THE VISIBILITY NUMBER OF A GRAPH

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Abstract. We introduce the *visibility number* $b(G)$ of a graph G, which is the minimum t such that G can be represented by assigning each vertex a union of at most t horizontal segments in the plane so that vertices u, v are adjacent if and only if some point assigned to u sees some point assigned to v via a vertical segment unobstructed by other assigned points. We prove the following:

1) every planar graph has visibility number at most 2, which is sharp.
2) $r \le b(K_{m,n}) \le r+1$, where $r = \lceil (mn+4)/(2m+2n) \rceil$.
3) $\lceil n/6 \rceil \le b(K_n) \le \lceil n/6 \rceil + 1$.
4) When G has n vertices, $b(G) \le \lceil n/6 \rceil + 2$.

1. INTRODUCTION

Researchers in computational geometry have studied the use of graphs to model visibility relations in the plane. For example, in a polygon in the plane we say that two vertices "see" each other if the segment joining them lies inside the polygon. Letting vertices that see each other be adjacent defines the *visibility graph* of the polygon. Similarly, we can define a visibility graph on a set of line segments in the plane, where two segments see each other if some segment joining them crosses no other segment. The literature on these models has dozens of papers, mostly concerning the computation and the recognition of visibility graphs. Also there are applications to search problems and motion planning.

We consider a simpler model in which visibility is vertical only. Let S be a family of horizontal bars in the plane. Tamassia and Tollis [5] defined the *bar visibility graph* of S to be the graph with vertex set S in which two vertices are adjacent if and only if there is

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some unobstructed vertical segment joining them. They characterized bar visibility graphs as the planar graphs having a planar embedding in which all cut-vertices lie on a common face. (Graphs generated by horizontal and vertical visibility of rectangles in the plane are studied in [2].)

Realistically, one would like visibility to occur along a channel of positive width. This enables two bars $[(a,y),(b,y)]$ and $[(b,z),(c,z)]$ to block visibility at b without seeing each other. We obtain this effect by letting bars be half-open segments of the form $((a,y),(b,y)]$.

We study problems for visibility graphs analogous to those studied for intersection graphs. The *interval graph* of a family S of intervals on the real line is the graph with vertex set S in which two vertices are adjacent if and only if as intervals they intersect. Bar visibility graphs provide a geometric analogue of interval graphs; visibility replaces intersection as the adjacency relation, and we place the intervals at various heights. The models yield different families of graphs because intervening bars can block visibility, whereas intervals having a common point on the horizontal line are pairwise intersecting.

The interval graph model has been generalized to permit multi-interval representations of all graphs. A *t-interval* is a union of (at most) t intervals on the real line. A *t-interval representation* of G is an assignment of t-intervals to vertices of G so that vertices are adjacent if and only if their t-intervals intersect. The *interval number* $i(G)$ of a graph G is the minimum t such that G has a t-interval representation.

Here we similarly generalize the bar visibility model. A *t-bar* is a union of (at most) t horizontal bars in the plane. A *t-bar representation* of G is an assignment of t-bars to vertices of G so that vertices are adjacent if and only if some vertical segment links their t-bars without intersecting any other t-bar in the representation. The *visibility number* $b(G)$ of a graph G is the minimum t such that G has a t-bar representation. When t is unspecified, we use the term *multibar*.

For graphs without large cliques, visibility number tends to be smaller than interval number, because the upper and lower "sides" of a bar can be used independently to establish edges. Using the result of [5], we show that every planar graph has visibility number at most two. (This compares with interval number at most three.)

For other families, our lower bounds arise from an easy lemma involving the maximum number of edges in N-vertex planar graphs. Combining this with constructions tells us (within 1) the visibility number for complete bipartite graphs (bicliques) and for cliques. The visibility number of a biclique $K_{m,n}$ is roughly half its interval number, but the clique K_n has interval number 1 and visibility number roughly $n/6$.

