Math Meth Oper Res (2004) 60:379–397
DOI 10.1007/s001860400378

Mathematical Methods
of Operations Research

Characterizing the idle time of a nonexponential server system

Hsing Luh and Kai-Hung Tseng

Department of Mathematical Sciences, National ChengChi University, Taiwan R.O.C.

Manuscript received: February 2003/Final revision received: January 2004

Abstract. Understanding the behavior of an idle time of a limited resource is the key to increase productivity in service operations. When the system consists of nonexponential properties of time distributions it becomes difficult to provide results for the general case. We derive the MacLaurin series for the moments of the idle time with respect to the parameters in the service time and interarrival time distributions for a $GI/G/1$ queue. The light traffic derivatives are obtained to investigate the quality of a well-known MacLaurin series. The expected error bound under this approach is identified. The coefficients in these series are expressed in terms of the derivatives of the interarrival time density function evaluated at zero and the moments of the service time distribution, which can be easily calculated through a simple recursive procedure. The result for the idle period is easily taken as input to the calculation of other performance measures of the system, e.g., cycle time or interdeparture time distributions. Numerical examples are given to illustrate these results.

Key words: Queueing theory, Idle time distribution, Light traffic

1. Research background

The overall objective in designing and operating a service line is to produce a high efficiency throughput at the lowest possible cost per job. Although it is usually quite easy to determine the costs associated with a proposed service line design, it is often quite difficult to predict the impact on output

---

This research was partially supported in project number NSC-88-2115-M-004-004 by National Science Council, Taiwan.

production rate of changes in design and operating practices. In the context of production scheduling or output analysis, idle time analysis occurs whenever a resource is deliberately kept idle in the face of waiting jobs. For example, a scheduling problem concerns estimating job completion times to obtain an estimate of the total cost of a marked schedule where a decision point corresponds to the event that a resource has become available. A decision alternative corresponds to the selection of a waiting job from the queue. At each decision point, the total cost of an extended schedule corresponds to each decision alternative is estimated. Kanet and Sridharan (2000) indicated many research areas in idle time scheduling might prove beneficial. In seeking to increase the productivity in service operations, further development of algorithms and dominance properties is needed to better understand these issues in a variety of environments.

Our objective here is not to provide any scheduling rule in operations management, but rather to cast some light on idle time distributions by giving some explicit formulas that characterize it. That our purpose here is to develop simple explicit formulas for the moments of the idle time distribution as a function of the moments of the intarrarrival times and the service times in standard queueing models.

The $GI/G/1$ queueing model has been of much interest to queueing theory researchers and practitioners since the 1950's. The study of this model is significant since it has wide applications in computers, communication systems and networks of queues. Among those applications, we are often interested in numerical computations of the steady-state distributions of many aspects of this queue, for example, the steady-state distribution of the waiting time, the idle time and the interdeparture time. The analytic results for particular cases have been available for some time in the literature, but the exact computational analysis, in many cases, has been difficult to obtain. Several references may be found in the work of Gross and Harris (1985), Kleinrock (1975), and in a number of other textbooks or similar research papers. Only recently some studies have been undertaken in this direction. For example, Li (1997) introduces a new method for the analysis of the $GI/G/1$ queue that is based on GH approximations which are represented by a linear combination of exponential functions. He showed that once a $GI/G/1$ queue is approximated by a $GH/G/1$ queue, it can be solved without resorting to the embedded-Markov-chain method. He uses a linear approximation of a nonlinear integral equation of the distribution of idle time. However, to approximate any specified function by a GH distribution in practice is not so simple as it describes in theory.

The recent results of Sridharan and Zhou (1996) suggest that the value of idle time is indeed a function of utilization, with marked improvement when the machine is not heavily loaded. Our strategy is to consider the idle time instead of the waiting time. We assume the interarrival time is approximated by an analytic function. We derive a simple recursive formula in the MacLaurin series for the moments of the idle time in a $GI/G/1$ queue in this paper. Since the result for the idle period is easily taken as input to the calculation of other performance measures of the system, the moments of interdeparture time are derived as well. These series are expressed in terms of the derivatives of the interarrival time probability density function (p.d.f.) evaluated at zero and the moments of the service time. However, we shall point out that some of the relationships used in this paper are known and have also been used by others. For instances, Gong and Hu (1992) used these MacLaurin series to obtain the waiting time distribution. Hu (1996) uses it to obtain the entire response curves of the moments of the departure process but without taking the idle time into account. The main method of Hu (1995) is to investigate the recursive Lindley equation. He finds that the kth moment of the delay depends only on the $(k+1)$th and higher moments of the system time. This leads to a recurrence relation for the moments of the system time and the waiting time.

So far, analytic methods that have been proposed for presenting moments of the waiting time distribution in $GI/G/1$ assume that the error term will vanish eventually and ignore it, i.e., a convergence assumption. However, this is usually not the case in actual implementation of queueing systems. Indeed an error is often associated with this approximation. For example, the convergent speed is very slow or the computation must stop in certain time intervals. It is thus important to estimate an error (remainder) so that we may know how far the approximating point is away from the true value when applying a MacLaurin series of finite terms to a queueing system. In this paper, we investigate the quality of a well-known MacLaurin series for performance measures. We derive the MacLaurin series for the moments of the idle time with respect to the parameters in the service time and interarrival time distributions. The expected error bound under this approach is discussed. It will be shown that because the expected error is independent of the number of terms which are used for estimation in light traffic, this approximation is very accurate in light traffic.

This paper is organized as follows: In Section 2, we review how the MacLaurin series is used to derive the recursive formulas for the moments of the waiting time distribution and estimate its error. In Section 3, we derive the approximation of the moments of the idle time and discuss the error estimation of this approach. In Section 4, we calculate the light traffic derivatives for both the interdeparture time and the idle time distributions and present numerical examples in Section 5. Finally, summary and concluding remarks are drawn in Section 6.

