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Blow-up for a semilinear reaction-diffusion system coupled in both equations and boundary conditions

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Abstract

We study the blow-up behavior for a semilinear reaction-diffusion system coupled in both equations and boundary conditions. The main purpose is to understand how the reaction terms and the absorption terms affect the blow-up properties. We obtain a necessary and sufficient condition for blow-up, derive the upper bound and lower bound for the blow-up rate, and find the blow-up set under certain assumptions.

1. Introduction

In this paper, we study the problem for the following parabolic system

$$u_t = u_{xx} + v^{p_1}, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$v_t = v_{xx} + u^{p_2}, \quad 0 < x < 1,\ t > 0, \tag{1.2}$$

with boundary conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = v^{q_1}(1,t), \quad t > 0, \tag{1.3}$$

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$$v_x(0,t)=0, \quad v_x(1,t)=u^{q_2}(1,t), \quad t>0, \tag{1.4}$$

and initial conditions

$$u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad 0\leqslant x\leqslant 1, \tag{1.5}$$

where p_1, p_2, q_1, q_2 are positive constants, and $u_0(x), v_0(x)$ are positive smooth functions satisfying the compatibility conditions

$$u_0'(0)=v_0'(0)=0, \quad u_0'(1)=v_0^{q_1}(1), \quad v_0'(1)=u_0^{q_2}(1).$$

The local (in time) existence and uniqueness of classical solutions of the problem (1.1)–(1.5) can be derived easily by standard parabolic theory.

We say that the solution (u,v) of the problem (1.1)–(1.5) blows up in finite time if

$$T:=\sup\bigl\{\tau>0 \mid \text{ both } u \text{ and } v \text{ are bounded in } [0,1]\times[0,\tau]\bigr\}<\infty.$$

In this case, T is called the blow-up time. If $T=+\infty$, then (u,v) is said to exist globally.

Blow-up problems for the following systems:

$$\begin{cases} u_t=\Delta u+v^p, \quad v_t=\Delta v+u^q, \quad x\in\Omega,\ t>0,\\ u=v=0, \quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\Omega, \end{cases} \tag{1.6}$$

$$\begin{cases} u_t=\Delta u, \quad v_t=\Delta v, \quad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu}=v^p, \quad \frac{\partial v}{\partial\nu}=u^q, \quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\Omega, \end{cases} \tag{1.7}$$

and

$$\begin{cases} u_t=\Delta u+v^p, \quad v_t=\Delta v, \quad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu}=0, \quad \frac{\partial v}{\partial\nu}=u^q, \quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\Omega, \end{cases} \tag{1.8}$$

have been studied very extensively over past years. Here $p,q>0$, ν is the outer normal, and Ω is a bounded (or unbounded) domain in R^n. They studied the global and non-global existence, the blow-up set, and the blow-up rate for the above three systems (see, for example, [1–17] and the references cited therein). Blow-up results for other parabolic systems, we refer the readers to the survey paper [18] and the references cited therein. See also [19–22].

Recently, Lin and Wang in [23] considered the following problem for a single semilinear heat equation:

$$u_t=u_{xx}+u^p, \quad 0<x<1,\ t>0, \tag{1.9}$$

$$u_x(0,t)=0, \quad u_x(1,t)=u^q(1,t), \quad t>0, \tag{1.10}$$

$$u(x,0)=u_0(x), \quad 0\leqslant x\leqslant 1, \tag{1.11}$$

where $p, q > 0$. They studied how the reaction term u^p and the absorption term u^q affect the blow-up properties of the solution of (1.9)–(1.11). They obtained a necessary and sufficient condition for blow-up, derived the upper and lower bounds for the blow-up rate, and obtained the blow-up set under some assumptions. The authors in [24] then studied the blow-up set, described the time asymptotic behavior of blow-up solutions, and proved that the blow-up is complete under certain conditions for (1.9)–(1.11).

The main purpose of this paper is to understand how the reaction terms and the boundary absorption terms affect the blow-up properties for the problem (1.1)–(1.5). Some similar results to [23] and [24] are established for (1.1)–(1.5). This paper is organized as follows. We first study the global existence and blow-up results for the problem (1.1)–(1.5) in Section 2. After proving some blow-up criteria for problems in half real line in Section 3, we derive the blow-up rate estimates for (1.1)–(1.5) in Section 4. Finally, in Section 5 we deal with the blow-up set.

2. Global and non-global existence

Definition 2.1. A pair of functions (u, v) is called a supersolution of (1.1)–(1.5) in $[0,1] \times [0,T)$, if $u, v \in C^{2,1}([0,1] \times [0,T))$ and (u, v) satisfies

$$
\begin{aligned}
&u_t \geqslant u_{xx} + v^{p_1}, \quad (x,t) \in (0,1) \times (0,T),\\
&v_t \geqslant v_{xx} + u^{p_2}, \quad (x,t) \in (0,1) \times (0,T),\\
&u_x(0,t) \leqslant 0, \quad u_x(1,t) \geqslant v^{q_1}(1,t), \quad t \in (0,T),\\
&v_x(0,t) \leqslant 0, \quad v_x(1,t) \geqslant u^{q_2}(1,t), \quad t \in (0,T),\\
&u(x,0) \geqslant u_0(x), \quad v(x,0) \geqslant v_0(x), \quad x \in [0,1].
\end{aligned}
$$

Subsolution is defined by reversing the inequalities.

We shall use the following comparison principle to prove some global and non-global existence results.

Lemma 2.1. *Let $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ be a positive supersolution and a nonnegative subsolution of (1.1)–(1.5) in $[0,1] \times [0,T)$, respectively. Then $\bar{u} \geqslant \underline{u}$ and $\bar{v} \geqslant \underline{v}$ in $[0,1] \times [0,T)$.*

Proof. Let $w = \bar{u} - \underline{u}$ and $z = \bar{v} - \underline{v}$. Then

$$
\begin{aligned}
&w_t \geqslant w_{xx} + a(x,t)z, \quad z_t \geqslant z_{xx} + b(x,t)w, \quad 0 < x < 1,\ 0 < t < T,\\
&w_x(0,t) \leqslant 0, \quad z_x(0,t) \leqslant 0, \quad 0 < t < T,\\
&w_x(1,t) \geqslant c(t)z(1,t), \quad z_x(1,t) \geqslant d(t)w(1,t), \quad 0 < t < T,
\end{aligned}
$$

$$w(x,0) \geqslant 0, \quad z(x,0) \geqslant 0, \quad 0 \leqslant x \leqslant 1,$$

where

$$a(x,t) = \frac{\bar{v}^{p_1}(x,t) - \underline{v}^{p_1}(x,t)}{\bar{v}(x,t) - \underline{v}(x,t)}, \quad \text{if } \bar{v} \neq \underline{v}; \quad = 0, \text{ otherwise},$$
$$b(x,t) = \frac{\bar{u}^{p_2}(x,t) - \underline{u}^{p_2}(x,t)}{\bar{u}(x,t) - \underline{u}(x,t)}, \quad \text{if } \bar{u} \neq \underline{u}; \quad = 0, \text{ otherwise},$$
$$c(t) = \frac{\bar{v}^{q_1}(1,t) - \underline{v}^{q_1}(1,t)}{\bar{v}(1,t) - \underline{v}(1,t)}, \quad \text{if } \bar{v} \neq \underline{v}; \quad = 0, \text{ otherwise},$$
$$d(t) = \frac{\bar{u}^{q_2}(1,t) - \underline{u}^{q_2}(1,t)}{\bar{u}(1,t) - \underline{u}(1,t)}, \quad \text{if } \bar{u} \neq \underline{u}; \quad = 0, \text{ otherwise}.$$