We conjecture that, over graphs with n vertices, visibility number is maximized by K_n. We provide a construction for arbitrary n-vertex graphs that always uses at most $\lceil n/6 \rceil + 2$ bars for each vertex. This solves the extremal problem for n-vertex graphs with an error of at most two. The construction uses the result of Lovász [4] that every m-vertex graph can be decomposed into at most $\lfloor m/2 \rfloor$ paths and cycles.

2. PLANAR GRAPHS

We solve the extremal problem for planar graphs by expressing an arbitrary planar graph as the union of two bar visibility graphs.

REMARK 1. $b(G \cup H) \leq b(G) + b(H)$.

Proof: Bar visibility representations of G and H can be placed in disjoint vertical strips to represent $G \cup H$. ■

THEOREM 2. Every planar graph has a 2-bar representation in which all vertices other than cut-vertices are assigned 1-bars.

Proof: If H is a disjoint union of planar graphs having at most one cut-vertex in each component, then the result of Tamassia and Tollis [5] yields $b(H) = 1$. We express an arbitrary planar graph G as the union of two such graphs, which we call G_0 and G_1.

Begin with G_0 and G_1 empty. Choose an arbitrary vertex $v \in V(G)$ as a root. Place the union of all blocks containing v into G_0, and mark v *finished.* Proceed iteratively as follows. For each unfinished vertex added to G_i on the previous step, add to G_{1-i} the union of all blocks of G that have not yet been placed, and mark the vertex finished. Continuing in this breadth-first manner through the blocks of G decomposes G into two subgraphs.

At each phase when a new subgraph consisting of pairwise disjoint "stars of blocks" is added to G_j, the new subgraph is disjoint from the earlier subgraphs added to G_j. Thus each component of G_j has at most one cut-vertex, and the two graphs G_0, G_1 are bar visibility graphs. ■

The minimal planar graphs that are not embeddable with every vertex on a single face are K_4 and $K_{2,3}$. Adding a pendant edge at each vertex of such a graph produces a planar graph that is not a bar visibility graph. Thus Theorem 2 is sharp.

3. BICLIQUES (COMPLETE BIPARTITE GRAPHS)

Our subsequent lower bounds use an easy counting argument.

LEMMA 3. The visibility number of a graph G with n vertices and e edges is at least $(e+6)/(3n)$. If the graph is triangle-free, then $b(G) \geq (e+4)/(2n)$.

Proof: Consider a t-bar representation of G. The total number of bars used is $N \leq nt$. In the plane, add one vertical segment joining each pair of bars that see each other. Now shrink each bar so that it becomes a single point. The added segments remain, covering the edges of G. The result is a planar graph G' with N vertices and at least e edges. Since it also has at most $3N - 6$ edges, we have the desired bound.

If G is triangle-free, then the graph G' will also be simple and triangle-free after we contract all edges joining bars for the same vertex of G. Now G' has at most $2N - 4$ edges, and again these cover all edges of G. ■

Lemma 3 yields $b(K_{m,n}) \geq \lceil \frac{mn+4}{2m+2n} \rceil$. Trotter and Harary [6] proved that $i(K_{m,n}) = \lceil \frac{mn+1}{m+n} \rceil$. Our lower bound for $b(K_{m,n})$ always equals $\lceil i(K_{m,n})/2 \rceil$ or $\lceil i(K_{m,n})/2 \rceil + 1$. By using the tops and bottoms of bars separately, we prove constructively that the visibility number of $K_{m,n}$ is within one of our lower bound. Our construction is motivated by the Trotter-Harary construction for $i(K_{m,n})$.

THEOREM 4. If $r = \lceil \frac{mn+4}{2m+2n} \rceil$, then $r \le b(K_{m,n}) \le r+1$.

Proof: We construct an $r+1$-bar representation of $K_{m,n}$. We may assume that $m \ge n$, with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Let $s = \lfloor (n-1)/2 \rfloor - r$.

As m grows, r increases to $\lceil n/2 \rceil$. We construct a representation using $\lceil n/2 \rceil$ bars for each x_i and one bar for each y_j by arranging the bars for Y as a horizontal sequence of vertical pairs and separating each pair vertically by a set of bars for X.