Below, we introduce the definition for some notations, so that it's clear and convenient for discussion. The following notations are established:

A_n-the arrival time of the n-th custom,
S_n- the service time of the nth customer,
τ_n- the interarrival time between $(n-1)$th and nth customer arrival,
T_n- the flow time of the nth customer,
W_n- the waiting time of the nth customer,
S - a generic service time,
τ- a generic interarrival time,
T - a generic steady-state flow time,
W - a generic steady-state waiting time,
Δ- a generic interdeparture time,
$f(x)$ - a bounded probability density function of τ,
$\alpha_j \triangleq f^{(j)}(0^+)$ is the jth right-hand-side derivative of $f(x)$ at $x=0$,
$\beta_k \triangleq \dfrac{E[S^k]}{k!}$
$\gamma_k \triangleq \dfrac{E[\tau^k]}{k!}$

2. Power series approach

In this section, we first define an expected error bound and review the moments of the waiting time approximated by a MacLaurin series. Then, we propose a general error estimation for an approximation under this development.

Expected error bound

Suppose $f(x)$ is approximated by a polynomial of degree n. Then $R_n(x)$ is defined as the difference between the polynomial and $f(x)$. According to the theory of Taylor expansion, we have a formal definition of $R_n(x)$.

Definition 1 *If $f(x)$ has derivatives of order up to and including n in a neighborhood (a, b) of x_0 then for any $x \in (a, b)$, we define*

$$R_n(x) \triangleq f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \tag{1}$$

Thus if f has a derivative of order n at $x_0 = 0$, we may write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x), \tag{2}$$

which is the standard definition of a MacLaurin series.

It is easy to prove that if $f(x)$ has a continuous $(n+1)$ derivation $f^{(n+1)}(x)$ in the same neighbor, we have

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t) t^n dt. \tag{3}$$

Furthermore, it suffices to choose an appropriate point $\bar{x}$ between x and x_0 and we obtain a Lagrange's form of the remainder,

$$R_n(x) = f^{(n+1)}(\bar{x}) \frac{(x - x_0)^{n+1}}{(n+1)!} \tag{4}$$

If $\lim_{n\to\infty} R_n(x) = 0$, we can define an analytic function at x_0 in the following.

Definition 2 *A function $f(x)$ is said to be analytic at a point x_0 if there is an open interval (a, b) about x_0 on which $f(x)$ is infinitely differentiable and such that* $\lim_{n\to\infty} R_n(x) = 0$ *for each $x \in (a, b)$.*

Suppose $f(x)$ is analytic at $x_0 = 0$, which can be expanded as

$$f(x) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} \, . \, x^j, \ \textit{for } x \in (a, b), \tag{5}$$

where $\alpha_j \triangleq f^{(j)}(0^+)$ assuming it exists, is the jth right-hand-side derivative of $f(x)$ at $x = 0$. Numerically, the assumption $\lim_{n\to\infty} R_n(x) = 0$ is too strong to follow if the series converges slowly. In particular, this is the case when the MacLaurin series is applied to calculate the moments of waiting time in queueing problems. In Gong and Hu (1992), they indicate that it needs 0.6 seconds to compute 40 MacLaurin coefficients and 194.25 seconds to compute 500 MacLaurin coefficients. The slowness of convergence rate is clearly a curse in this approach. For this reason, it is intuitive to ask how far away from the limit when n MacLaurin coefficients has been calculated. Thus, to consider $R_n(x)$ as an error by any approximation in a finite term n is important for n is not large enough.

We turn now to a discussion of the error in the approximation of p.d.f., $f(x)$. Clearly, if $f(x)$ is a p.d.f. of a random variable X, $R_n(X)$ is also a random variable with its p.d.f. $f(x)$. If upper and lower bounds for $f^{(n+1)}(x)$ are known for each x, we can compute corresponding expected upper and lower bounds for $R_n(x)$, as described in the next theorem. Before the theorem is presented, we define the expected error.

Definition 3 *The expected error of approximating $f(x)$ by a polynomial of degree n is given by*

$$E_n(f) \triangleq \int_0^\infty R_n(x)f(x)dx. \tag{6}$$

Now we present the first result of this paper.

Theorem 1 *Suppose $f(x)$ which is analytic at x_0 on interval $(0,\infty)$ is a probability density function for a random variable X. If $|f^{(n+1)}(x)| \le M$ for all $x \in (0,\infty)$, then*

$$|E_n(f)| \le M\sum_{k=0}^{n+1}\frac{E[X^k]}{k!(n+1-k)!}x_0^{n+1-k}. \tag{7}$$

Proof
First, according to (4) and (6) for every $x \in (0,\infty)$, we compute

$$\begin{aligned}\int_0^\infty (x-x_0)^{n+1}f(x)dx &= \int_0^\infty \sum_{k=0}^{n+1}\binom{n+1}{k}x^k x_0^{n+1-k}f(x)dx\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}x_0^{n+1-k}E[X^k].\end{aligned}$$

By (6) again, we have

$$E_n(f) = f^{(n+1)}(\bar{x})\cdot\frac{1}{(n+1)!}\sum_{k=0}^{n+1}\frac{(n+1)!}{k!(n+1-k)!}x_0^{n+1-k}E[X^k]$$

which implies

$$|E_n(f)| \le M\sum_{k=0}^{n+1}\frac{E[X^k]}{k!(n+1-k)!}x_0^{n+1-k}$$

Corollary 1

If $f(x)$ is an exponential distribution with parameter $1/\lambda$, $\lambda > 0$ and analytic at $x_0 = 0$, then $|E_n(f)| = \frac{\lambda}{2}$ for all n.

Proof
Since $f(x)$ is exponential distributed, its nth derivative and $(n+1)$th moment are

$$f^{(n)}(x) = \lambda^{n+1}e^{-\lambda x}(-1)^n \text{ and } E[X^{n+1}] = \frac{(n+1)!}{\lambda^{n+1}}$$

If $x_0 = 0$, then we have

$$E_n(f) = f^{(n+1)}(x_0)\frac{E[x^{n+1}]}{(n+1)!} = \frac{\lambda^{n+2}}{2}(-1)^{n+1}\cdot\frac{(n+1)!}{\lambda^{n+1}}\cdot\frac{1}{(n+1)!} = \frac{\lambda}{2}(-1)^{n+1}$$

This corollary indicates that if $f(x)$ is exponential then $|E_n(f)|$ is only a function of λ no matter how large n is. It also says that $|E_n(f)|$ can only be reduced by λ.

