Measurement of campaign efficiency using data envelopment analysis

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Abstract

Data Envelopment Analysis (DEA) is proposed as a method for measuring the efficiency of presidential campaigns. Campaign efficiency is defined as the success of incumbent party candidates in converting the economic conditions and presidential popularity of July into actual votes in November. DEA is described and illustrated using a case study (US presidential elections 1948–1996). Differences between outcome and efficiency and reasons for inefficiencies are explored.

Keywords: Elections; Data envelopment analysis (DEA); Efficiency; Presidency

1. Introduction

Intuitively, we know that once every four years, during an August–October presidential election season, some incumbent parties are extremely effective in translating the initial conditions of July—typically, the underlying state of the economy and the president approval rating—into votes for their presidential candidate in November. Others are not. This is not to say that effectiveness during the campaign season necessarily leads to electoral success. Underlying conditions may be so bleak that no matter how effective the incumbent party, the vote share received may be insufficient to retain control of the White House. But the effectiveness question does raise issues that have not been addressed analytically by students of elections. How might we begin to think comparatively about the effectiveness of political campaigns in the light of underlying economic conditions? When have incumbent parties been effective in translating initial conditions into votes and when have they not? How might campaign effectiveness be measured?

We propose that data envelopment analysis (DEA) be used to answer these questions, and we offer a simple case study of its application. DEA is a mathematical programming technique originally developed by operations research workers studying business firms and not-for-profit organizations to identify best-practice efficiency frontiers and to measure shortfalls from the frontiers. In business applications, focus is upon input–output relations: firm A is said to be more efficient than firm B if it produces one more unit of output using identical inputs, or if it produces the same

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level of output using one less unit of one input (Koopmans, 1951). By analogy, we argue that campaign A is more efficient than campaign B if it secures a greater share of the presidential vote given identical July input conditions, or if it secures the same vote share proceeding from less favorable conditions. DEA is most commonly used retrospectively, eliciting the best-practice or efficiency frontier from actual observations. In this sense, it is not unlike conventional regression-based electoral outcome analysis. It also has prospective potential if used to identify the circumstances when July bellwethers accurately predict the November outcome, and when they do not.

2. Differences between DEA and regression analysis

The similarity between DEA and conventional electoral outcome analysis should not be pushed too far. In electoral outcome analysis a relationship is postulated between a dependent variable, typically partisan share of the two-party vote, and a set of independent or causal variables designed to capture both longer-term determinants of the outcome, such as economic conditions and presidential popularity, and short-run campaign-season factors. A multiple regression equation is fitted, producing parameters that best represent the on-average relationships between the dependent variable and each of the independent variables for the selected sample of observations. To fit such a model, the functional form has to be specified, the expected outcomes must be estimated without bias, and the residuals (the differences between actual and expected outcomes, which are relegated to the error variance term) must be independently and normally distributed. According to this logic, if an actual outcome exceeds that predicted by the model, the excess is attributed to random causes.

DEA proceeds differently, making no such assumptions. DEA uses mathematical programming methods rather than statistical inference, focuses on individual observations in contrast to population averages, can simultaneously utilize multiple outputs and multiple inputs with each being stated in different units of measurement, can incorporate categorical (dummy) variables, places no restriction on the functional form of the input–output relationship, focuses on revealed best-practice frontiers rather than on central-tendency properties, and satisfies strict equity criteria in the relative evaluation of each observation (Charnes et al., 1994, 8). A performance measure is calculated for each observation relative to all other observations in the observed population, with the sole requirement that each observation lie on or below the extremal (i.e. most efficient) frontier (Charnes et al., 1994, 5–6). Each observation not on the frontier is scaled relative to the observations on the segment of the frontier closest to it. The frontier is thus the revealed best-practice production frontier— the maximum output empirically obtainable from any observation in the observed population, given its level of inputs—and the scaling provides a measurement of the relative efficiency of each observation not on the frontier.

In regression analysis, observations lie both above and below a surface that is assumed to have a particular functional form, and the deviations are expected to be randomly generated and normally distributed. In DEA, observations lie on or beneath the maximum-efficiency frontier. Location relative to the frontier has a substantive interpretation, relative efficiency in the conversion of inputs into outputs, a major conceptual gain over the notion that residuals from regression are simply part of the error term.