Therefore, we may assume that $r \le \lceil n/2 \rceil - 1 = \lfloor (n-1)/2 \rfloor$. Since $(mn+4)/(2m+2n)$ increases strictly with m (for $n > 2$) and equals $(n-1)/2$ when $m = n^2 - n - 4$, we may assume that $m < n^2 - n - 3$.

We construct a multibar representation using (up to) $r+1$ pairs of rows of bars for vertices of Y. In each pair of rows, the top row has bars for $y_1, \ldots, y_{\lceil n/2 \rceil}$, and the bottom row has bars for the remainder of Y. In each row, the ith bar extends horizontally from $i-1$ to i, except that when n is odd the bar for y_n extends from $(n-3)/2$ to $(n+1)/2$.

The bars for X also form rows, with a row for some of X between successive rows for Y. In Fig. 1, the bars have been shrunk for clarity within rows; the overlap of half-open bars is such that each bar sees only bars for vertices of the opposite partite set in the two nearest rows. Each bar for x_i sees two bars above it and two below it, except that when n is odd the rightmost bar in each row for X sees bars for only three vertices of Y.

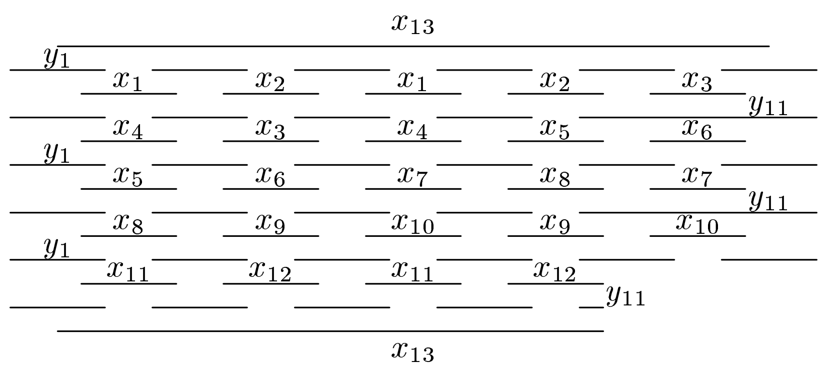

Fig. 1. Part of a 4-bar representation of $K_{13,11}$; $s = 2$ and $r = 3$.

Reading from left to right within successive rows for X from top to bottom, we alternate x_1, x_2 until we have s bars for each, then we alternate x_3, x_4, etc. These bars need $2s \lfloor m/2 \rfloor$ positions. In each of up to $2(r+1)-1$ rows, there are $\lceil n/2 \rceil - 1$ positions available. We thus require that $2s \lfloor m/2 \rfloor \le \lfloor (n-1)/2 \rfloor (2r+1)$. Proving that $ms \le (s+r)(2r+1)$ will show that there are enough locations for these bars.

Because $mn + 4 \le 2r(m+n)$, we have $m(n/2 - r) \le rn - 2$. When n is odd, this inequality becomes $m(s+1/2) \le r(2s+2r+1) - 2$. Since $r(2s+2r+1) < (s+r)(2r+1)$, the desired inequality holds. When n is even, the known inequality becomes $m(s+1) \le 2r(s+r+1) - 2$. Since $2r(s+r+1) - 2 = (s+r)(2r+1) + r - s - 2$ and $m > r$, again the desired inequality holds.

If the available postions are not all needed, we discard the later bars for Y to avoid visibilities between vertices of Y. If m is odd, we add one long bar for x_m at the top and another at the bottom; together these see all of Y. If m is odd and the bottom row of intervals for Y was deleted, then we can put it above the top bar for x_m instead.

For $x_1, \ldots, x_{2\lfloor m/2 \rfloor}$, we have established visibility to $4s$ or to at least $4s-1$ (if n is odd) vertices of Y. These visibilities involve distinct vertices of Y, because $m \geq n$ implies that $r \geq \lceil n/4 \rceil$, and hence $s \leq \lfloor (n-3)/4 \rfloor$. With $r+1$ bars allowed per vertex, we have $r+1-s = \lceil n/2 \rceil - 2s$ bars remaining to be assigned to x_i. There remain $n-4s$ (or at most $n+1-4s$ if n is odd) vertices in Y that x_i must see. Picking up two of these with each remaining bar completes the representation.