Error bounds on waiting time

We now turn to discuss a $GI/G/1$ queue. Consider the First Come First Serve (F.C.F.S.) $GI/G/1$ queue where the renewal arrival process and the service times are independently and identically distributed (i.i.d.). The service times are also independent of the arrival process. We assume the $GI/G/1$ queue is stable. Let A_n be the arrival time of the nth customer, S_n be the service time of the nth customer, and $\tau_n = A_n - A_{n-1}$ is the interarrival time between the $(n-1)$th and nth customer arrivals, $n = 1, 2, \ldots$, where $A_0 = 0$. Let τ be a generic interarrival time with a bounded probability density function $f(x)$ and S be a generic service time. Assume $E[S^k]$ and $E[\tau^k]$ exist, for any $k > 0$. Denote by T_n and W_n the flow time and delay of the nth customer, respectively. Let T be a generic steady-state flow time, and W be a generic steady-state waiting time. Note that T_n and W_n are equal in distribution to T and W, respectively.

First we introduce a scale parameter θ into the service time. i.e., we consider $GI/G/1$ queue with interarrival time τ and service time θS. It is clear that parameterized queue reduces to the original queue when $\theta = 1$. In order to derive the two moments of the flow time and waiting time, we must discuss the relation in (8) shown in Figure 1. If $T_{n-1} \geq \tau_n$ then $W_n = T_{n-1} - \tau_n$. If $T_{n-1} < \tau_n$ in which τ'_n, A'_n and S'_n denote the interarrival time, arrival instant and service time, then $W_n = 0$. Since the flow time equals service time plus waiting time, we have

$$T_n = \theta \cdot S_n + W_n \quad \text{and} \quad W_n = (T_{n-1} - \tau_n)^+ \quad \text{where } x^+ \triangleq \max(x, 0). \tag{8}$$

Therefore in steady-state we have

$$T = \theta \cdot S + W \stackrel{d}{=} \theta \cdot S + (T - \tau)^+ \tag{9}$$

where $\stackrel{d}{=}$ means equal in distribution.

Note that in (8), T_{n-1}, τ_n, and S_n are independent of each other; therefore, T, τ and S in (9) are independent of each other as well. According to the result of (9) and binomial expansion of T, it immediately follows as:

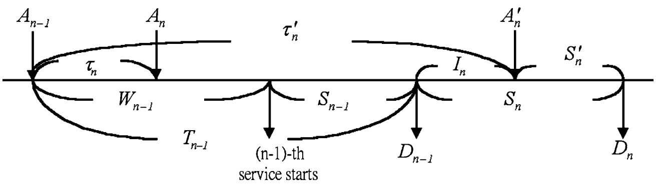

Fig. 1. Successive flow times

$$\frac{E[T^k]}{k!} = \sum_{j=0}^{k} \beta_{k-j} \frac{E[W^j]}{j!} \theta^{k-j} \tag{10}$$

where $\beta_k \triangleq \frac{E[S^k]}{k!} \quad k = 1, 2, \ldots$

In order to derive the MacLaurin Series for the waiting time distribution, we adopt the following two conditions proposed in Hu (1996):

(A1) All the moments of τ and S are finite, and $E[S^k] \leq k!(C_s)^k$ for $k = 0, 1, 2 \ldots$, where $C_s > 0$ is a constant;

(A2) $f(x)$ is an analytic at $x = 0^+$, i.e., there exists $y > 0$ such that $f(x) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} \cdot x^j$, for $0 < x < y$

where $|\alpha_j| \leq (C_f)^{j+1}$, $j = 1, 2, \ldots$ and $f(x) \leq C_f$, for $x \geq 0$.

Hu (1996) has proved that (A1) is equivalent to the condition that S has finite generating functions, i.e., there is a $r > 1$ such that $E[e^{rs}] < \infty$. According to (A2) $f^{(n+1)}$ is bounded and there exists $M > 0$ such that $f(\bar{x}) < M$, for $0 < \bar{x} < x$. We shall always assume $E[S] < E[A]$ in this paper.

Depending upon (2) and (8), we then have

$$\begin{aligned}
\frac{E[W^k]}{k!} &= E\int_0^T \frac{(T-x)^k}{k!} f(x)dx \\
&= E\int_0^T \sum_{j=0}^{n} \frac{\alpha_j}{j!} \frac{(T-x)^k}{k!} x^j dx + E\int_0^T \frac{(T-x)^k}{k!} R_n(x)dx \\
&= \sum_{j=0}^{n} \alpha_j \frac{E[T^{k+j+1}]}{(k+j+1)!} + \phi_n \frac{E[T^{k+n+2}]}{(k+n+2)!} \\
&\quad \text{for } k = 1, 2, \ldots
\end{aligned} \tag{11}$$

The expression is derived based on Gone and Hu (1992). In order to compute the error while estimating the moment of waiting time, we impose a correct term in (11), namely,

$$\phi_n \frac{E[T^{k+n+2}]}{(k+n+2)!} \tag{12}$$

which occurs as the interarrival time is approximated by a polynomial of degree n. Under assumption (A2) and $\lim_{n\to\infty} R_n(x) = 0$ the second term in (11) will tend to zero for n tends to very large, which implies $\lim_{n\to\infty} \phi_n = 0$.

However, for a finite n, ϕ_n is a constant depending on n and the point at which we evaluate for the interarrival time distribution.

For computing $E[W^k]/k!$, (11) defines a correct term that depends upon the expected $(k+n+2)$th moment of flow time and the function f. Thus (11) provides an error estimation for $E[W^k]/k!$. Resorting to the derivation of the moments of waiting and flow times, we make use of the transformation of $E[T^k]/k!$ and $E[W^k]/k!$ written in a MacLaurin series in the following Lemma.

Lemma 1 *Under (A1) and (A2), $E[T^k]$ and $E[W^k]$ exist, for $k = 1, 2, \ldots$, which can be expressed as follows: for* $0 < \theta < 1$ then

$$\frac{E[T^k]}{k!} = \sum_{m=0}^{\infty} t_{km}\theta^m, \tag{13}$$

and

$$\frac{E[W^k]}{k!} = \sum_{m=0}^{\infty} \omega_{km}\theta^m. \tag{14}$$

The proof of this lemma, is given in Gong and Hu (1992).

