continuous function, and the constant parts of $h_{\text{top}}(t)$ is open and dense in parameter space, i.e., $\{t\in[0,1]|h_{\text{top}}(t)$ is constant$\}$ is open and dense in* $[0,1]$.

Then, the main theorem of this work is as follows.

Theorem A. *If f is a unimodal map with $h_{\text{top}}(f)>0$ and let F_t be the gap map induced by f, then*

(i) $\underline{\dim}_B(t), \overline{\dim}_B(t)$ *and* $\dim_H(t)$ *are monotonic increasing functions.*

(ii) *The union of the constant parts of each of the three dimensions are open and dense in parameter space; that is,* $\{t\in[0,1]|\underline{\dim}_B(t)$ ($\dim_H(t)$ *and* $\overline{\dim}_B(t)$, *respectively*) *is constant*$\}$ *is open and dense in* $[0,1]$.

Furthermore, If f is a tent map with slope $\lambda>1$, then

(iii) $\underline{\dim}_B(t)=\overline{\dim}_B(t)=\dim_H(t)$ *and is devil's staircase function with respect to t.*

(iv) *there exist a measure μ (which is defined in Theorem 4) such that $h_\mu(t)$ of the gap map F_t forms a devil's staircase function with respect to t.*

Theorem B. *Under the same assumption as in Theorem A. Let V be some maximal interval of the constant part of $h_{\text{top}}(t)$; V is also a the maximal constant part in $\underline{\dim}_B$, $\overline{\dim}_B$ and $\dim_H$.*

Furthermore, there exist $\bar{t} \in \partial V$ such that for all t in V,

$$\underline{\dim}_B(t) \leqslant \frac{h_{\text{top}}(\bar{t})}{\log \lambda} \overline{\lim_{n\to\infty}} \left(\frac{\log \bar{\varepsilon}_n^{-1}}{\log \hat{\varepsilon}_n^{-1}} \right), \tag{2.1}$$

where $\lambda = \lim_{n\to\infty} |Df^n(\bar{t})|^{1/n}$ the Lyapunov exponent, $\bar{\varepsilon}_n(t)$ is the minimum length of the elements in $C_n(t)$, and $\hat{\varepsilon}_n(t)$ is the maximum length of the elements in $C_n(t)$.

In addition, if f is a tent map with slope $\lambda > 1$, the inequality is actually equality; that is,

$$\underline{\dim}_B(t) = \frac{h_{\text{top}}(\bar{t})}{\log \lambda}.$$

3. MORAN-LIKE GEOMETRIC CONSTRUCTIONS

In this section, we recall the Moran-like geometric construction [6] given by Pesin and Weiss [7,8], and some theorems are stated without proof.

Let $\sum_p^+$ be the set of all one-sided infinite sequences $(i_0, i_1, \dots)$ of p symbols, and σ be the shift map, i.e., $\sigma(w)_i = w_{i+1}$ if $w = (i_0, i_1, \dots)$. The basic geometric construction is considered first. Starting from p arbitrary closed subsets $\Delta_1, \dots, \Delta_p$ of R^m, a Cantor-like set is defined by

$$F = \bigcap_{n=0}^{\infty} \bigcup_{\substack{(i_0,\dots,i_n) \\ Q-\text{admissible}}} \Delta_{i_0,\dots,i_n}, \tag{3.1}$$

where the basic sets on the nth step of the geometric construction, $\Delta_{i_0,\dots,i_n}$, $i_k = 1, \dots, p (n \geqslant 0)$ are closed, and $Q \subset \sum_p^+$ is compact set invariant under the shift map σ.