For any fixed $\tau \in (0,T)$, we will show that $w \geqslant 0$ and $z \geqslant 0$ for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant \tau$. For contradiction, we assume that w has a negative minimum in $[0,1] \times [0,\tau]$ and $\min_{[0,1]\times[0,\tau]} w \leqslant \min_{[0,1]\times[0,\tau]} z$. Let $\widetilde{w} = e^{-Mt-Lx^2} w$ and $\tilde{z} = e^{-Mt-Lx^2} z$, where

$$L = \max_{0\leqslant t\leqslant\tau} c(t)/2, \quad M = 2L + 4L^2 + \max_{[0,1]\times[0,\tau]} a(x,t) + \max_{[0,1]\times[0,\tau]} b(x,t).$$

Then

$$\widetilde{w}_t \geqslant \widetilde{w}_{xx} + 4Lx\widetilde{w}_x + \big(2L + 4L^2x^2 - M\big)\widetilde{w} + a(x,t)\tilde{z},$$
$$0 < x < 1,\ 0 < t < \tau, \tag{2.1}$$
$$\tilde{z}_t \geqslant \tilde{z}_{xx} + 4Lx\tilde{z}_x + b(x,t)\widetilde{w} + \big(2L + 4L^2x^2 - M\big)\tilde{z},$$
$$0 < x < 1,\ 0 < t < \tau. \tag{2.2}$$

Since $\widetilde{w} \geqslant -\delta$ and $\tilde{z} \geqslant -\delta$ on the boundary $([0,1] \times \{0\}) \cup (\{0,1\} \times (0,\tau])$, where $-\delta := \min_{[0,1]\times[0,\tau]} \widetilde{w} < 0$, it follows from the strong maximum principle for weakly coupled parabolic systems (cf. Theorem 15 of Chapter 3 in [25]) that $\widetilde{w}$ cannot assume its negative minimum in the interior. Hence $\widetilde{w} > -\delta$ in $(0,1) \times (0,\tau]$. Let (x_0,t_0) be a minimum point on the boundary $\{0,1\} \times (0,\tau]$. Since $\widetilde{w}_x(0,t) \leqslant 0$, $0 < t \leqslant \tau$, the same strong maximum principle implies that $x_0 = 1$ and $\widetilde{w}_x(x_0,t_0) < 0$. But,

$$\widetilde{w}_x(1,t_0) \geqslant -\big(c(t_0) - 2L\big)\delta \geqslant 0,$$

a contradiction. This completes the proof. □

Theorem 2.2. *Suppose that* $\max\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\} \leqslant 1$. *Then the solution* (u,v) *of* (1.1)–(1.5) *exists globally.*

Proof. Since $\max\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\} \leqslant 1$, there exists a positive number l such that $p_2 \leqslant l \leqslant 1/p_1$ and $q_2 \leqslant l \leqslant 1/q_1$. Let

$$\bar{u} = Ce^{Kt+Lx^2}, \qquad \bar{v} = Ce^{l(Kt+Lx^2)},$$

where C, K, L are positive constants satisfying

$$C \geqslant \max\{\|u_0\|_\infty, \|v_0\|_\infty\},$$
$$L \geqslant \frac{1}{2}C^{q_1-1}, \qquad L \geqslant \frac{1}{2l}C^{q_2-1},$$
$$K \geqslant 2LC + 4L^2 + C^{p_1-1}, \qquad K \geqslant 2L + 4lL^2 + \frac{1}{l}C^{p_2-1}.$$

It is easy to verify that $(\bar{u}, \bar{v})$ is a supersolution of (1.1)–(1.5). Then, by Lemma 2.1, we get $u \leqslant \bar{u}$ and $v \leqslant \bar{v}$. Hence the theorem follows. □

Theorem 2.3. *Suppose that* $\max\{p_1p_2, q_1q_2, p_1q_2, p_2q_1\} > 1$. *Then the solution* (u, v) *of* (1.1)–(1.5) *blows up in finite time.*

Proof. Set $l_1 = \inf_{0\leqslant x\leqslant 1} u_0(x)$ and $l_2 = \inf_{0\leqslant x\leqslant 1} v_0(x)$.

Suppose that $p_1p_2 > 1$. Let

$$\underline{u} = A(S-t)^{-\alpha}, \qquad \underline{v} = B(S-t)^{-\beta},$$

where $\alpha = (p_1+1)/(p_1p_2-1)$, $\beta = (p_2+1)/(p_1p_2-1)$, and A, B, S are positive constants satisfying

$$B \geqslant (\alpha^{p_2}\beta)^{1/(p_1p_2-1)},$$
$$(\beta B)^{1/p_2} \leqslant A \leqslant \alpha^{-1}B^{p_1},$$
$$AS^{-\alpha} \leqslant l_1, \qquad AS^{-\beta} \leqslant l_2.$$

Then $(\underline{u}, \underline{v})$ is a subsolution of (1.1)–(1.5). Thus, by Lemma 2.1, we obtain that $u \geqslant \underline{u}$ and $v \geqslant \underline{v}$ as long as both $(\underline{u}, \underline{v})$ and (u, v) exist. Therefore, (u, v) blows up in finite time.

For $q_1q_2 > 1$, we let

$$\underline{u} = \left(M - \eta t - \eta x^2\right)^{-\alpha}, \qquad \underline{v} = \left(M - \eta t - \eta x^2\right)^{-\beta},$$

where $\alpha = (q_1+1)/(q_1q_2-1)$, $\beta = (q_2+1)/(q_1q_2-1)$, and M, η are positive constants satisfying

$$\eta \leqslant \min\{1/(2\alpha), 1/(2\beta)\},$$
$$M \geqslant \eta + \max\{l_1^{-1/\alpha}, l_2^{-1/\beta}\}.$$

Then $(\underline{u}, \underline{v})$ is a subsolution of (1.1)–(1.5). It follows from Lemma 2.1 that $u \geqslant \underline{u}$ and $v \geqslant \underline{v}$ as long as both $(\underline{u}, \underline{v})$ and (u, v) exist. Hence (u, v) blows up before $(\underline{u}, \underline{v})$ does.

For $p_1q_2 > 1$ or $p_2q_1 > 1$, the conclusion follows from Theorem 2.3 of [6] and Lemma 2.1. This completes the proof. □

3. Blow-up criteria

In this section, we first derive the comparison principles for the following two problems

$$u_t = u_{xx} + v^p, \quad v_t = v_{xx}, \quad x > 0,\ t > 0, \tag{3.1}$$

$$-u_x(0,t) = 0, \quad -v_x(0,t) = u^q(0,t), \quad t > 0, \tag{3.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \geqslant 0, \tag{3.3}$$

and

$$u_t = u_{xx}, \quad v_t = v_{xx}, \quad x > 0,\ t > 0, \tag{3.4}$$

$$-u_x(0,t) = v^p(0,t), \quad -v_x(0,t) = u^q(0,t), \quad t > 0, \tag{3.5}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \geqslant 0, \tag{3.6}$$

where p and q are positive constants. For completeness, we shall give the proof here. To this end, we need the following lemma.