By repeatedly substituting (13) and (14) into (10) and (11), and comparing the coefficients t_{km}, ω_{km} of θ^m in (10) and (11), it leads to

$$t_{km} = \begin{cases} \beta_k & m = k \\ \sum_{i=1}^{k} \beta_{k-i}\omega_{i(m-k+i)} & m > k \\ 0 & m < k \end{cases} \tag{15}$$

and

$$\omega_{km} = \begin{cases} \sum_{j=0}^{m-k-1} \alpha_j \; t_{(k+j+1)m} + \phi_n \cdot t_{(k+n+2)m} & m > k, \\ 0 & m \le k \end{cases} \tag{16}$$

Since $t_{(k+n+2)m} = 0$ if $m < k+n+2$, ω_{km} is affected by ϕ_n only when $n \le m-k-2$. By repeatedly using (15) and (16), it is observed that both ω_{km} and t_{km} can be estimated by ϕ_n when $n < m-k-1$. If a polynomial of degree n is used for approximating the interarrival time, the errors in $E[T^k]$ and $E[W^k]$ should be estimated respectively. For example, if k and n are given, the error for calculating $E[W^k]/k!$ with a fixed $\theta = \theta_0$ is estimated by

$$\sum_{m=n+k+2}^{\infty} \omega_{km}\theta_0^m \tag{17}$$

which has a upper bound written as a power series of ϕ_n, $\sum_{p=0}^{\infty} a_p\phi_n^p$ for some $a_p > 0$. Clearly, a_p may be computed in terms of ω_{ij} and t_{ij} for some i and j.

3. Idle time approximation

Moments of the idle time

In this section we discuss how to obtain the MacLaurin series of the moments of the idle time. This derivation is presented in the following theorem.

Theorem 2 *Under assumptions (A1) and (A2) and the result of Lemma 1, we obtain the MacLaurin series of the moments of idle time.*

$$\frac{E[I^k]}{k!} = \sum_{m=0}^{\infty} \delta_{km}\theta^m \tag{18}$$

where

$$\delta_{km} = \begin{cases} \sum_{i=0}^{k}(-1)^{2k-i}\gamma_{k-i}t_{im} & m \le k \\ \sum_{i=0}^{k}(-1)^{2k-i}\gamma_{k-i}t_{im} - (-1)^k\omega_{km} & m > k. \end{cases} \tag{19}$$

Proof.

$$\text{Since } \; I_n = (\tau_n - T_{n-1})^+ \quad n = 1, 2, \ldots \tag{20}$$

and $\{I_n\}$, $\{\tau_n\}$ and $\{T_n\}$ converge in distribution, we have

$$I \overset{d}{=} (\tau - T, 0)^+$$

We also have,

$$((\tau - T, 0)^+)^j \overset{d}{=} (-1)^j((T-\tau)^j - ((T-\tau,0)^+)^j) \overset{d}{=} (-1)^j((T-\tau)^j - W^j) \tag{21}$$

for $j = 1, 2, \ldots, k$.

Taking results from (21), we attain the MacLaurin series of the moment of idle time of the $GI/G/1$ queue,

$$\begin{aligned} \frac{E[I^k]}{k!} &= \frac{(-1)^k}{k!}(E[(T-\tau)^k] - E[W^k]) \\ &= (-1)^k E\left[\sum_{i=0}^{k}\frac{T^i \cdot (-\tau)^{k-i}}{i! \cdot (k-i)!}\right] - \frac{(-1)^k}{k!}E[W^k] \end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{k}(-1)^{2k-i}\gamma_{k-i} \cdot \sum_{m=0}^{\infty} t_{im}\theta^m - (-1)^k \sum_{m=0}^{\infty}\omega_{km}\theta^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{k}(-1)^{2k-i}\gamma_{k-i}t_{im} - (-1)^k\omega_{km}\right)\theta^m. \end{aligned}$$

Since the value of δ_{km} depends upon ω_{km} and t_{im} which are defined in (15) and (16), ω_{km}, t_{im} and γ_{k-i} are bounded and δ_{km} is determined by (19).

Theorem 3 *For any $k > 0$, $E[I^k]$ is decreasing of θ.*

Proof. By (8), we have

$$W_{n+1} = [W_n + \theta S_n - \tau_n]^+.$$

On the other hand, by (20), we have

$$I_n = [-(W_n + \theta S_n - \tau_n)]^+.$$

Together with these two equations, it leads to

$$W_{n+1} - I_n = W_n + \theta S_n - \tau_n.$$

Under the stationary assumption and W_n and W_{n+1} with the same distribution, it reduces to

$$I \stackrel{d}{=} \tau - \theta S.$$

Hence, for any $k > 0$, and I, S, τ, $\theta \geq 0$ the following equation holds

$$E[I^k] = E[(\tau - \theta S)^k].$$

Thus, the theorem is proved.

It is easy to verify (19) with the existing formulas for $k = 1$ and 2. When $k = 1$, (18) becomes

$$E[I] = \sum_{m=0}^{\infty} \delta_{1m}\theta^m = \delta_{10} + \delta_{11}\theta + \sum_{m=2}^{\infty} \delta_{1m}\theta^m.$$

By substituting the values of ω_{km}, t_{im} and γ_{k-i}, we have

$$\delta_{10} = E[\tau]$$
$$\delta_{11} = -E[S]$$
$$\delta_{1m} = 0 \quad \text{for } m \geq 2$$

$$\text{Therefore} \quad E[I] = E[\tau] - E[S] \cdot \theta. \tag{22}$$

When $k=2$, (18) becomes

$$\frac{E[I^2]}{2!} = \sum_{m=0}^{\infty} \delta_{2m}\theta^m = \delta_{20} + \delta_{21}\theta + \delta_{22}\theta^2 + \sum_{m=3}^{\infty} \delta_{2m}\theta^m$$

in which

$$\delta_{20} = \frac{E[\tau^2]}{2}$$
$$\delta_{21} = -E[\tau] \cdot E[S]$$
$$\delta_{22} = -E[\tau]\left(\frac{\alpha_0 E[S^2]}{2}\right) + \frac{E[S^2]}{2}$$
$$\delta_{2m} = -E[\tau]\omega_{1m} + E[S]\omega_{1(m-1)} \text{ for } m \geq 3.$$