In 1996 [8], Pesin and Weiss established the following geometric construction whose basic sets satisfy the following conditions:

(i) $\Delta_{i_0,\dots,i_n j} \subset \Delta_{i_0,\dots,i_n}$ for $j = 1, \dots, p$;
(ii) $\underline{B}_{i_0,\dots,i_n} \subset \Delta_{i_0,\dots,i_n} \subset \bar{B}_{i_0,\dots,i_n}$, where $\underline{B}_{i_0,\dots,i_n}$ and $\bar{B}_{i_0,\dots,i_n}$ are closed balls of radii $\underline{r}_{i_0,\dots,i_n}$ and $\bar{r}_{i_0,\dots,i_n}$;
(iii) $\underline{r}_{i_0,\dots,i_n} = K_1 \prod_{j=0}^{n} \lambda_{i_j}, \bar{r}_{i_0,\dots,i_n} = K_2 \prod_{j=0}^{n} \lambda_{i_j}$, where $\lambda_{i_j} \in \{\lambda_1, \dots, \lambda_k\}$ with $0 < \lambda_k < 1$ for $k = 1, \dots, p$, $K_1 > 0$, and $K_2 > 0$ are constants;

(iv) int $B_{i_0,\ldots,i_n} \bigcap \text{int}\, B_{j_0,\ldots,j_m} = \emptyset$ for any $(i_0,\ldots,i_n) \neq (j_0,\ldots,j_m)$ and $m \geqslant n$.

Let (Q,σ) be the dynamical systems generated by the above construction, where $Q \subset \sum_p^+$ is a compact shift invariant set and σ is shift map. Pesin and Weiss were able to prove following results for F and (Q,σ), [8] and [7].

Theorem 4. *Let F be the limit set for a geometric construction* ((i)–(iv)) *modeled by a symbolic dynamical system* (Q,σ). *Then*

(1) $\underline{\dim}_B F = \overline{\dim}_B F = \dim_H F = s_\lambda$, *where* s_λ *is such that*

$$P_Q(s_\lambda \log \lambda_{i_0}) = 0,$$

P_Q *denote the topological pressure on Q with respect to the shift map* σ

(2) $s_\lambda = -h_{\mu_\lambda}(\sigma|Q)/\int_Q \log \lambda_{i_0} d_{\mu_\lambda}$, *where* μ_λ *be an equilibrium measure for the function* $(i_0,\ldots,i_n) \mapsto s_\lambda \log \lambda_{i_0}$ *on Q*
(Ref Theorem 13.1 of [7], p. 125).

Theorem 5. *Let F be the limit set for a geometric construction* ((i)–(iv)) *modeled by a symbolic dynamical system* (Q,σ). *Then*

(1) $s_\lambda \leqslant h_Q(\sigma)/-\log \lambda_{\max}$, *where* $\lambda_{\max} = \max\{\lambda_k : 1 \leqslant k \leqslant p\}$ *and* $h_Q(\sigma)$ *is the topological entropy of* σ *on Q; equality occurs if* $\lambda_i = \lambda$ *for* $i = 1,\ldots,p$; *in particular, if* $h_Q(\sigma) = 0$, *then*

$$\underline{\dim}_B F = \overline{\dim}_B F = \dim_H F = 0,$$

(2) *if* $\lambda_i = \lambda$ *for* $i = 1,\ldots,p$, *then*

$$\underline{\dim}_B F = \overline{\dim}_B F = \dim_H F = s_\lambda = \frac{h_Q(\sigma)}{-\log \lambda}.$$

(Ref Theorem 13.2 of *[7],* p. 129)

4. PROOF OF THE MAIN THEOREM

This section presents detail proofs of the Theorem A and Theorem B. f is assumed throughout to be a unimodal map with positive topological entropy.

We first prove (iv) of Theorem A.

Lemma 6. *If is a tent map with slope* $\lambda > 1$, *let* F_t *be the gap maps induced by f, then the measure-theoretical entropy* $h_{\mu_\lambda}(t)$ *with respect to* *the equilibrium measure defined in Theorem 4 is monotonic, continuous and the constant part of* $h_{\mu_\lambda}(t)$ *is open and dense in parameter space; that is,* $\{s|h_{\mu_\lambda}(t)=constant\}$ *is open and dense in* $[0,1]$.