Lemma 3.1. *Let* $\tau^* > 0$ *and let* $u \in C^{2,1}((0,\infty) \times (0,\tau^*))$ *be a bounded continuous function in* $[0,\infty) \times [0,\tau^*)$ *satisfying*

$$u_t \leqslant u_{xx}, \quad x > 0,\ 0 < t < \tau^*, \tag{3.7}$$

$$u(0,t) \leqslant 0, \quad 0 < t < \tau^*, \tag{3.8}$$

$$u(x,0) \leqslant 0, \quad x \geqslant 0. \tag{3.9}$$

Then $u \leqslant 0$ *in* $[0,\infty) \times [0,\tau^*)$.

Proof. Given any fixed $\tau \in (0,\tau^*)$. Let χ be a $C_0^\infty(\mathbf{R})$ function satisfying $0 \leqslant \chi \leqslant 1$ and $\operatorname{supp}\chi \subset [0,\infty)$. For any $R > 1$ such that $\operatorname{supp}\chi \subset [0, R-1]$, let φ be the solution of the following backward problem

$$\varphi_t + \varphi_{xx} = \varphi, \quad 0 < x < R,\ 0 < t < \tau, \tag{3.10}$$

$$\varphi(0,t) = \varphi(R,t) = 0, \quad 0 < t < \tau, \tag{3.11}$$

$$\varphi(x,\tau) = e^{-x}\chi(x), \quad 0 \leqslant x \leqslant R. \tag{3.12}$$

It follows from the maximum principle that

$$0 \leqslant \varphi \leqslant e^{-x}, \quad 0 \leqslant x \leqslant R,\ 0 \leqslant t \leqslant \tau. \tag{3.13}$$

Set

$$\psi(x) = K\left(e^{-x} - e^{x-2R}\right), \quad K = \frac{e}{e - 1/e}.$$

It is easy to see that ψ satisfies

$$\psi'' = \psi, \quad R-1 < x < R,$$
$$\psi(R-1) = e^{-(R-1)}, \quad \psi(R) = 0.$$

Applying the maximum principle, we obtain that $\varphi \leqslant \psi$ for $R-1 \leqslant x \leqslant R$ and $0 \leqslant t \leqslant \tau$. Since $\varphi(R,t) = \psi(R) = 0$, we conclude that

$$0 \leqslant -\varphi_x(R,t) \leqslant -\psi'(R) = 2Ke^{-R}, \quad 0 < t < \tau. \tag{3.14}$$

Multiplying both sides of (3.7) by φ and integrating it over $[0,R] \times [0,\tau]$, by (3.8)–(3.14), we deduce that

$$\int_0^R u(x,\tau)e^{-x}\chi(x)\,dx \leqslant \int_0^\tau \int_0^R u^+ e^{-x}\,dx\,dt + 2KM\tau e^{-R},$$

where $M = \sup_{[0,\infty)\times[0,\tau^*]} |u|$. Letting $R \to \infty$, we get

$$\int_0^\infty u(x,\tau)e^{-x}\chi(x)\,dx \leqslant \int_0^\tau \int_0^\infty u^+ e^{-x}\,dx\,dt. \tag{3.15}$$

Note that (3.15) holds for any $\chi \in C_0^\infty(\mathbf{R})$ satisfying $0 \leqslant \chi \leqslant 1$ and $\operatorname{supp}\chi \subset [0,\infty)$.

Now, for each $k \in \mathbf{N}$, let $\chi_k = g_k h_k$, where g_k is a $C^\infty(\mathbf{R})$ function satisfying $0 \leqslant g_k \leqslant 1$ and

$$g_k(x) = \begin{cases} 1 & \text{if } u(x,\tau)e^{-x} \geqslant 1/k \text{ and } 0 \leqslant x \leqslant 3k, \\ 0 & \text{if } u(x,\tau)e^{-x} \leqslant 0 \text{ or } x \leqslant 0 \end{cases}$$

(notice that such function g_k exists, since the set $\{x \mid u(x,\tau)e^{-x} \geqslant 1/k \text{ and } 0 \leqslant x \leqslant 3k\}$ is compact, the set $\{x \mid u(x,\tau)e^{-x} \leqslant 0 \text{ or } x \leqslant 0\}$ is closed, and they are disjoint), and h_k is a $C_0^\infty(\mathbf{R})$ function satisfying $0 \leqslant h_k \leqslant 1$ and

$$h_k(x) = \begin{cases} 1 & \text{if } x \leqslant k, \\ 0 & \text{if } x \geqslant 2k. \end{cases}$$

Clearly, $\chi_k \in C_0^\infty(\mathbf{R})$, $0 \leqslant \chi_k \leqslant 1$, and $\operatorname{supp}\chi_k \subset [0,\infty)$ for any $k \in \mathbf{N}$. Replacing χ by χ_k in (3.15) and applying the Lebesgue dominated convergence theorem, we obtain that

$$\int_0^\infty u^+(x,\tau)e^{-x}\,dx \leqslant \int_0^\tau \int_0^\infty u^+ e^{-x}\,dx\,dt.$$

Then from the Gronwall's inequality it follows that

$$\int_0^\tau \int_0^\infty u^+ e^{-x}\,dx\,dt \leqslant 0.$$

Hence $u^+ = 0$ in $[0,\infty) \times [0,\tau]$. Since τ is arbitrary, the lemma follows. $\square$

Definition 3.1. A pair of functions $(\bar{u}, \bar{v})$ is called a (nonnegative) supersolution of (3.1)–(3.3) in $[0, \infty) \times [0, T)$, if $\bar{u}, \bar{v} \in C^{2,1}((0, \infty) \times (0, T)) \cap C([0, \infty) \times [0, T))$ and $(\bar{u}, \bar{v})$ satisfies

$$\bar{u}_t \geqslant \bar{u}_{xx} + \bar{v}^p, \quad \bar{v}_t \geqslant \bar{v}_{xx}, \quad x > 0,\ 0 < t < T, \tag{3.16}$$

$$-\bar{u}_x(0, t) \geqslant 0, \quad -\bar{v}_x(0, t) \geqslant \bar{u}^q(0, t), \quad 0 < t < T, \tag{3.17}$$

$$\bar{u}(x, 0) \geqslant u_0(x), \quad \bar{v}(x, 0) \geqslant v_0(x), \quad x \geqslant 0. \tag{3.18}$$

Subsolution is defined by reversing the inequalities in (3.16)–(3.18). Similarly, we can define supersolution and subsolution of (3.4)–(3.6).