Now we compare these two expressions with existing formulas associated with them in queueing models respectively. First, we illustrate that for Poisson arrival process, (22) will lead to the exact expression for the mean idle time. When the interarrival times are exponentially distributed, $\{I_n | I_n > 0\}$ which is equal to $\{\tau_n - T_{n-1} | \tau_n - T_{n-1} > 0\}$ has an exponential distribution with mean $E[\tau]$. Therefore, (22) approaches to $E[\tau]$ as $\theta \to 0$, and approaches to $E[I]$ as $\theta \to 1$. Second, in order to write out ω_{1m} and $\omega_{1(m-1)}$ in terms of $E[\tau]$ and $E[S]$, we'll let $\alpha_j = \lambda(-\lambda)^j$ and $E[\tau] = 1/\lambda$. Therefore, based on (15) and (16), we have

$$\begin{aligned}\omega_{1m} &= \sum_{i=0}^{m-2} \alpha_i t_{(2+i)m} \\ &= \alpha_0 E[S]\omega_{1(m-1)} + \alpha_0 \omega_{2m} + \sum_{i=1}^{m-2} \alpha_i t_{(2+i)m} \\ &= \alpha_0 E[S] W_{1(m-1)} \quad m = 3, 4, \ldots\end{aligned}$$

and $\delta_{2m} = 0$, for $m \geq 3$.

Thus For the $M/G/1$ queue, we have

$$E[I^2] = 2(\frac{1}{\lambda^2} - \frac{1}{\lambda} E[S] \cdot \theta). \tag{23}$$

It tends to the exact expression as $\theta \to 1$ and tends to $E[\tau^2]$ which is equivalent to $E[I^2|I > 0]$ as $\theta \to 0$.

When a non-exponential interarrival time is approximated by a polynomial of degree n, . $E[I^k]$ has no a simple expression like (22) and (23). Clearly, it will result in an error which depends on n and f. In Section 3.2, we derive an upper bound to estimate it.

Error estimation of the idle time

To compute $E[I^k]$ when the interarrival time is general, we start with (18). Suppose we have computed only $n+1$ terms and the remaining terms are defined as an error,

$$\frac{E[I^k]}{k!} - \sum_{m=0}^{n} \delta_{km}\theta^m. \tag{24}$$

Namely, (24) is equivalent to

$$\sum_{m=n+1}^{\infty} \delta_{km}\theta^m. \tag{25}$$

According to (19) and for a fixed k, (25) leads to

$$\sum_{m=n+1}^{\infty} \sum_{i=0}^{k} (-1)^{2k-i} \gamma_{k-i} t_{im} \theta^m - \sum_{m=n+1}^{\infty} (-1)^k \omega_{km} \theta^m. \tag{26}$$

Rearranging (26), it may be written as

$$\sum_{i=0}^{k} (-1)^i \gamma_{k-i} \sum_{m=n+1}^{\infty} t_{im}\theta^m - (-1)^k \sum_{m=n+1}^{\infty} \omega_{km}\theta^m. \tag{27}$$

Thus, to estimate the error in computing $E[I^k]$ is equivalent to estimate the errors occur in $E[T^k]$ and $E[W^k]$. Based on (15) and (16), it is sufficient to estimate (25) by computing the error which is taken in approximating the interarrival time distribution. To be specific, the error bound of $E[I^k]$ may be again written as a power series expressed by ϕ_n. This describes how we can estimate and restrain the error of the idle time. The observation is stated in the following theorem.

Theorem 4 For any given n, the upper bound of the error of $E[I^k]$ with $\theta = \theta_0$ may be written as

$$\sum_{p=0}^{\infty} b_p \phi_n^p \quad \text{for some } b_p.$$

Proof. The proof is immediate from (17) and (27).

Since ϕ_n is determined by R_n the error bound of the idle time is also affected by the interarrival time distribution. Because R_n is considered as the error calculated by the interarrival time approximation which is written by a polynomial of degree n, the question is: in general when will ϕ_n go to zero for any finite n? From (4), notice R_n approaches to zero as x approaches to x_0. Therefore, we need to derive the MacLaurin series for the moments of the idle time with respect to λ and let the interarrival time distribution be evaluated at $\lambda = 0$. Apparently, this is the approach called the light traffic derivatives in queueing theory.

4. The light traffic derivation

In this section, we shall derive the moments of idle time and interdeparture time in light traffic.

There is an important reason to consider the performance measures in light traffic. For a heavy traffic queueing model, i.e., λ is very close to μ, the departure process of the $GI/G/1$ is theoretically a renewal process whose interdeparture time apparently is approximated by the service time distribution. However, it is not clear how the departure behaves when λ is very small. To answer this question, we shall first derive the idle time in light traffic since the idle time may be used directly to derive the interdeparture time. Based on the results developed in Section 3, we can derive the idle time analysis easily and obtain the idle time approximation without taking the error into account.

At first, consider the light traffic derivatives of flow time and waiting time. We focus on the case where $1/\lambda$ is a scale parameter of the interarrival time distribution. Let $Y \stackrel{d}{=} \lambda \cdot \tau$ where τ is independent of the parameter λ. Let $\theta = 1$, $\widetilde{W} = \lambda W$ and $\widetilde{T} = \lambda T$. We have

$$\widetilde{T} \stackrel{d}{=} \lambda \cdot S + \widetilde{W} \quad \text{where} \quad \widetilde{W} \stackrel{d}{=} (\widetilde{T} - Y)^+.$$

According to the result of Lemma 1, we obtain the MacLaurin series of $E[\widetilde{T}^k]/k!$ and $E[\widetilde{W}^k]/k!$ as functions of λ. We write

$$\frac{E[\widetilde{T}^k]}{k!} = \sum_{m=0}^{\infty} t_{km} \lambda^m \quad \text{and} \quad \frac{E[\widetilde{W}^k]}{k!} = \sum_{m=0}^{\infty} \omega_{km} \lambda^m.$$

Since $\widetilde{T} = \lambda T$ and $\widetilde{W} = \lambda W$, we have

$$E[\widetilde{T}^k] = \lambda^k E[T^k] \quad \text{and} \quad E[\widetilde{W}^k] = \lambda^k E[W^k].$$

Hence, $E[T^k]$ and $E[W^k]$ are analytic at $\lambda = 0$ and their MacLaurin series can be written respectively as

$$E[T^k] = k!\sum_{m=k}^{\infty} t_{km}\lambda^{m-k} \quad \text{and} \quad E[W^k] = k!\sum_{m=k}^{\infty} \omega_{km}\lambda^{m-k}.$$

Readers may refer to Hu (1995) for details .