Proof. The geometric construction of Pesin and Weiss can be easily demonstrated as follows:

(1) Take $p=2$, and let $\Delta_1=C_{1,1}=C_{1,1}(t)$ be the left part and $\Delta_2=C_{1,2}=C_{1,2}(t)$ be the right part of $[0,\ 1]\backslash U_t$, where $C_{m,i}$ are the elements of C_m. Now $\Delta_{i_0\ldots i_{n-1}j}\in C_{n+1}$ is such that $f(\Delta_{i_0\ldots i_{n-1}j})=\Delta_{i_1\ldots i_{n-1}j}$, where $\Delta_{i_1\ldots i_{n-1}j}$ is some element of C_n.
(2) Conditions (ii) and (iv) clearly hold for C_m and C_{m+1},
(3) Condition (iii) holds by showing $K_1=\mathrm{diam}C_{1,1}=\mathrm{diam}\Delta_1$ or $K_2=\mathrm{diam}C_{1,2}=\mathrm{diam}\Delta_2$ and $\lambda_k=\lambda$, for $k=1,2$.

Thus for such a measure μ_λ defined in Theorem 4, and by applying Theorem 5 we have

$$s_\lambda=-\frac{h_{\mu_\lambda}(\sigma|Q)}{\int_Q\log\lambda_{i_0}\mathrm{d}\mu_\lambda}=-\frac{h_{\mu_\lambda}(\sigma|Q)}{\log\lambda}.$$

Since $\lambda_k=\lambda$, for $k=1,2$,

$$\int_Q\log\lambda_{i_0}\mathrm{d}\mu_\lambda=\log\lambda.$$

Therefore, we obtain

$$h_{\mu_\lambda}(t)=h_{\mu_\lambda}(\sigma|Q)=h_Q(\sigma)=h_{\mathrm{top}}(t).$$

The result follow by applying. Theorem 3 to $h_{\mathrm{top}}(t)$. The proof is complete. □

Next, (iii) of Theorem A is proven.

Lemma 7. *If f is a tent map with slope* $\lambda>1$, *then the lower and upper box dimension and Hausdorff dimension of the gap map* F_t *form devil's staircase functions with respect to t. Furthermore, these three dimensions are equal.*

Proof. By Theorem 5 and the proof of Lemma 6, we have

$$\underline{\dim}_B C(t)=\overline{\dim}_B C(t)=\dim_H C(t)=\frac{h_Q(\sigma)}{-\log\lambda}=\frac{h_{\mathrm{top}}(F_t)}{-\log\lambda}$$

and from the definition of F_t, λ is independent of t. The results follow. □

Next, we prove (i) of Theorem A.

Lemma 8. (*Monotonicity*). *If* $s < t$, $\underline{\dim}_B(s) \leqslant \underline{\dim}_B(t), \overline{\dim}_B(s) \leqslant \overline{\dim}_B(t)$ *and* $\dim_H(s) \leqslant \dim_H(t)$.

Proof. We only prove the lemma for $\underline{\dim}_B(t), \overline{\dim}_B(t)$ and $\dim_H(t)$ can be treated analogously. If $s < t, C(s) \subseteq C(t)$ is easily checked. If κ is a cover of $C(t)$ with diam $\kappa = \varepsilon$, then κ is also a cover of $C(s)$, and a subcover $\hat{\kappa}$ of κ can be chosen that, covers $C(s)$; with card$(\hat{\kappa}) \leqslant$ card(κ). Hence $\underline{\dim}_B C(s) \leqslant \underline{\dim}_B C(t)$, and the proof is completed. □

To prove (ii) of Theorem A, we need to introduce the kneading sequence $K(t)$ of $t \in [0, 1]$ with respect to F_t [3].

Definition 8. Let f be a unimodal map with turning point c.

(i) The itinerary of x with respect to f, defined by $I(x)$ is the sequence $i = (i_0(x), i_1(x), \ldots, i_n(x), \ldots)$, where

$$i_j(x) = \begin{cases} 0 & \text{if} \quad f^j(x) < c, \\ 1 & \text{if} \quad f^j(x) > c, \\ c & \text{if} \quad f^j(x) = c. \end{cases}$$

(ii) A signed lexicographic ordering $\prec$ on $\sum = \{0, c, 1\}^N$ is defined as follows. Let $s_i = t_i$ for $i = 1$ to $n - 1$, then $s \prec t$ if either

(a) $\tau_{n-1}(s)$ is even and $s_n < t_n$ or
(b) $\tau_{n-1}(s)$ is odd and $s_n > t_n$,

where $\tau_k(s) = \sum_{i=0}^{k} s_i$. We also write $s \preceq t$ if $s \prec t$ or $s = t$.