Theorem 3.2. *Let* $(\bar{u}, \bar{v})$ *and* $(\underline{u}, \underline{v})$ *be a supersolution and a subsolution of* (3.1)–(3.3) *in* $[0, \infty) \times [0, T)$*, respectively. Suppose that* $(\bar{u}, \bar{v})$ *and* $(\underline{u}, \underline{v})$ *are bounded in* $[0, \infty) \times [0, T)$*. If* $\bar{u}(0, 0) > \underline{u}(0, 0)$ *and* $\bar{v}(0, 0) > \underline{v}(0, 0)$*, then* $\bar{u} \geqslant \underline{u}$ *and* $\bar{v} \geqslant \underline{v}$ *in* $[0, \infty) \times [0, T)$*.*

Proof. For contradiction, we assume that

$$t_0 := \sup\bigl\{\sigma \geqslant 0 \mid \bar{u} \geqslant \underline{u} \text{ and } \bar{v} \geqslant \underline{v} \text{ in } [0, \infty) \times [0, \sigma]\bigr\} < T.$$

Since $\bar{u}(0, 0) > \underline{u}(0, 0)$ and $\bar{v}(0, 0) > \underline{v}(0, 0)$, there exists $\tau^* \in (0, T)$ such that $\bar{u}(0, t) > \underline{u}(0, t)$ and $\bar{v}(0, t) > \underline{v}(0, t)$ for $t \in [0, \tau^*]$. From Lemma 3.1, we obtain that $\bar{v} \geqslant \underline{v}$ in $[0, \infty) \times [0, \tau^*]$. Thus

$$(\bar{u} - \underline{u})_t \geqslant (\bar{u} - \underline{u})_{xx} + \bar{v}^p - \underline{v}^p \geqslant (\bar{u} - \underline{u})_{xx} \quad \text{in } (0, \infty) \times (0, \tau^*).$$

Again, by Lemma 3.1, we obtain that $\bar{u} \geqslant \underline{u}$ in $[0, \infty) \times [0, \tau^*]$. Hence $t_0 \geqslant \tau^* > 0$. The definition of t_0 implies that there exists $x_0 \geqslant 0$ such that either $\bar{u}(x_0, t_0) = \underline{u}(x_0, t_0)$ or $\bar{v}(x_0, t_0) = \underline{v}(x_0, t_0)$. By the strong maximum principle, $x_0 = 0$. Then, by applying the Hopf's boundary point lemma, either $\bar{u}_x(0, t_0) > \underline{u}_x(0, t_0)$ or $\bar{v}_x(0, t_0) > \underline{v}_x(0, t_0)$, a contradiction. Hence $t_0 = T$ and the proof is complete. □

Now, we consider the problem

$$\begin{cases} \varphi_s = \varphi_{yy} + \psi^{p_1}, \quad \psi_s = \psi_{yy} + \mu_1 \varphi^{p_2}, \quad y > 0,\ s > 0, \\ \varphi_y(0, s) = -\mu_2 \psi^{q_1}(0, s), \quad \psi_y(0, s) = -\varphi^{q_2}(0, s), \quad s > 0, \\ \varphi(y, 0) = \varphi_0(y), \quad \psi(y, 0) = \psi_0(y), \quad y \geqslant 0, \end{cases} \tag{3.19}$$

where $\mu_i \in \{0, 1\}$, $i = 1, 2$. Set

$$\alpha = \frac{p_1 + 2}{2(p_1 q_2 - 1)}, \qquad \beta = \frac{2q_2 + 1}{2(p_1 q_2 - 1)}.$$

Theorem 3.3. *Suppose that* $p_1 q_2 > 1$*. Under the assumption that either* $\max\{\alpha, \beta\} > 1/2$*, or,* $\max\{\alpha, \beta\} = 1/2$ *and* $\min\{p_1, q_2\} \geqslant 1$*, every nontrivial nonnegative solution* (φ, ψ) *of* (3.19) *blows up in finite time.*

Proof. This theorem is just the main Theorem of [8] when $\mu_1 = \mu_2 = 0$.

In general, we may without loss of generality assume that $\varphi_0(0) > 0$ and $\psi_0(0) > 0$, since $\varphi(0,s) > 0$ and $\psi(0,s) > 0$ as long as φ, ψ exist and $s > 0$. Now, let (u, v) be a solution of (3.1)–(3.3) with $p = p_1, q = q_2$, and initial functions $u_0 = \varphi_0/2$, $v_0 = \psi_0/2$. Then by the comparison principle (Theorem 3.2) we have $\varphi \geqslant u$ and $\psi \geqslant v$ as long as u, v, φ, ψ are bounded. Since (u, v) blows up in finite time, the theorem follows. $\square$

Using a similar argument as in the proof of Theorem 3.2, we can also prove the following theorem.

Theorem 3.4. *Let $(\bar u, \bar v)$ and $(\underline{u}, \underline{v})$ be a supersolution and a subsolution of* (3.4)–(3.6) *in* $[0,\infty) \times [0,T)$, *respectively. Suppose that $(\bar u, \bar v)$ and $(\underline{u}, \underline{v})$ are bounded in* $[0,\infty) \times [0,T)$. *If $\bar u(0,0) > \underline{u}(0,0)$ and $\bar v(0,0) > \underline{v}(0,0)$, then $\bar u \geqslant \underline{u}$ and $\bar v \geqslant \underline{v}$ in* $[0,\infty) \times [0,T)$.

Using Theorem 2.1 of [5] and Theorem 3.4, we can prove the following blow-up result for solutions of the system:

$$\begin{cases} \varphi_s = \varphi_{yy} + \mu_1 \psi^{p_1}, \quad \psi_s = \psi_{yy} + \mu_2 \varphi^{p_2}, \quad y > 0,\ s > 0, \\ \varphi_y(0,s) = -\psi^{q_1}(0,s), \quad \psi_y(0,s) = -\varphi^{q_2}(0,s), \quad s > 0, \\ \varphi(y,0) = \varphi_0(y), \quad \psi(y,0) = \psi_0(y), \quad y \geqslant 0, \end{cases} \tag{3.20}$$

where $\mu_i \in \{0,1\}$, $i = 1,2$.

Theorem 3.5. *Suppose that $q_1 q_2 > 1$. Set*

$$\alpha = \frac{q_1 + 1}{2(q_1 q_2 - 1)}, \qquad \beta = \frac{q_2 + 1}{2(q_1 q_2 - 1)}.$$

Under the assumption that $\max\{\alpha, \beta\} \geqslant 1/2$, *every nontrivial nonnegative solution (φ, ψ) of* (3.20) *blows up in finite time.*

Finally, we consider the following problem:

$$\begin{cases} \varphi_s = \varphi_{yy} + \psi^{p_1}, \quad \psi_s = \psi_{yy} + \varphi^{p_2}, \quad y > 0,\ s > 0, \\ \varphi_y(0,s) = -\mu_1 \psi^{q_1}(0,s), \quad \psi_y(0,s) = -\mu_2 \varphi^{q_2}(0,s), \quad s > 0, \\ \varphi(y,0) = \varphi_0(y), \quad \psi(y,0) = \psi_0(y), \quad y \geqslant 0, \end{cases} \tag{3.21}$$

where $\mu_i \in \{0,1\}$, $i = 1,2$.