3. A simple case study

Because DEA is based on mathematic programming methods that will be unfamiliar to political scientists more accustomed to using multiple regression analysis, a full exposition of the method is provided in Appendix A. The balance of the paper is devoted to an illustration of its application using a very simple case with only one output and two inputs. Should investigators want to replicate our example, the data we use are listed in columns three to five of Table 1.

The 'output' variable selected for the case study is the percentage of the popular vote received by the incumbent party in the 13 presidential elections held from 1948 to 1996 (Py). The two 'input' variables are the incumbent president's July approval rating Ay, and the state of the economy in July as indicated by the growth rate of employment in the preceding 12 months (1 July of the preceding year to 30 June of the election year), Ey. Controlling for these initial conditions, the question is which campaigns have yielded the greatest outputs.

A word on the choice of these variables is appropriate. The typical electoral outcome analysis uses partisan share of the two-party vote as the dependent variable, thereby eliminating the campaign effects of third parties from consideration, presumably because Rosenstone et al. (1984) concluded that third parties, on average, take votes equally from both major party candidates. We chose to look at the incumbent party vote share as the output variable because we wanted to be able to revisit the question of third party campaign effects, as well as to ask whether one of the major parties has consistently been a more efficient campaigner than the other when campaigning from a position of incumbency.

The July presidential approval rating is as commonly used by electoral outcome analysts, but such investigators tend to use the growth rate of real per capita

Table 1

Reference data

1	2	3	4	5	6	7

1948	Truman	50.0	37.0	2.20	Dewey, Thurmond	5.4
1952	Stevenson	44.4	32.0	0.49	Eisenhower	0.4
1956	Eisenhower	57.4	69.0	2.62	Stevenson	1.5
1960	Nixon	49.5	61.0	1.78	Kennedy	1.9
1964	Johnson	61.1	74.0	2.28	Goldwater	1.4
1968	Humphrey	42.7	40.0	2.08	Nixon, Wallace	15.5
1972	Nixon	60.7	56.0	2.60	McGovern	1.7
1976	Ford	48.0	45.0	2.40	Carter	1.8
1980	Carter	41.0	31.0	0.48	Reagan, Anderson	9.4
1984	Reagan	58.8	52.0	4.14	Mondale	0.7
1988	Bush	53.4	51.0	2.25	Dukakis	1.0
1992	Bush	37.4	32.0	0.20	Clinton, Perot	19.0
1996	Clinton	49.1	56.0	0.87	Dole, Perot	9.6

1. Election year y.
2. Incumbent party's candidate.
3. Percentage of popular vote Py.
4. Incumbent president's July approval rating Ay.
5. Employment growth rate 1 July–30 June Ey.
6. Opposing candidates.
7. Third party vote.

disposable personal income as the indicator of the state of the economy, whereas we use the growth rate of employment. There are several reasons for our choice. The personal income variable requires monthly estimates of the gross domestic product, the inflation rate, and the population of the USA, but preliminary GDP estimates are frequently adjusted in succeeding months, rendering the variable of little value as a pre-campaign bellwether. Employment, on the other hand, is reported by the Department of Labor with less than a month's lag, is closely watched by economic forecasters, by markets, and by the Federal Reserve, is experienced directly by the voting public, and performs just as well as the income variable in conventional electoral outcome models. For example, in electoral outcome analysis mode, a regression model fitted using variables three to five for the first 12 elections reported in Table 1 yields Py = 29.11 + 3.34 Ey + 0.30 Ay. Py is the percentage of the popular vote cast for the incumbent party's presidential candidate in year y, Ey is the July y－1 to June y growth rate of employment, and Ay is the July approval rating of the

Table 2
Statistics

	Standard error	t	P-value
Intercept	3.82	7.60	0.000
Ey coefficient	1.25	2.67	0.025
Ay coefficient	0.09	3.22	0.010

incumbent president in year y. The model has an R^2 of 0.83 (adjusted 0.79) and a standard error of 3.6 with nine degrees of freedom. The related diagnostic statistics are shown in Table 2. An out-of-model forecast of the 1996 election (election 13 in Table 1) yields P_{1996} = 29.11 + 3.34 (0.8657) + 0.30 (56) = 48.8 per cent. The incumbent party candidate, Clinton, received 49.1 per cent of the total popular vote. Clinton's performance was average, given initial conditions.

Employment/unemployment issues are, of course salient in public political psychology and, as a consequence, central to electoral choice models in the post-New Deal/post-war era (Lewis-Beck and Rice, 1992; Kiewiet and Udell, 1998). (Un)employment issues are more visible, and hence more easily covered by the media, so people more readily identify with them, and thus they have relatively unambiguous effects (Conover et al., 1986).