To what follows, we consider the derivatives of the interdeparture and idle time moments with respect to the arrival rate λ at $\lambda = 0$. Let $\tilde{\Delta} = \lambda \cdot \Delta$ and $\tilde{I} = \lambda \cdot I$.

Since

$$\Delta \overset{d}{=} I + S \quad \text{where} \quad I \overset{d}{=} (\tau - T)^{+} \tag{28}$$

Now we multiply both sides of (28) by λ and we have

$$\lambda\Delta \overset{d}{=} \lambda I + \lambda \cdot S$$

and equivalently

$$\tilde{\Delta} \overset{d}{=} \tilde{I} + \lambda \cdot S.$$

We write the MacLaurin series of $E[\tilde{I}^k]/k!$ and $E[\tilde{\Delta}^k]/k!$ as functions of λ.

Theorem 5

We write

$$\frac{E[\tilde{I}^k]}{k!} = \sum_{m=0}^{\infty} c_{km}\lambda^m \tag{29}$$

and

$$\frac{E[\tilde{\Delta}^k]}{k!} = \sum_{m=0}^{\infty} e_{km}\lambda^m \tag{30}$$

where

$$c_{km} = \begin{cases} 0 & m < k \\ \gamma_k + \sum\limits_{i=1}^{k}(-1)^{2k-i}\gamma_{k-i} \cdot \beta_i & m = k \\ \sum\limits_{i=1}^{k}(-1)^{2k-i}\gamma_{k-i} \cdot t_{i(m-k+i)} - (-1)^k\omega_{km} & m > k \end{cases} \tag{31}$$

and

$$e_{km} = \begin{cases} 0 & m < k \\ \beta_k + \sum\limits_{i=1}^{k} \beta_{k-i}c_{ii} & m = k \\ \sum\limits_{i=1}^{k} \beta_{k-i}c_{i(m-k+i)} & m > k. \end{cases} \tag{32}$$

The proof of this theorem is given in the Appendix.

Since $\tilde{I} = \lambda I$ and $\tilde{\Delta} = \lambda\Delta$, we have

$$E[\tilde{I}^k] = \lambda^k E[I^k] \; \textit{and} \; E[\tilde{\Delta}^k] = \lambda^k E[\Delta^k].$$

The MacLaurin series for $E[I^k]$ and $E[\Delta^k]$ now can be written as

$$E[I^k] = k! \sum_{m=k}^{\infty} c_{km} \lambda^{m-k} \tag{33}$$

and

$$E[\Delta^k] = k! \sum_{m=k}^{\infty} e_{km} \lambda^{m-k}. \tag{34}$$

Now we will find the nth derivative of $E[I^k]$ and $E[\Delta^k]$ with respect to λ at the value $\lambda = 0$. The same method was applied to study in Hu (1995) where he has discussed the analyticity properties in light traffic in a general queueing model. However, he only obtain $E[W^k]$ and $E[T^k]$ in light traffic.

Theorem 6 *Under (A1) and (A2), $E[I^k]$ and $E[\Delta^k]$ $(k = 1, 2, \ldots)$ have derivatives of any order at $\lambda = 0^+$. Furthermore, we have*

$$\left[\frac{d^p}{d\lambda^p} E[I^k]\right]_{\lambda=0} = c_{k(p+k)} k! \cdot p! \tag{35}$$

and

$$\left[\frac{d^p}{d\lambda^p} E[\Delta^k]\right]_{\lambda=0} = e_{k(p+k)} k! \cdot p! \tag{36}$$

where $c_{k(p+k)}$ and $e_{k(p+k)}$ are defined by (31) and (32).

The proof is immediately derived from the discussion above.

Note that we see from (31) and (32) that the moments that are higher than $m \geq 2$ of service times do not affect the coefficients of the terms $\lambda^k, \cdots, \lambda^{m-2+k}$. Moreover, $f^{(m)}(0)$, for $m \geq 1$, only affects the coefficients of the terms $\lambda^{m+k+1}, \lambda^{m+k+2}, \cdots$ in the MacLaurin series of $E[I^k]$ and $E[\Delta^k]$.

5. Numerical examples

In this section, we illustrate the numerical results based on (35) and (36) by considering two examples.

The first example is taken from a typical $M/E_2/1$ queueing model. Suppose the p.d.f. of the interarrival time is $f(y) = \lambda \cdot e^{-\lambda y}$, for $y \geq 0$ where λ is the average arrival rate and the p.d.f. of the service time is $h(y) = 4\mu^2 y \cdot e^{-2\mu y}$, for $y \geq 0$ where $\mu = 1$ is the service rate. The results computed by using (35) and (36) under different traffic intensities $\lambda/\mu = \lambda$ are presented in Tables 1 and 2.

Let $I[k,p]$ denote $\left[\frac{d^p}{d\lambda^p} E[I^k]\right]_{\lambda=0}$ and $D[k,p]$ denote $\left[\frac{d^p}{d\lambda^p} E[\Delta^k]\right]_{\lambda=0}$. The results presented in Tables 1 and 2 show the rates of change of the moments in light traffic with respect to idle times and interdeparture times respectively.

The second example is taken from Li (1997) where the interarrival time distribution with $E[\tau] = 11/6$ is so called the GH distribution given by $f(x) = 3e^{-x} - 6e^{-2x} + 3e^{-3x}$, $x \geq 0$.