Theorem 3.6. *Suppose that $p_1 p_2 > 1$. Set*

$$\alpha = \frac{p_1 + 1}{p_1 p_2 - 1}, \qquad \beta = \frac{p_2 + 1}{p_1 p_2 - 1}.$$

Under the assumption that $\max\{\alpha, \beta\} \geqslant 1/2$, *every nontrivial nonnegative solution (φ, ψ) of* (3.21) *blows up in finite time.*

Proof. Let

$$G(x,y,t) = (4\pi t)^{-1/2}\exp\left(-\frac{(x-y)^2}{4t}\right),$$
$$g(t)w(x,\cdot) = \int_0^\infty \bigl[G(x,y,t)+G(x,-y,t)\bigr]w(y,\cdot)\,dy.$$

Then the solution (φ,ψ) of (3.21) can be represented by

$$\varphi(\cdot,s) = g(s)\varphi_0 + \int_0^s g(s-t)\psi^{p_1}(\cdot,t)\,dt + 2\mu_1\int_0^s G(\cdot,0,s-t)\psi^{q_1}(0,t)\,dt,$$
$$\psi(\cdot,s) = g(s)\psi_0 + \int_0^s g(s-t)\varphi^{p_2}(\cdot,t)\,dt + 2\mu_2\int_0^s G(\cdot,0,s-t)\varphi^{q_2}(0,t)\,dt.$$

The theorem can be proved by following the proof of Theorem 2 in [7] step by step. □

4. Blow-up rate

In this section, we always assume that $u_0' \geqslant 0$, $v_0' \geqslant 0$, and the solution (u,v) of (1.1)–(1.5) blows up in finite time T. Then by the maximum principle we have $u_x \geqslant 0$ and $v_x \geqslant 0$ in $[0,1]\times[0,T)$. Notice that $u(1,t) = \max_{0\leqslant x\leqslant 1} u(x,t)$ and $v(1,t) = \max_{0\leqslant x\leqslant 1} v(x,t)$. Motivated by [26] for scalar equations and [1] for systems, we shall use a scaling method (cf. [27]) to derive the blow-up rate.

For convenience, we let

$$p_1^* := \frac{2q_1q_2+q_1-1}{q_2+1}, \qquad p_2^* := \frac{2q_1q_2+q_2-1}{q_1+1},$$
$$q_1^* := \frac{p_1p_2+2p_1+1}{2(p_2+1)}, \qquad q_2^* := \frac{p_1p_2+2p_2+1}{2(p_1+1)}$$

for given positive constants p_1, p_2, q_1, q_2. It is easy to check that $\max\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\} \leqslant 1$ if one of the following conditions holds:

(1) $p_1q_2 \leqslant 1$, $p_1 \geqslant p_1^*$, and $q_2 \geqslant q_2^*$;
(2) $p_2q_1 \leqslant 1$, $p_2 \geqslant p_2^*$, and $q_1 \geqslant q_1^*$;
(3) $q_1q_2 \leqslant 1$, $p_1 \leqslant p_1^*$, and $p_2 \leqslant p_2^*$;
(4) $p_1p_2 \leqslant 1$, $q_1 \leqslant q_1^*$, and $q_2 \leqslant q_2^*$.

Since (u, v) blows up in finite time, it follows from the above observation and Theorem 2.2 that

(1) $p_1q_2 > 1$, if $p_1 \geqslant p_1^*$ and $q_2 \geqslant q_2^*$;
(2) $p_2q_1 > 1$, if $p_2 \geqslant p_2^*$ and $q_1 \geqslant q_1^*$;
(3) $q_1q_2 > 1$, if $p_1 \leqslant p_1^*$ and $p_2 \leqslant p_2^*$;
(4) $p_1p_2 > 1$, if $q_1 \leqslant q_1^*$ and $q_2 \leqslant q_2^*$.

We also define

$$(\alpha, \beta) = \begin{cases} \left(\frac{p_1+2}{2(p_1q_2-1)}, \frac{2q_2+1}{2(p_1q_2-1)}\right), & \text{if } p_1 \geqslant p_1^* \text{ and } q_2 \geqslant q_2^*; \\ \left(\frac{p_2+2}{2(p_2q_1-1)}, \frac{2q_1+1}{2(p_2q_1-1)}\right), & \text{if } p_2 \geqslant p_2^* \text{ and } q_1 \geqslant q_1^*; \\ \left(\frac{q_1+1}{2(q_1q_2-1)}, \frac{q_2+1}{2(q_1q_2-1)}\right), & \text{if } p_1 \leqslant p_1^* \text{ and } p_2 \leqslant p_2^*; \\ \left(\frac{p_1+1}{p_1p_2-1}, \frac{p_2+1}{p_1p_2-1}\right), & \text{if } q_1 \leqslant q_1^* \text{ and } q_2 \leqslant q_2^*. \end{cases} \tag{4.1}$$

Theorem 4.1. *Suppose that $p_1 \geqslant p_1^*$, $q_2 \geqslant q_2^*$, and that either $\max\{\alpha, \beta\} > 1/2$, or, $\max\{\alpha, \beta\} = 1/2$ and $\min\{p_1, q_2\} \geqslant 1$. Then there exist positive constants C_i, $i = 1, 2, 3, 4$, such that*

$$C_1(T-t)^{-\alpha} \leqslant \sup_{0<\tau<t} u(1, \tau) \leqslant C_2(T-t)^{-\alpha}, \quad \forall t \in [0, T), \tag{4.2}$$

$$C_3(T-t)^{-\beta} \leqslant \sup_{0<\tau<t} v(1, \tau) \leqslant C_4(T-t)^{-\beta}, \quad \forall t \in [0, T), \tag{4.3}$$

where (α, β) is defined by (4.1).

Proof. We shall divide the proof into the following four steps.

Step 1: *Scaling.* Let $M_u(t) = \sup_{\tau \in (0,t)} u(1, \tau)$ and $M_v(t) = \sup_{\tau \in (0,t)} v(1, \tau)$. Without loss of generality we may assume that $M_u(t) \to +\infty$ as $t \to T$. Given $t \in (0, T)$ such that $M_u(t) > \|u_0\|_\infty$, there exists $\hat{t} \in (0, t]$ such that

$$u(1, \hat{t}) = M_u(t). \tag{4.4}$$

Take $\lambda = M_u^{-1/(2\alpha)}(t)$. Let

$$\varphi^\lambda(y, s) = \lambda^{2\alpha} u\left(1 - \lambda y, \hat{t} + \lambda^2 s\right), \tag{4.5}$$

$$\psi^\lambda(y, s) = \lambda^{2\beta} v\left(1 - \lambda y, \hat{t} + \lambda^2 s\right), \tag{4.6}$$

for any $(y, s) \in [0, 1/\lambda] \times [-\hat{t}/\lambda^2, (T - \hat{t})/\lambda^2)$. It is easy to see that $(\varphi^\lambda, \psi^\lambda)$ is the solution of the problem (P^λ):