DEA solutions are readily obtainable using one of the several software packages now available. For a review of these packages see Charnes et al., 1994, 89–94.

The key tabulation produced by an output-oriented DEA is shown in Table 3. The 'ICs' are the units of analysis, the incumbent party campaigns. Omicron is an efficiency score that takes on a value of 1 when an IC lies on the efficiency frontier and a value exceeding 1 if the IC lies behind the frontier and could have utilized the available inputs to produce greater outputs. Thus, IC number 3 has an omicron value of 1.063, which reveals that the July inputs were capable of producing 1.063 units of output (an increase of 6.3 per cent). The inefficiency score converts this into the shortfalls of the popular vote share from that which might have been achieved. If the incumbent party's campaign had converted the initial July 1956 conditions into November 1956 votes with maximal efficiency, an additional 3.61 per cent of the popular vote would have been secured by Eisenhower in that election (57.4 3 0.063). The cost of inefficient input usage, compared with the campaigns lying on the bestpractice data envelopment frontier, was - 3.61 per cent.

Table 3
Output-oriented DEA

IC	Election	Omicron	Inefficiency
1	1948	1.000	0.00
2	1952	1.000	0.00
3	1956	1.063	- 3.61
4	1960	1.228	- 11.31
5	1964	1.000	0.00
6	1968	1.212	- 9.06
7	1972	1.000	0.00
8	1976	1.139	- 6.69
9	1980	1.000	0.00
10	1984	1.000	0.00
11	1988	1.090	- 4.81
12	1992	1.187	- 7.00
13	1996	1.236	- 11.60

This result may be visualized by examining Fig. 1, which simplifies by plotting the output variable (percentage of the popular vote received by the incumbent party) on the vertical axis, and one of the input variables (the incumbent president's July approval voting) on the horizontal axis. Each campaign appears as a point in the graph. The DEA analysis places the incumbent party campaigns of 1948, 1952, 1964, 1972, 1980, and 1984 on the efficiency frontier. The corresponding points are linked in Fig. 1 to reveal the frontier's location. Sharply diminishing returns in the form of declining increments of the percentage of the popular vote received to each percentage increase in the approval rating are revealed: there is very little vote gain for approval ratings in excess of 55 per cent. Likewise (not graphed), the share of the popular vote also displays diminishing responsiveness to employment growth rates as they increase beyond 2.2 per cent.

For elections lying behind the frontier, their vertical distance from the efficiency frontier in Fig. 1 indicates the greater output that might have been achieved had the campaigns built on initial July conditions as efficiently as those lying on the frontier. Thus, the 1956 outcome lies 3.61 percentage points beneath the frontier, and that for 1996 11.6 per cent, revealing that, while controlling for initial conditions, DEA provides a way to quantitatively compare the effectiveness or efficiency of different presidential campaigns.

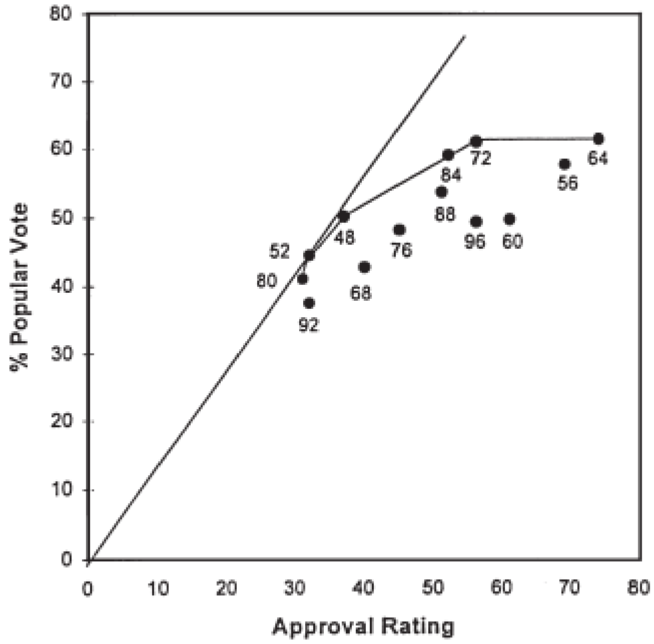

Fig. 1. The efficiency frontier with variable returns to scale. Campaigns lying on the arc from 1980 to 1964 are adjusted by the BCC model of Banker et al. (1984) to be 'efficient' but subject to diminishing returns to scale of inputs, and the campaigns lying behind this frontier are 'inefficient'. The CCR model of Charnes et al. (1978) identifies the 1952 incumbent party campaign as the most efficient of all—i.e. the campaign that secured the greatest popular vote share per unit of July inputs. If all campaigns had been equally efficient, and if there had been constant returns to scale, they would have been located on the diagonal axis extending from the origin to 1952 and beyond. Because there are diminishing returns to level of inputs, the actual maximally efficient data envelopment arc is 80–52–48–84–72–64. The BCC and CCR models are described in Appendix A.