The visibilities established so far for x_i consist of two strings A, B of consecutive entries in the list $y_1, \ldots, y_n$, viewed cyclically. The lists may have length $2s$ each, or one may have length $2s-1$ if n is odd and it includes y_n. The ending positions of A and B are separated by $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$. If the last bar placed for x_i sees y_j and y'_j at its right end, then we add a small bar for x_i by shrinking the next two bars for X between y_{j+1} and $y_{j'+1}$. Continuing in this fashion extends A and B to create the remaining visibilities. When n is odd, we need two visibilities for each new interval assigned to x_i if and only if x_i already sees y_n in the first phase; otherwise x_i starts with $4s$ visibilities and one of the small intervals seeing y_n picks up only one. ■

4. CLIQUES

A surprisingly simple construction for $b(K_n)$ produces a representation using at most one more bar per vertex than the counting bound from Lemma 3.

THEOREM 5. $\lceil n/6 \rceil \leq b(K_n) \leq \lceil n/6 \rceil + 1$.

Proof: Lemma 3 yields $b(K_n) \geq \left\lceil \frac{n-1}{6} + \frac{2}{n} \right\rceil = \lceil n/6 \rceil$. For the upper bound, we first reduce to the case where n is divisible by 6. In a visibility representation of a clique, deleting the bars for a vertex cannot introduce unwanted visibilities, since all visibilities are wanted. Deletion also cannot destroy visibilities. Hence $b(K_{n-1}) \leq b(K_n)$.

Now let $n = 6m$. We partition the vertex set into three sets A_1, A_2, A_3, each of size $2m$. A clique with $2m$ vertices has a decomposition into m spanning paths, consisting of the m rotations of a zig-zag path when the vertices are placed around a circle (see Fig. 2).

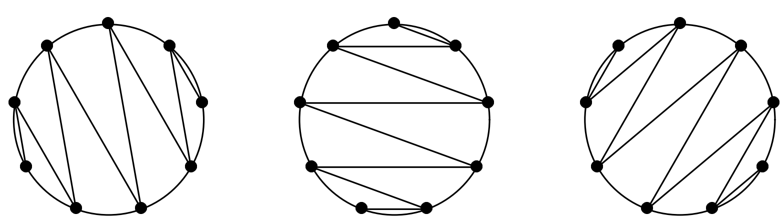

Fig. 2. Path decomposition of K_n.

Our visibility representation of K_n consists of $3m$ isomorphic modules. Each module is a bar visibility representation of the join $P_m \vee 2K_1$, using one bar per vertex. We represent the path by a staircase of bars, each of which sees the bar before and after it. We add one long bar above and one long bar below; each sees the entire path (see Fig. 3).

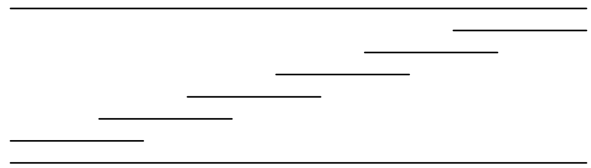

Fig. 3. Bar visibility representation of $P_m \vee 2K_1$.

To each of the m paths in the decomposition of A_i, we assign two vertices in A_{i+1} (indices modulo 3). This produces m pairwise edge-disjoint copies of $P_m \vee 2K_1$. Their union covers A_i and all edges from A_i to A_{i+1}. Doing this for each A_i yields $3m$ modules whose union represents K_n.

A vertex of A_i appears in each path drawn from A_i, and it appears once as a top or bottom bar in a module for a path in A_{i-1}. Thus each vertex is assigned $m+1$ bars. ■

Example 6. $b(K_9) = 2$. A graph represented with one bar per vertex must be planar, so the upper bound of Theorem 5 cannot be improved when $5 \leq n \leq 6$. However, it can be improved when $7 \leq n \leq 9$.