$$\begin{cases} \varphi_s = \varphi_{yy} + \psi^{p_1}, \quad \psi_s = \psi_{yy} + \lambda^{\gamma_1}\varphi^{p_2}, \quad (y, s) \in \left(0, \frac{1}{\lambda}\right) \times \left(-\frac{\hat{t}}{\lambda^2}, \frac{T-\hat{t}}{\lambda^2}\right), \\ \varphi_y\left(\frac{1}{\lambda}, s\right) = 0, \quad \psi_y\left(\frac{1}{\lambda}, s\right) = 0, \quad s \in \left(-\frac{\hat{t}}{\lambda^2}, \frac{T-\hat{t}}{\lambda^2}\right), \\ \varphi_y(0, s) = -\lambda^{\gamma_2}\psi^{q_1}(0, s), \quad \psi_y(0, s) = -\varphi^{q_2}(0, s), \quad s \in \left(-\frac{\hat{t}}{\lambda^2}, \frac{T-\hat{t}}{\lambda^2}\right), \\ \varphi(0, 0) = 1 \end{cases}$$

and satisfies

$$\begin{aligned}&0 \leqslant \varphi^\lambda \leqslant 1, \quad 0 \leqslant \psi^\lambda \leqslant M_u^{-\beta/\alpha}(t) M_v(t),\\ &\quad \forall (y,s) \in \left[0, \frac{1}{\lambda}\right] \times \left[-\frac{\hat{t}}{\lambda^2}, 0\right],\end{aligned} \tag{4.7}$$

where $\gamma_1 := 2\beta + 2 - 2\alpha p_2 \geqslant 0$ and $\gamma_2 := 2\alpha + 1 - 2\beta q_1 \geqslant 0$, since $p_1 q_2 > 1$, $p_1 \geqslant p_1^*$, and $q_2 \geqslant q_2^*$. Moreover, $\gamma_1 = 0$ if and only if $q_2 = q_2^*$; $\gamma_2 = 0$ if and only if $p_1 = p_1^*$.

Step 2: *Claim that there exists* $\delta > 0$ *such that*

$$\delta \leqslant M_u^{-1/(2\alpha)}(t) M_v^{1/(2\beta)}(t) \leqslant \delta^{-1}, \quad \forall t \in [0, T). \tag{4.8}$$

If the lower bound estimate in (4.8) does not hold, then there exists a sequence $\{t_j\} \nearrow T$ such that $M_u(t_j) > \|u_0\|_\infty$, $\forall j$, and

$$M_u^{-1/(2\alpha)}(t_j) M_v^{1/(2\beta)}(t_j) \to 0 \quad \text{as } j \to \infty. \tag{4.9}$$

For each j, we define $\hat{t}_j, \lambda_j$, and $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ as in *Step* 1 such that the solution $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ of the corresponding problem (P^{λ_j}) satisfies

$$\begin{aligned}&0 \leqslant \varphi^{\lambda_j} \leqslant 1, \quad 0 \leqslant \psi^{\lambda_j} \leqslant M_u^{-\beta/\alpha}(t_j) M_v(t_j),\\ &\quad \forall (y,s) \in \left[0, \frac{1}{\lambda_j}\right] \times \left[-\frac{\hat{t}_j}{\lambda_j^2}, 0\right].\end{aligned} \tag{4.10}$$

Note that $\hat{t}_j \to T$ and $\lambda_j \to 0$ as $j \to \infty$. For any $m \in N$, from (4.9) and (4.10) it follows that

$$0 \leqslant \varphi^{\lambda_j} \leqslant 1, \quad 0 \leqslant \psi^{\lambda_j} \leqslant 1, \quad \forall (y,s) \in [0,m] \times [-m^2, 0], \tag{4.11}$$

if j is sufficiently large. Then applying the standard parabolic estimate for scalar equations (cf. [28] or [29]), we obtain that

$$\left\| \varphi^{\lambda_j} \right\|_{C^{2+\sigma,1+\sigma/2}([0,m]\times[-m^2,0])} \leqslant C, \tag{4.12}$$

$$\left\| \psi^{\lambda_j} \right\|_{C^{2+\sigma,1+\sigma/2}([0,m]\times[-m^2,0])} \leqslant C \tag{4.13}$$

for some $0 < \sigma < \min\{1, p_1, p_2, q_1, q_2\}$ and $C = C(m,\sigma) > 0$. Using (4.12), (4.13), and a diagonal process, we can get a subsequence (still denoted by $(\varphi^{\lambda_j}, \psi^{\lambda_j})$) such that $\varphi^{\lambda_j} \to \varphi$ and $\psi^{\lambda_j} \to \psi$ uniformly on each compact subset of $[0,\infty) \times (-\infty, 0]$ for some (φ, ψ) satisfying

$$\begin{cases} \varphi_s = \varphi_{yy} + \psi^{p_1}, \quad \psi_s = \psi_{yy} + \mu_1 \varphi^{p_2}, \quad 0 < y < \infty,\ -\infty < s < 0,\\ \varphi_y(0,s) = -\mu_2 \psi^{q_1}(0,s), \quad \psi_y(0,s) = -\varphi^{q_2}(0,s),\\ \quad -\infty < s < 0,\\ \varphi(0,0) = 1, \end{cases} \tag{4.14}$$

where $\mu_i \in \{0,1\}$, $i = 1,2$, $\mu_1 = 1$ if and only if $q_2 = q_2^*$, and $\mu_2 = 1$ if and only if $p_1 = p_1^*$. But, by (4.10), $\psi \equiv 0$, a contradiction. Hence there exists $\delta > 0$ such that the lower bound estimate in (4.8) holds.

If the upper bound estimate in (4.8) does not hold, then there exists a sequence $\{t_j\} \nearrow T$ such that

$$M_u^{-1/(2\alpha)}(t_j)M_v^{1/(2\beta)}(t_j) \to +\infty. \tag{4.15}$$

Clearly, $M_v(t_j) \to +\infty$. Choose $j^* \in N$ such that $M_v(t_j) > \|v_0\|_\infty$, $\forall j \geqslant j^*$. For any $j \geqslant j^*$, we take $\hat{t_j} \in (0, t_j]$ such that $v(1, \hat{t_j}) = M_v(t_j)$. Let $\lambda_j = M_v^{-1/(2\beta)}(t_j)$. Define $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ by (4.5) and (4.6) with $\lambda = \lambda_j$. Then $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ is the solution of (P^{λ_j}) such that

$$\begin{aligned}&0 \leqslant \varphi^{\lambda_j} \leqslant M_u(t_j)M_v^{-\alpha/\beta}(t_j), \quad 0 \leqslant \psi^{\lambda_j} \leqslant 1,\\ &\quad \forall (y, s) \in [0, 1/\lambda_j] \times \left[-\hat{t_j}/\lambda_j^2, 0\right].\end{aligned}$$

Proceeding as before, we will get a contradiction. Thus (4.8) is established.