4. Discussion

Even a simplistic case such as that just presented reveals that DEA affords investigators an opportunity to analyse elections in different ways. The nonparametric programming procedure compares each IC with every other to determine which lie on the efficiency frontier. The frontier reveals what the best campaigns achieved and what the less-than-best might have achieved. This means that the frontier identifies those campaigns that were most effective in converting the July baseline into their popular vote shares in November, those that were less effective and by how much. In turn, the investigator is encouraged to think why efficiency occurs for some elections and not for others, why outcome and efficiency are not congruent, and what role the candidates and the campaigns played, since the DEA has controlled for the macroeconomic considerations captured by the July baseline variables.

4.1. Efficiency vs. inefficiency

The incumbent party candidates lying on the efficiency frontier were Truman (1948), Stevenson (1952), Johnson (1964), Nixon (1972), Carter (1980) and Reagan (1984). No campaigns were more successful than those of these candidates in converting the July conditions that confronted them into November results, even though Stevenson and Carter lost. DEA challenges us to ask why. Complementing these six campaigns are seven others than may be ranked by their relative inefficiency or 'shortfalls' from what might have been achieved, according to the July bellwethers (see Table 4). Immediately, other questions pop to mind. The shortfall list includes three incumbent presidents, four vice-presidents, and one appointee. To what extent was the candidate a source of the electoral shortfall? To

Table 4
Efficiency vs. inefficiency

Year	Incumbent party candidate	Popular vote received	Inefficiency	Efficient vote
1996	Clinton	49.1	- 11.6	60.7
1960	Nixon	49.5	- 11.3	60.8
1968	Humphrey	42.7	- 9.1	51.8
1992	Bush	37.4	- 7.0	44.4
1976	Ford	48.0	- 6.7	54.7
1988	Bush	53.4	- 4.8	58.2
1956	Eisenhower	57.4	- 3.6	61.0

what extent did dissatisfaction with both candidates and parties lead to third party challenges, and how did these affect relative efficiency? What about the ability of vice-presidents to capitalize on their president's popularity (Hibbs, 1987)? What about the effect of late-breaking economic disruptions not reported until August, September or October? How about the unexpected event such as a heart attack?

We hazard a few guesses, to illustrate how DEA results provide a different focus for qualitative campaign evaluations. In the list of inefficient campaigns, the poor showings of Nixon in 1960 and Humphrey in 1968 are as expected, but the result for Clinton in 1996 is a surprise. There was, of course, the third party challenge from Perot, but perhaps there is also, in the numbers, a previously unmeasured confirmation of the electorate's doubts about the former Arkansas governor's character. Much has been said about the effectiveness of Clinton's use of the media, yet he and Richard Nixon share the unenviable position of being the incumbent party candidates most suspected of malfeasance, and least able to convert their July baselines into November outcomes. Dare we conclude that their untrustworthiness cost the incumbent party up to 11 per cent of the popular vote?

4.2. Outcome vs. efficiency

Campaign efficiency does not guarantee electoral success. The converse appears to be more to the point. Our DEA case study reveals that only three incumbent party candidates were in no-win situations, according to their July baselines, Stevenson in 1952, Carter in 1980, and Bush in 1992. This suggests that except under the most adverse macroeconomic conditions, presidential elections are for the incumbent party to lose through its choice of candidate and appeal to the electorate during the campaign.

A comparison of electoral outcome and campaign efficiency is given in Table 5.