The service times are assumed to be uniform over $[0, b)$, with b varying from 0 to $2E[\tau]$. Again, the solution obtained from (35) and (36) under dif-

Table 1. Light Traffic Derivatives for Idle times in $M/E_2/1$

λ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
I[1,0]	19.00	9.00	5.67	4.00	3.00	2.33	1.857	1.50
I[2,0]	761.50	181.50	77.06	41.50	25.50	17.06	12.11	9.00
I[2,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[2,2]	−2.85	−2.70	−2.55	−2.40	−2.25	−2.10	−1.95	−1.80
I[2,3]	0.28	0.54	0.76	0.96	1.12	1.26	1.36	1.44
I[2,4]	−1.74	−3.37	−4.87	−6.24	−7.45	−8.50	−9.38	−10.08
I[3,0]	45687.0	5442.00	1538.11	619.50	303.00	167.56	100.81	64.50
I[3,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[3,2]	−171.33	−81.67	−52.01	−37.35	−28.68	−23.02	−19.07	−16.2
I[3,3]	34.23	32.53	30.90	29.34	27.84	26.41	25.05	23.76
I[3,4]	−107.08	−106.14	−105.21	−104.31	−103.46	−102.70	−102.04	−100.52

ferent traffic intensities $\rho = \frac{b}{2E[\tau]}$ are presented in Tables 3 and 4. Observe that from $D[1,0]$ in Tables 2 and 4, the mean of the interdeparture time is equal to $E[\tau]$. It is theoretically true since the system is stable with $E[S] < E[\tau]$.

6. Concluding remarks

We see that from (16), (33) and (34), ϕ_n will appear with the coefficients of the terms $\lambda^{n+k+2}, \lambda^{n+k+3}, \cdots$ in the MacLaurin series of $E[I^k]$ and $E[\Delta^k]$ for $n \leq m-k-2$. However, those terms with power greater than $n+k+2$ may be ignored since λ is close to zero. Thus to compute $E[I^k]$ and $E[\Delta^k]$ in (33) and (34), we only need a finite number of terms up to $n+k+1$. The same arguments are also applied to compute $E[T^k]$ and $E[W^k]$.

In summary, we have derived a simple procedure to calculate the coefficients of the MacLaurin series of the moments of the idle process of $GI/G/1$ queue, given all the moments of τ and S are finite and $f(x)$ is an analytic at $x=0$. Under some mild conditions we show that the moments of the idle time and interdeparture time have derivatives of any order at $\lambda = 0$, a scale parameter in the interarrival time, and can be calculated based on the coefficients we obtain.

It has previously been established that using idle time analysis in an exponential service system can essentially yield accurate calculation for a high

Table 2. Light Traffic Derivatives for interdeparture times in $M/E_2/1$

λ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
D[1,0]	20.00	10.00	6.67	5.00	4.00	3.33	2.857	2.50
D[2,0]	801.00	201.00	88.89	51.00	33.00	23.22	17.33	13.50
D[2,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[2,2]	−2.85	−2.70	−2.55	−2.40	−2.25	−2.10	−1.95	−1.80
D[2,3]	0.28	0.54	0.76	0.96	1.12	1.26	1.36	1.44
D[2,4]	−1.74	−3.37	−4.87	−6.24	−7.45	−8.50	−9.38	−10.08
D[3,0]	48060.0	6030.00	1797.78	765.00	396.00	232.22	148.51	101.25
D[3,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[3,2]	−179.88	−89.77	−59.66	−44.55	−35.43	−29.32	−24.92	−21.60
D[3,3]	35.08	34.15	33.20	32.22	31.21	30.19	29.14	28.08
D[3,4]	−112.32	−116.26	−119.84	−123.03	−125.82	−128.21	−130.20	−131.76

Table 3. Light Traffic Derivatives for idle time in $GH/G/1$

ρ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
I[1,0]	1.83	1.83	1.83	1.83	1.83	1.83	1.83	1.83
I[1,n]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[2,0]	4.72	4.74	4.77	4.81	4.86	4.92	5.00	5.08
I[2,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[2,2]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[2,3]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[2,4]	−0.004	−0.07	−0.34	−1.01	−2.33	−4.50	−7.75	−12.21
I[3,0]	16.00	16.10	16.25	16.47	16.74	17.08	17.48	17.94
I[3,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[3,2]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[3,3]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
I[3,4]	−0.019	−0.30	−1.54	−4.82	−11.64	−23.826	−43.43	−72.69

efficiency throughput. In situation with processing times varying substantially, care must be taken to design a performance evaluation procedure to characterize the behavior of the system and to maximize the throughput rate by incorporating optimal parameter values. This paper adds to the understanding of idle time and departure process for better operations management. Further development of an optimization model needs to be studied for service scheduling in more general cases.

Appendix

Proof of Theorem 5

To prove $\frac{E[\tilde{I}^k]}{k!} = \sum_{m=0}^{\infty} c_{km}\lambda^m$.

Consider $\tilde{I} = \lambda I = \max(Y - \widetilde{T}, 0)$ and where $Y = \lambda\tau$ and $\widetilde{T} = \lambda T$ and

Table 4. Light Traffic Derivatives for interdeparture time in $GH/G/1$

ρ	0.05	0.1	0.15	0.20	0.25	0.30	0.35	0.40
D[1,0]	1.83	1.83	1.83	1.83	1.83	1.83	1.83	1.83
D[1,n]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[2,0]	4.72	4.74	4.77	4.81	4.86	4.92	5.00	5.08
D[2,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[2,2]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[2,3]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[2,4]	−0.004	−0.07	−0.34	−1.01	−2.33	−4.50	−7.75	−12.21
D[3,0]	16.00	16.10	16.25	16.47	16.74	17.08	17.48	17.94
D[3,1]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[3,2]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[3,3]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D[3,4]	−0.019	−0.30	−1.54	−4.82	−11.64	−23.826	−43.43	−72.69