(iii) The kneading sequence of f, denoted by Kf, is defined as follows,

(a) If c is aperiodic point,

$$Kf = I(f(c)),$$

(b) If c is periodic point of f with period n,

$$Kf = \lim_{x \to c^+} I(f(x)).$$

Moreover, $K(t) = KF_t \equiv I(f(t))$, and $K(t)$ is called even (odd) if $\tau_{n-1}(K(t))$ is even (odd).

We need some lemma from [2].

Lemma 9. *If f is a continuous unimodal map, and F_t is induced by f and $t \in [0, 1]$ is a periodic point of f with period n; that is, $f^n(t) = t$, then* *there exists a neighborhood* V_t *of* t *in* [0,1] *such that if* $K(t)$ *is even (odd), then* t *is the left (right) end point of* V_t *such that for all* $s \in V_t$, $C(s) = C(t)$. *Furthermore,*

$$h_{\text{top}}(s) = h_{\text{top}}(t), \forall s \in V_t.$$

Proof. Refer [2]. □

Lemma 10. *If* $h(f) > 0$, *then the union of the constant part of* $\underline{\dim}_B(t)(\overline{\dim}_B(t)$ *and* $\dim_H(t))$ *of the gap maps is dense in parameter space.*

Proof. The lemma is proven by contradiction. Suppose there exist open interval K of [0, 1] such that for all s, t in K, and $s < t, \underline{\dim}_B(s) < \underline{\dim}_B(t)$.

First, we may assume $0 < h_{\text{top}}(s) \leqslant h_{\text{top}}(t)$ for some s, t in K, and $s < t$, thus there exist a periodic orbit $p \in (s, t)$ with periodic n. We assume $K(p)$ is even (odd), then there exists a neighborhood $V_p \subset K$ of p from Lemma 9 with p is the left (right) end point of V_p such that for all $q \in V_p, C(q) = C(p)$, which implies $\underline{\dim}_B(q) = \underline{\dim}_B(p)$, this leads a contradiction. $\overline{\dim}_B(t)$ and $\dim_H(t)$ can be treated analogously. □

Combined with results of Lemmas 6–10, the proof Theorem A is complete.

We first prove the first part of Theorem B.

Lemma 11. *If* V *is a maximal interval of the constant part of* $h_{\text{top}}(t)$, *then* V *is a maximal constant part in* $\underline{\dim}_B(s), \overline{\dim}_B(t)$ *and* $\dim_B(t)$.

Proof. Let $t \in [0, 1]$, and V_t be such that $h_{\text{top}}(t)$ is constant in V_t. V_t is assumed to be maximal; that is $\exists \hat{t} \in [0, 1]$ such that $\hat{t}$ is a periodic point of period n, and if $K(\hat{t})$ is even (odd), then $\hat{t}$ is the left (right) end point of V_t by Lemma 9, thus we have

$$C(s) = C(\hat{t}), \quad \forall_s \in V_t.$$

It implies $\underline{\dim}_B(s) = \underline{\dim}_B(\hat{t}), \forall s \in V_t$. Similar arguments can apply to $\underline{\dim}_B(t)$ and $\dim_H(t)$. The proof is complete. □

To prove the second part of Theorem B, we need following lemmas.

Lemma 12. *If* f *is a unimodal map, and* F_t *are the family of gap maps induced by* f, *we have,*

$$\lim_{n\to\infty} \frac{\log N_n(t)}{n} = h_{\text{top}}(t),$$

where $N_n(t)$ *is the cardinality of the maximal interval of* $C_n(t)$.

Proof. Refer [4]. □

Lemma 13. *If f is a unimodal map, and F_t are the family of gap maps induced by f, we have*

$$\lim_{n\to\infty} \frac{\log \bar{\varepsilon}_n^{-1}}{n} \geqslant \lim_{n\to\infty} \log |DF^n(t)|^{1/n} \tag{4.1}$$

for all $t \in C$.