When we allow only two bars per vertex, it is possible that we use the first bar for each vertex in one vertical strip and the second bar in another. This would express the graph as the union of two planar subgraphs: thickness 2. The graph K_9 has thickness 3. However, the flexibility of putting the representation for one subgraph above the other and thereby obtaining an extra edge yields $b(K_9) = 2$ (see Fig. 4).

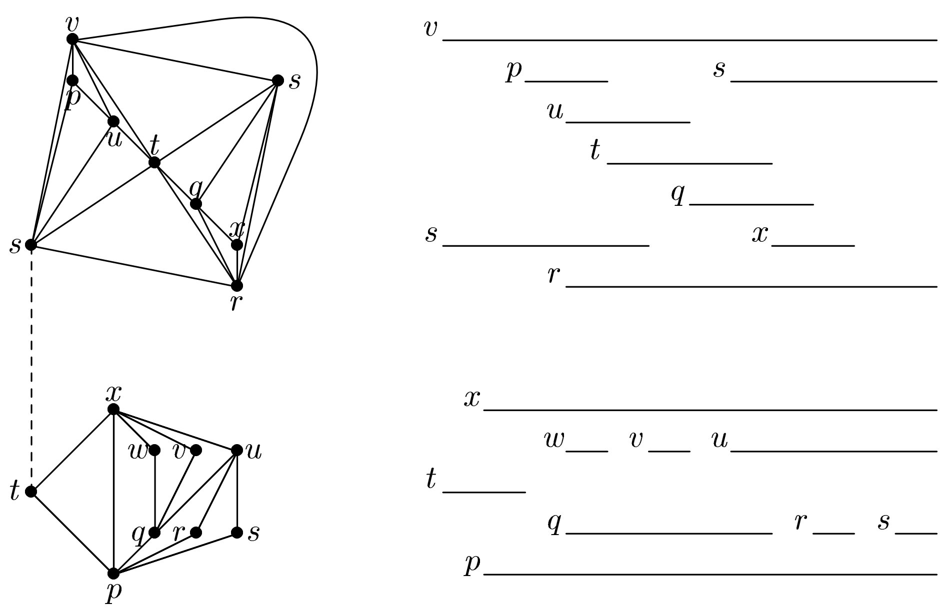

Fig. 4. 2-bar representation of K_9.

5. n-VERTEX GRAPHS

When bounding the visibility number of an n-vertex graph G, it is tempting to use Remark 1 and express G as a union of planar graphs, since planar graphs have visibility at most 2. When n is not 9 or 10, every n-vertex graph is the union of at most $\lfloor (n+7)/6 \rfloor$ planar graphs, with equality for cliques (see [1] for $n \not\equiv 4 \bmod 6$, with the remaining case settled through the work of many authors). This yields about $n/3$ as an upper bound on visibility number of n-vertex graphs. Using planar graphs whose cut-vertices lie on one face, which seems feasible, could eliminate the factor of 2 between this and the lower bound in Theorem 5.

Instead, we generalize the construction in Theorem 5 to prove directly that $b(G) \le \lceil n/6 \rceil + 2$. The construction in Theorem 5 establishes each edge once, but it is difficult to modify it to delete an arbitrary set of edges. For example, let u, v, w appear consecutively on some path in the decomposition of A_1, and let y, z be the vertices of A_2 whose bars surround this path. By extending u or w, it is possible to block v from seeing y or z. By deleting the bar for v and extending those for u and w to the same vertical line, we can delete all these edges. However, how can we delete vy, vz, uv and keep vw?

If all edges of the path in A were present, then we could delete arbitrary edges to y and z by extending the bars for vertices on the path. If t_k is the maximum number of paths needed to partition the edges of a k-vertex graph, we could thus obtain $b(G) \le t_{n/3} + 1$. Gallai [3] conjectured that $t_k = \lceil k/2 \rceil$, which would yield $b(G) \le \lceil n/6 \rceil + 1$.

We do almost as well by using the result of Lovász [4] that every k-vertex graph can be decomposed into $\lfloor k/2 \rfloor$ paths and cycles. Each vertex of odd degree must be an endpoint of some path in such a decomposition. Thus the decomposition must consist entirely of paths when G has at most one vertex of even degree.