Step 3: *Estimate the lower bounds.* Given any $t \in (0, T)$ such that $M_u(t) > \|u_0\|_\infty$. Let $\hat{t}$, λ, and $(\varphi^\lambda, \psi^\lambda)$ be defined as in *Step* 1. Since φ^λ blows up in finite time, there exists positive number s_λ such that

$$\max_{0 \leqslant y \leqslant 1/\lambda} \varphi_\lambda(y, s) < 2 \quad \text{for } -\hat{t}/\lambda^2 \leqslant s < s_\lambda \tag{4.16}$$

and $\max_{0 \leqslant y \leqslant 1/\lambda} \varphi_\lambda(y, s_\lambda) = 2$. From (4.7), (4.8), and (4.16), one can easily show that

$$0 \leqslant \varphi^\lambda \leqslant 2, \quad 0 \leqslant \psi^\lambda \leqslant 2^{\beta/\alpha}\delta^{-2\beta}, \quad \forall (y, s) \in [0, 1/\lambda] \times \left[-\hat{t}/\lambda^2, s_\lambda\right].$$

Then by applying the standard parabolic estimate for scalar equations (cf. [28] or [29]), we get

$$\begin{aligned}&\|\varphi^\lambda\|_{C^{2+\sigma,1+\sigma/2}([0,1/\lambda]\times[0,s_\lambda])} \leqslant C,\\ &\|\psi^\lambda\|_{C^{2+\sigma,1+\sigma/2}([0,1/\lambda]\times[0,s_\lambda])} \leqslant C\end{aligned}$$

for some $0 < \sigma < \min\{1, p_1, p_2, q_1, q_2\}$ and a positive constant C independent of λ. This implies that $s_\lambda \geqslant c > 0$ for some positive constant c independent of λ, or, equivalently independent of t. Let $t_0 = \hat{t}$ and $t_1 = t_0 + \lambda^2 s_\lambda$. Then $M_u(t_1) = 2M_u(t_0)$ and $M_u^{1/\alpha}(t_0)(t_1 - t_0) = s_\lambda \geqslant c$.

Replacing t by t_1, defining the corresponding $\hat{t}$, λ, and $(\varphi^\lambda, \psi^\lambda)$ as in *Step* 1, and by the same process as above, we obtain a new s_λ such that $M_u(t_2) = 2M_u(t_1)$ and $M_u^{1/\alpha}(t_1)(t_2 - t_1) = s_\lambda \geqslant c$, where $t_2 = t_1 + \lambda^2 s_\lambda$. Continuing in this process, we can get a sequence $\{t_j\} \nearrow T$ such that

$$\begin{aligned}&M_u^{1/\alpha}(t_{j-1})(t_j - t_{j-1}) \geqslant c, \quad \forall j \in N,\\ &M_u(t_j) = 2M_u(t_{j-1}), \quad \forall j \in N.\end{aligned}$$

Using a similar argument as Lemma 3.1 in [26], we derive that $M_u(t_1) \geqslant c(T - t_1)^{-\alpha}$. Since $M_u(t_1) = 2M_u(t_0) = 2M_u(t)$ and $t_1 > t$, it follows that $M_u(t) \geqslant c(T - t)^{-\alpha}$. Hence the lower bound for u in (4.2) holds, i.e.,

$$\sup_{0<\tau<t} u(1,\tau) \geqslant C_1(T-t)^{-\alpha}, \quad \forall t \in [0,T). \tag{4.17}$$

Then the lower bound for v in (4.3) follows from (4.17) and (4.8).

Step 4: *Estimate the upper bounds.* To this end, we claim that there exists a positive number C such that $s_\lambda \leqslant C$ for all sufficiently small λ, where s_λ is defined as in *Step* 3. For contradiction, we suppose that there exists a sequence $\{\lambda_j\}$ with $\lambda_j \to 0$ such that $s_{\lambda_j} \to +\infty$. Take $t_j = \max\{t \mid M_u(t) = \lambda_j^{-2\alpha}\}$. As before, we can define $\hat{t_j}$, λ_j, and $(\varphi^{\lambda_j}, \psi^{\lambda_j})$, the solution of (P^{λ_j}) in $[0, 1/\lambda_j] \times [-\hat{t_j}/\lambda_j^2, (T-\hat{t_j})/\lambda_j^2)$, such that

$$\begin{aligned} &0 \leqslant \varphi^{\lambda_j} \leqslant 2, \quad 0 \leqslant \psi^{\lambda_j} \leqslant \lambda_j^{2\beta} M_v\big(\hat{t_j} + \lambda_j^2 s_{\lambda_j}\big), \\ &\quad \forall (y,s) \in [0, 1/\lambda_j] \times \big[-\hat{t_j}/\lambda_j^2, s_{\lambda_j}\big]. \end{aligned} \tag{4.18}$$

By using (4.8) and (4.18), we obtain that

$$0 \leqslant \psi^{\lambda_j} \leqslant 2^{\beta/\alpha}\delta^{-2\beta}, \quad \forall (y,s) \in [0, 1/\lambda_j] \times \big[-\hat{t_j}/\lambda_j^2, s_{\lambda_j}\big],$$

if j is sufficiently large. As before, we can find a subsequence of $\{(\varphi^{\lambda_j}, \psi^{\lambda_j})\}$ converging to a solution of

$$\begin{cases} \varphi_s = \varphi_{yy} + \psi^{p_1}, \quad \psi_s = \psi_{yy} + \mu_1 \varphi^{p_1}, \quad (y,s) \in (0,\infty) \times (-\infty,\infty), \\ \varphi_y(0,s) = -\mu_2 \psi^{q_1}(0,s), \quad \psi_y(0,s) = -\varphi^{q_2}(0,s), \\ \quad s \in (-\infty,\infty), \\ \varphi(0,0) = 1, \end{cases} \tag{4.19}$$

where μ_1 and μ_2 are defined as in (4.14). In addition, we have

$$0 \leqslant \varphi \leqslant 2, \quad 0 \leqslant \psi \leqslant 2^{\beta/\alpha}\delta^{-2\beta}, \quad y > 0,\ s > 0.$$

However, by Theorem 3.3 the nontrivial solution (φ, ψ) of (4.19) must blow up in finite time, a contradiction. Hence $s_\lambda \leqslant C$ for all sufficiently small λ for some $C > 0$.

Let $t_0 = t$ and $t_1 = \lambda^2 s_\lambda + \hat{t}$. Then $M_u(t_1) = 2M_u(t_0)$ and $M_u^{1/\alpha}(t_0)(t_1 - t_0) \leqslant M_u^{1/\alpha}(t_0)(t_1 - \hat{t}) = s_\lambda \leqslant C$. Continuing in this process, we can get a sequence $\{t_j\} \nearrow T$ such that

$$\begin{aligned} &M_u^{1/\alpha}(t_{j-1})(t_j - t_{j-1}) \leqslant C, \quad \forall j \in N, \\ &M_u(t_j) = 2M_u(t_{j-1}), \quad \forall j \in N. \end{aligned}$$

Again, from Lemma 3.1 in [26] and (4.8), the upper bounds for u and v follow. This completes the proof. □

Similarly, we have the following theorem.

Theorem 4.2. *Suppose that* $p_2 \geqslant p_2^*$, $q_1 \geqslant q_1^*$, *and that either* $\max\{\alpha, \beta\} > 1/2$, *or,* $\max\{\alpha,\beta\} = 1/2$ *and* $\min\{p_2, q_1\} \geqslant 1$. *Then there exist positive constants* C_i, $i = 1,2,3,4$, *such that* (4.2) *and* (4.3) *hold, where* (α, β) *is defined by* (4.1).