Furthermore, if f is a tent map with constant slope $\lambda > 1$, then we have

$$\lim_{n\to\infty} \frac{\log \bar{\varepsilon}_n^{-1}}{n} = \log \lambda. \tag{4.2}$$

Proof. By the definition of $C_n(t)$, $F_t|c_n(t)$ is homeomorphism; Then, by the mean value theorem, for each $A \in C_n(t)$, there exists $\bar{t} \in A$ such that

$$\mu(F_t^n(A)) = |DF_t^n(\bar{t})|\mu(A),$$

where μ is Lebesgue measure of [0, 1].

Let $k_1 = \min\{\mu(C_{1,1}), \mu(C_{1,2})\}$ and $k_2 = \min\{\mu(C_{1,1}), \mu(C_{1,2})\}$, thus we have

$$k_1 \leqslant \mu(F_t^n(A)) \leqslant k_2.$$

By the definition, $0 < \bar{\varepsilon}_n \leqslant \mu(A)$ for all $A \in C_n$, this implies

$$\frac{\log \bar{\varepsilon}_n^{-1}}{n} \geqslant \frac{\log |DF_s^n(\bar{t})| - \log k_1}{n}.$$

Since k_1 is constant independent of n and $\cap_{n\geqslant 1} C_n = C$. Taking $n \to \infty$, then (4.1) holds.

If f is tent map with slope $\lambda > 1$, then $\mu(A) = k_1\lambda^{n-1}$ or $k_2\lambda^{n-1}$ for all $A \subset C_n(t)$. Hence, for all $A, B \in C_n(t)$,

$$\frac{\mu(A)}{\mu(B)} \leqslant \frac{k_2}{k_1} = k_3,$$

where k_3 is a constant independent the choice of A, B and n. Hence,

$$k_3\mu(A) \geqslant \bar{\varepsilon}_n \geqslant \mu(A).$$

This implies

$$\frac{\log k_3 + \log |DF_s^n(\bar{t})| - \log k_2}{n} \geqslant \frac{\log \bar{\varepsilon}_n^{-1}}{n} \geqslant \frac{\log |DF_s^n(\bar{t})| - \log k_1}{n}.$$

Since k_1, k_2 and k_3 are constants independent of the subcover of C_n, and $\log |DF_s^n(\bar{t})|/n = \lambda$. Letting $n \to \infty$, then (4.2) follows. The proof is complete. □

Lemma 14. *If f is a unimodd map, and F_t are the family of gap maps induced by f, then*

$$\liminf_{n\to\infty} \frac{\log N_n(t)}{\log \hat{\varepsilon}_n^{-1}(t)} \geqslant \underline{\dim}_B(t)(\overline{\dim}_B(t), \dim_H(t)).$$

Furthermore, if f is a tent map with constant slope $\lambda > 1$, then we have

$$\liminf_{n\to\infty} \frac{\log N_n(t)}{\log \hat{\varepsilon}_n^{-1}(t)} = \underline{\dim}_B(t) = \overline{\dim}_B(t) = \dim_H(t).$$

Proof. Since C_n is a cover of C with diam $C_n = \hat{\varepsilon}_n$, thus we have $N_n \geqslant N_C(\hat{\varepsilon}_n)$, which implies

$$\begin{aligned}
\underline{\dim}_B(t) &= \liminf_{\varepsilon\to\infty} \frac{\log N_C(\varepsilon)}{\log \varepsilon^{-1}} \\
&= \liminf_{n\to\infty} \frac{\log N_C(\hat{\varepsilon}_n^{-1})}{\log \hat{\varepsilon}^{-1}} \\
&\leqslant \liminf_{n\to\infty} \frac{\log N_n(t)}{\log \hat{\varepsilon}^{-1}}.
\end{aligned}$$