THEOREM 7. If G has n vertices, then $b(G) \le \lceil n/6 \rceil + 2$.

Proof: By adding isolated vertices, we may assume that n is divisible by 6. Let $n = 6m$, and again partition $V(G)$ into sets A_1, A_2, A_3 of size $2m$. To $G[A_i]$, add one vertex w adjacent to all vertices with even degree in $G[A_i]$; call this graph G_i'. Since G_i' has at most one vertex of even degree, G_i' has a decomposition into $\lfloor (2m+1)/2 \rfloor = m$ paths.

To each such path P, we assign two vertices of A_{i+1}. We design a module for our representation that establishes all the edges of P contained in G and all the edges of G joining A_i with these two vertices of A_{i+1}. In each such module, we use one bar for each vertex of A_i and two bars for each of the two special vertices of A_{i+1}. Doing this for each i and each P in the decomposition of G_i' produces a visibility representation with $m+2$ bars per vertex (see Fig. 5).

Let I_v be the bar (or two) to be assigned to v in such a module. Let y, z be the two special vertices of A_{i+1}. We begin by representing P as a staircase of bars, with a bar for y underneath and a bar for z above. The edges on P that don't involve the dummy vertex w belong to G, so we never need to block these visibilities. Erasing I_w will produce a gap that may cause us to break I_y and I_z.

Begining at the upper right end of P, which is also initially the right end of I_y, we block visibilities between y and P as needed. When the current vertex v of P is not adjacent to y, we extend the bar for the next lower vertex of P to the right end of the current I_v. If the last vertex v before w is not adjacent to y, then we cannot extend a lower bar to block I_v from I_y; instead, we break I_y and shorten the left end of the right portion to the right endpoint of I_v.

Having arrived at I_w, we delete it; the staircase was built so that the bar for the vertex u after w on P does not see the bar for the vertex v before w on P. We now continue blocking vertices in the rest of P from seeing I_y as needed, in the same manner as before. Visibilities up to I_z are corrected in the symmetric manner, working from the bottom left end of P.

We still must consider the vertices of A that do not belong to P. Each such vertex is adjacent to neither, one, or both of $\{y, z\}$. Vertices adjacent to neither can be ignored; we need add no bar. For each vertex adjacent only to y, we add a bar at the right of P; none of these see each other, and I_u extends to the right to see them all. Similarly, the bars for vertices of $A - V(P)$ adjacent only to z can be added at the left of the bars for P.

For vertices of $A - V(P)$ adjacent to both y and z, we add a bar in the gap between the left and right portions of P that was left by deleting w. Together they fill this gap so that I_y does not see I_z. The left portion of I_y and the right portion of I_z see these bars. If there are no such vertices adjacent to both y and z, then we shorten the left portion of I_y and the right portion of z so that they won't see each other.

We have established and/or deleted all the desired adjacencies, using the desired number of bars for each vertex. ∎

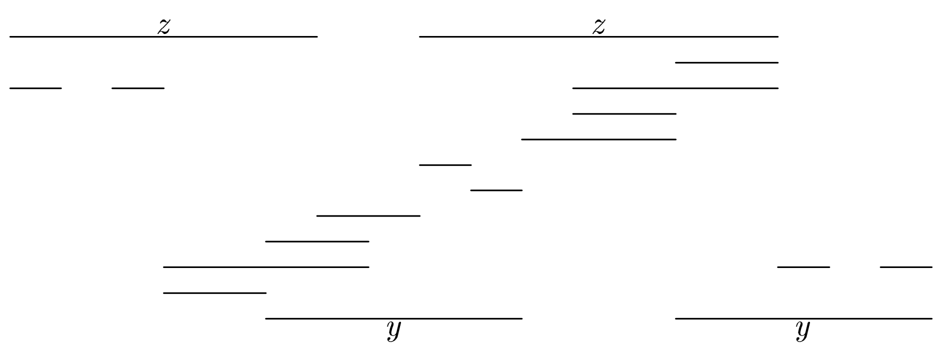

Fig. 5. Module for multibar representation of general graph.

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