Proceeding as the proof of Theorem 4.1 and using Theorem 3.5, we can prove the following theorem.

Theorem 4.3. *Suppose that* $p_1 \leqslant p_1^*$, $p_2 \leqslant p_2^*$, *and* $\max\{\alpha, \beta\} \geqslant 1/2$. *Then there exist positive constants* C_i, $i = 1, 2, 3, 4$, *such that* (4.2) *and* (4.3) *hold, where* (α, β) *is defined by* (4.1).

Finally, the following theorem can be deduced by using Theorem 3.6.

Theorem 4.4. *Suppose that* $q_1 \leqslant q_1^*$, $q_2 \leqslant q_2^*$, *and* $\max\{\alpha, \beta\} \geqslant 1/2$. *Then there exist positive constants* C_i, $i = 1, 2, 3, 4$, *such that* (4.2) *and* (4.3) *hold, where* (α, β) *is defined by* (4.1).

Remark. Notice that

(a) $q_1 \leqslant q_1^*$ and $q_2 \geqslant q_2^*$, if $p_1 \geqslant p_1^*$ and $p_2 \leqslant p_2^*$.
(b) $q_1 \geqslant q_1^*$ and $q_2 \leqslant q_2^*$, if $p_1 \leqslant p_1^*$ and $p_2 \geqslant p_2^*$.

Suppose that (u, v) blows up in finite time. Then we can also classify the exponents for the blow-up rates as follows.

$$(\alpha, \beta) = \begin{cases} \left(\frac{p_1+2}{2(p_1q_2-1)}, \frac{2q_2+1}{2(p_1q_2-1)}\right), & \text{if } p_1 \geqslant p_1^* \text{ and } p_2 \leqslant p_2^*; \\ \left(\frac{p_2+2}{2(p_2q_1-1)}, \frac{2q_1+1}{2(p_2q_1-1)}\right), & \text{if } p_1 \leqslant p_1^* \text{ and } p_2 \geqslant p_2^*; \\ \left(\frac{q_1+1}{2(q_1q_2-1)}, \frac{q_2+1}{2(q_1q_2-1)}\right), & \text{if } p_1 \leqslant p_1^* \text{ and } p_2 \leqslant p_2^*; \\ \left(\frac{p_1+2}{2(p_1q_2-1)}, \frac{2q_2+1}{2(p_1q_2-1)}\right), & \\ \quad \text{if } p_1 \geqslant p_1^*,\ p_2 \geqslant p_2^*, \text{ and } q_2 \geqslant q_2^*; & \\ \left(\frac{p_2+2}{2(p_2q_1-1)}, \frac{2q_1+1}{2(p_2q_1-1)}\right), & \\ \quad \text{if } p_1 \geqslant p_1^*,\ p_2 \geqslant p_2^*, \text{ and } q_1 \geqslant q_1^*; & \\ \left(\frac{p_1+1}{p_1p_2-1}, \frac{p_2+1}{p_1p_2-1}\right), & \\ \quad \text{if } p_1 \geqslant p_1^*,\ p_2 \geqslant p_2^*,\ q_1 < q_1^*, \text{ and } q_2 < q_2^*. & \end{cases} \tag{4.20}$$

5. Blow-up set

We shall modify the method of Hu and Yin [30] to study the blow-up set.

Lemma 5.1. *Suppose that* $q_1q_2 > 1$, $p_1 < p_1^*$, *and* $p_2 < p_2^*$. *Let* (u, v) *be the solution of* (1.1)–(1.5) *satisfying*

$$\max_{0 \leqslant x \leqslant 1} u(x, t) \leqslant C(T - t)^{-\alpha} \quad \textit{and}$$

$$\max_{0\leqslant x\leqslant 1} v(x,t) \leqslant C(T-t)^{-\beta}, \quad 0 \leqslant t < T,$$

for some positive constant C, where

$$\alpha = \frac{q_1+1}{2(q_1q_2-1)} \quad \text{and} \quad \beta = \frac{q_2+1}{2(q_1q_2-1)}.$$

Then the blow-up point occurs only at $x=1$.

Proof. Set $\eta(x) = (1-x^2)^2$. Let

$$\varphi(x,t) = \frac{AB^{\alpha}}{[\eta(x)+B(T-t)]^{\alpha}}, \qquad \psi(x,t) = \frac{DB^{\beta}}{[\eta(x)+B(T-t)]^{\beta}},$$

where A, B, D are positive constants satisfying

$$\begin{aligned}
&D \geqslant 2^{\beta}C, \qquad A \geqslant 2^{\alpha}C,\\
&B^{\alpha+1-\beta p_1} \geqslant 2^{\alpha+2-\beta p_1} D^{p_1}\alpha^{-1}A^{-1},\\
&B^{\beta+1-\alpha p_2} \geqslant 2^{\beta+2-\alpha p_2} A^{p_2}\beta^{-1}D^{-1},\\
&B \geqslant 32\alpha+16, \qquad B \geqslant 32\beta+16.
\end{aligned}$$

Note that $\alpha+1-\beta p_1 > 0$ and $\beta+1-\alpha p_2 > 0$ by assumptions. Choose t_0 such that $B(T-t_0)=1$. Then (φ,ψ) satisfies

$$\begin{aligned}
&\varphi_t \geqslant \varphi_{xx} + \psi^{p_1},\ \psi_t \geqslant \psi_{xx} + \varphi^{p_2}, \quad 0<x<1,\ t_0<t<T,\\
&\varphi_x(0,t)=0,\ \psi_x(0,t)=0, \quad t_0<t<T,\\
&\varphi(1,t) = A(T-t)^{-\alpha} \geqslant C(T-t)^{-\alpha} \geqslant u(1,t), \quad t_0<t<T,\\
&\psi(1,t) = D(T-t)^{-\beta} \geqslant C(T-t)^{-\beta} \geqslant v(1,t), \quad t_0<t<T,\\
&\varphi(x,t_0) \geqslant 2^{-\alpha}A(T-t_0)^{-\alpha} \geqslant C(T-t_0)^{-\alpha} \geqslant u(x,t_0), \quad 0\leqslant x\leqslant 1,\\
&\psi(x,t_0) \geqslant 2^{-\alpha}A(T-t_0)^{-\alpha} \geqslant C(T-t_0)^{-\beta} \geqslant v(x,t_0), \quad 0\leqslant x\leqslant 1.
\end{aligned}$$

By the maximum principle, we have $u \leqslant \varphi$ and $v \leqslant \psi$ in $[0,1]\times[t_0,T)$. Since (φ,ψ) does not blow up at any point in $[0,1)$, the lemma follows. □

Combining Theorem 4.3 and Lemma 5.1, we conclude that

Theorem 5.2. *Suppose that $q_1q_2>1$, $p_1<p_1^*$, and $p_2<p_2^*$. Suppose also that* $[\max\{q_1,q_2\}+1]/[2(q_1q_2-1)] \geqslant 1/2$. *Let (u,v) be the solution of* (1.1)–(1.5) *with $u_0' \geqslant 0$ and $v_0' \geqslant 0$. Then the blow-up point occurs only at $x=1$.*

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