If f is tent map with slope $\lambda > 1$, as in Lemma 13, we have

$$\frac{\hat{\varepsilon}_n}{\bar{\varepsilon}_n} \geqslant k_3.$$

Thus we have

$$N_C(k_3\bar{\varepsilon}_n) \geqslant N_n \geqslant N_C(\hat{\varepsilon}_n),$$

which implies

$$\frac{\log N_C(k_3\bar{\varepsilon}_n)}{\log(\hat{\varepsilon})^{-1}} \geqslant \frac{\log N_n}{\log(\hat{\varepsilon})^{-1}} \geqslant \frac{\log N_C(\hat{\varepsilon}_n)}{\log(\hat{\varepsilon})^{-1}}.$$

Since k_3 is independent of n, letting $n \to \infty$, implies

$$\underline{\dim}_B(t) \geqslant \liminf_{n\to\infty} \frac{\log N_n}{\log(\hat{\varepsilon})^{-1}} \geqslant \underline{\dim}_B(t)$$

the result follows. The proofs of Hausdoff and upper box dimensions are similar. □

To complete the proof of Theorem B. Given $\varepsilon_1 > 0$, choose $n_1 \in Z^+$ such that

$$\frac{\log N_n(t)}{\log \hat{\varepsilon}_n^{-1}(t)} \geqslant \underline{\dim}_B(t) + \varepsilon_1$$

whenever, $n \geqslant n_1$. Given $\varepsilon_2 > 0$, choose $n_2 \in Z^+$ such that

$$\frac{\log N_n}{n} \leqslant h_{\text{top}}(F_s) - \varepsilon_2,$$

whenever, $n \geqslant n_2$. Given $\varepsilon_3 > 0$, choose $n_3 \in Z^+$ such that

$$\frac{\log \bar{\varepsilon}_n^{-1}}{n} \geqslant \log |DF^n(t)|^{1/n} + \varepsilon_3,$$

whenever, $n \geqslant n_3$. Let $N = \max\{n_1, n_2, n_3\}$, as $n \geqslant N$,

$$\begin{aligned}
\underline{\dim}_B(t) + \varepsilon_1 &\leqslant \frac{\log N_n(t)}{\log \hat{\varepsilon}_n^{-1}(t)} \\
&= \left(\frac{\log N_n(t)}{n}\right)\left(\frac{n}{\log \bar{\varepsilon}_n^{-1}}\right)\left(\frac{\log \bar{\varepsilon}_n^{-1}}{\log \hat{\varepsilon}_n^{-1}}\right) \\
&\leqslant (h_{\text{top}}(F_t) - \varepsilon_2)(\log |DF^n(t)|^{1/n} + \varepsilon_3)^{-1}\left(\frac{\log \bar{\varepsilon}_n^{-1}}{\log \hat{\varepsilon}_n^{-1}}\right)
\end{aligned}$$

Since $\varepsilon_1, \varepsilon_2$ and ε_3 are arbitrary, now letting $n \to \infty$ yields

$$\lim_{n\to\infty}\left(\frac{\log \bar{\varepsilon}_n^{-1}}{\log \hat{\varepsilon}_n^{-1}}\right)\left(\frac{h_{\text{top}}(F_t)}{\lim\limits_{n\to\infty} \log |DF^n(t)|^{1/n}}\right) \geqslant \underline{\dim}_B(t).$$

If f is tent map with slope $\lambda > 1$, since

$$\bar{\varepsilon}_n \leqslant \hat{\varepsilon}_n \leqslant k_3 \bar{\varepsilon}_n,$$

where k_3 is defined in the proof of Lemma 13, then equality in Lemma 14 can be replaced by

$$\liminf_{n\to\infty} \frac{\log N_n(t)}{\log \bar{\varepsilon}_n^{-1}} = \underline{\dim}_B(t).$$

From Lemmas 12–14, we have

$$\underline{\dim}_B(t) = \frac{h_{\text{top}}(F_t)}{\log \lambda}.$$

Then the proof of Theorem B is complete.

Remark 1. Some relationship exists between $h_{\text{top}}(f)$, Hausdoff dimension, and $\text{lip}(f)$, where $\text{lip}(f)$ is the lipschitz constant of f, says

$$\underline{\dim}_H(f) \leqslant \frac{h_{\text{top}}(f)}{\log \text{lip}(f)},$$

see [5].

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