$$
\begin{aligned}
\widetilde{I}^k =& [\max(Y - \widetilde{T}, 0)]^k \\
=& (-1)^k [(\widetilde{T} - Y)^k - (\max(\widetilde{T} - Y, 0))^k] \\
=& (-1)^k [(\widetilde{T} - Y)^k - \widetilde{W})^k]
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{E[\widetilde{I}^k]}{k!} =& \frac{(-1)^k}{k!} \left(E[(\widetilde{T} - Y)^k] - E[\widetilde{W}^k] \right) \\
=& (-1)^k \left(E\left[\sum_{i=0}^{k} \frac{\widetilde{T}^i \cdot (-Y)^{k-i}}{i! \cdot (k-i)!} \right] - \frac{E[\widetilde{W}^k]}{k!} \right) \\
=& \sum_{i=0}^{k} (-1)^{2k-i} \gamma_{k-i} \lambda^{k-i} \frac{E[\widetilde{T}^i]}{i!} - (-1)^k \frac{E[\widetilde{W}^k]}{k!} \\
=& \sum_{i=0}^{k} (-1)^{2k-i} \gamma_{k-i} \lambda^{k-i} \sum_{m=0}^{\infty} t_{im} \lambda^m - (-1)^k \sum_{m=0}^{\infty} \omega_{km} \lambda^m \\
=& \sum_{m=0}^{\infty} \sum_{i=0}^{k} (-1)^{2k-i} \gamma_{k-i} t_{im} \lambda^{m+k-i} - \sum_{m=0}^{\infty} (-1)^k \omega_{km} \lambda^m \\
=& \gamma_k \lambda^k + \sum_{m=0}^{\infty} \sum_{i=1}^{k} (-1)^{2k-i} \gamma_{k-i} t_{im} \lambda^{m+k-i} - \sum_{m=0}^{\infty} (-1)^k \omega_{km} \lambda^m \\
=& \sum_{m=0}^{\infty} c_{km} \lambda^m
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{i=1}^{k} (-1)^{2k-i} \gamma_{k-i} t_{im} \lambda^{m+k-i} \\
&= (-1)^{2k-1} \gamma_{k-1} t_{1m} \lambda^{m+k-1} + (-1)^{2k-2} \gamma_{k-2} t_{2m} \lambda^{m+k-2} + \cdots + (-1)^k t_{km} \lambda^m
\end{aligned}
$$

and expanding it for $m = 0, 1, 2, \ldots$, we have

(1) m = 0
$$(-1)^{2k-1} \gamma_{k-1} t_{10} \lambda^{k-1} + (-1)^{2k-2} \gamma_{k-2} t_{20} \lambda^{k-2} + \cdots + (-1)^k t_{k0}$$

(2) m = 1
$$(-1)^{2k-1} \gamma_{k-1} t_{11} \lambda^{k} + (-1)^{2k-2} \gamma_{k-2} t_{21} \lambda^{k-1} + \cdots + (-1)^k t_{k1} \lambda$$

(3) m = 2
$$(-1)^{2k-1} \gamma_{k-1} t_{12} \lambda^{k+1} + (-1)^{2k-2} \gamma_{k-2} t_{22} \lambda^{k} + \cdots + (-1)^k t_{k2} \lambda^2$$

$\vdots$

Comparing them with

$$\sum_{m=0}^{\infty} c_{km} \lambda^m = c_{k0} + c_{k1} \lambda + c_{k2} \lambda^2 + \cdots$$

we have

$$\begin{aligned} c_{k0} &= (-1)^k t_{k0} - (-1)^k \omega_{k0} \\ c_{k1} &= (-1)^{k+1}\gamma_1 t_{(k-1)0} + (-1)^k t_{k1} - (-1)^k \omega_{k1} \\ &\vdots \\ c_{k(k-1)} &= (-1)^{2k-1}\gamma_{k-1} t_{10} + (-1)^{2k-2}\gamma_{k-2} t_{21} + \cdots + (-1)^k t_{k(k-1)} - (-1)^k \omega_{k(k-1)} \\ c_{kk} &= \gamma_k + (-1)^{2k-1}\gamma_{k-1} t_{11} + (-1)^{2k-2}\gamma_{k-2} t_{22} + (-1)^k t_{kk} - (-1)^k \omega_{kk} \\ &\vdots \end{aligned}$$

In general, it is written as

$$c_{km} = \begin{cases} 0 & m < k \\ \gamma_k + \sum_{i=1}^{k} (-1)^{2k-i} \gamma_{k-i} \, . \, t_{ii} & m = k \\ \sum_{i=1}^{k} (-1)^{2k-i} \gamma_{k-i} \, . \, t_{i(m-k+i)} - (-1)^k \omega_{km} & m > k \end{cases}$$

To prove $\frac{E[\tilde{\Delta}^k]}{k!} = \sum_{m=0}^{\infty} e^{k_m} \lambda^m$, we write

$\frac{E[\tilde{\Delta}^k]}{k!} = \sum_{j=0}^{k} \beta_{k-j} \frac{E[\tilde{I}^k]}{j!} \lambda^{k-j}$

since $\tilde{\Delta} = \tilde{I} + \lambda S$.

Thus consider

$$\begin{aligned} \sum_{m=0}^{\infty} e_{km} \lambda^m &= \sum_{j=0}^{k} \beta_{k-j} \sum_{m=0}^{\infty} c_{jm} \lambda^m \cdot \lambda^{k-j} \\ &= \sum_{m=0}^{\infty} \sum_{j=0}^{k} \beta_{k-j} c_{jm} \lambda^{m+k-j} \\ &= \beta_k \lambda^k + \sum_{m=0}^{\infty} \sum_{j=1}^{k} \beta_{k-j} c_{jm} \lambda^{m+k-j} \end{aligned}$$

which implies

$$e_{km} = \begin{cases} 0 & m < k \\ \beta_k + \sum_{i=1}^{k} \beta_{k-i} c_{ii} & m = k \\ \sum_{i=1}^{k} \beta_{k-i} c_{i(m-k+i)} & m > k \end{cases}$$

These complete the proof.

References

1. Gong WB, Hu JQ (1992) The MacLaurin series for the $GI/G/1$ queue, Journal of Applied Probability, 29:176–184

2. Gross D, Harris CM (1985) Fundamentals of Queueing Theory, 2^{nd} Edition, John Wiley & Sons, Inc..
3. Hu JQ (1995) Analyticity of single-server queues in light traffic, Queueing Systems, 19:63–80
4. Hu JQ (1996) The departure process of the $GI/G/1$ queue and its MacLaurin series, Operations Research, 44:810–815
5. Kanet JJ, Sridharan V (2000) Scheduling with inserted idle time: problem taxonomy and literature review, Operations Research, 48:99–110
6. Kleinrock L (1975) Queueing Systems Volume 1, John Wiley & Sons, Inc..
7. Li J (1997) An approximation method for the analysis of $GI/G/1$ queues, Operations Research, 45:140–144
8. Sridharan V, Zhou Z (1996) Dynamic non-preemptive single machine scheduling, Computer and Operations Research, 23(12):1183–1190