Table 5
Outcome vs. efficiency

	Campaign was efficient				Campaign was inefficient			
		a	b	c		a	b	c
Incumbent party candidate was elected	Truman (D) 48	37	2.2	5.4	Eisenhower (R) 56	69	2.6	1.5
	Johnson (D) 64	74	2.2	1.4	Bush (R) 88	51	2.2	1.0
	Nixon (R) 72	56	2.6	1.7	Clinton (D) 96	56	0.8	9.6
	Reagan (R) 84	52	4.1	0.7				
Incumbent party candidate was defeated	Stevenson (D) 52	31	0.4	9.4	Nixon (R) 60	61	1.7	1.9
	Carter (D) 80	32	0.4	0.4	Humphrey (D) 68	40	2.0	15.5
					Ford (R) 76	45	2.4	1.8
					Bush (R) 92	32	0.2	19.0

a. President’s July approval rating.
b. Rate of growth of employment.
c. Third parties’ vote share.

Efficient campaigns produced electoral victories for four incumbent presidents (Truman, Johnson, Nixon, and Reagan) but could not prevent losses by one incumbent (Carter) and by Truman’s successor (Stevenson). Four of the efficient campaigns were by incumbent Democrats, and two by Republicans. The Johnson landslide of 1964 was preceded by extremely strong July approval ratings and sound economic performance. The reelections of Nixon and Reagan were built on weaker July approval ratings but stronger economic growth. Truman remains the surprise victory, epitomizing the efficient campaigner: in July the economic indicators were good, but his approval rating was extremely low. Nonetheless Truman received 50 per cent of the vote and defeated Dewey, running what all acknowledge to be a remarkably effective campaign. In the cases of the two efficient campaigns that lost, Stevenson in 1952 and Carter in 1980, the July inputs signaled impossible and near-identical tasks: very low presidential approval votings (32 and 31) and very slow economic growth (0.4). The candidates did as well as might have been expected, but Stevenson was still able to capture only 44.4 per cent of the popular vote, and Carter 41 per cent. In two cases, Truman in 1948 and Carter in 1980, the candidates also faced third party challenges, but the DEA still placed them on the efficiency frontier, suggesting that Thurmond and Anderson cut into the Dewey and Reagan votes rather than those of Truman and Carter (compare with Rosenstone et al., 1984, who conclude that, on average, third parties take from both candidates equally).

Inefficient campaigns also produced winners and losers. Five of the inefficient campaigns were by incumbent Republicans, and two by Democrats. In the loser column, in 1960 and 1968 vice-presidents Nixon and Humphrey were unable to capitalize upon their presidents' July approval ratings and Humphrey was further hurt by conflict over Vietnam and by a split in the Democratic Party. In 1976, appointee Ford was wounded by his pardon of Nixon and by press portrayals of his lack of mental and physical acuity, despite strong economic expansion. In 1992, Republican complacency after the Gulf War was shattered by a sharp economic downturn, and a suddenly ineffective Bush proved incapable of combating the challenge by Ross Perot, but the July indicators reveal that he stood no chance of election in any case. Given the July inputs, had Nixon, Humphrey and Fords' campaigns been efficient, they would have secured 60.7, 51.8 and 54.7 per cent of the popular vote, so for them campaign efficiency did matter; inefficiency cost them 11.3, 9.1 and 6.7 per cent of the popular vote, respectively, and the elections. On the other hand, Bush would have secured only 44.4 per cent had his campaign been maximally efficient: his 1992 campaign inefficiency cost him 7.0 percentage points. As for the inefficient winners, in 1956 there was concern about Eisenhower's health in the aftermath of his heart attack, Nixon was the vice-presidential alternative, and Stevenson was able to activate a loyal following. In 1988, economic growth was good, but George Bush was not Ronald Reagan. Inefficiency cost Eisenhower and Bush 3.6 and 4.8 per cent of the popular vote, respectively. The most interesting case is that of Clinton, whose plurality was far less than his popular vote might have been. That he received a landslide majority in the Electoral College means that the Perot challenge hurt Dole, but Clinton's 11.6 per cent efficiency shortfall means that as many voters looked the other way as they did when Nixon ran in 1960.

5. Conclusions

Only a simplistic DEA case study was presented here, but the potentialities of this new method to raise new questions and to provide new measurements that might help integrate the quantitative and qualitative schools of electoral analysis should be evident. The form that integration takes should be noted: if outcomes are efficient, the controlling variables are the July inputs—the macroeconomy combined with incumbent popularity. If all campaigns were efficient, the July inputs would be perfect bellwethers. On the other hand, if the outcomes are less than efficient, DEA measures by how much, and should turn the investigator's attention to the sources of the inefficiency.

Future DEA applications should utilize the full multiple output-multiple input potentials that the method affords. One can envisage studies that include among the outputs partisan vote shares and Electoral College, house and senate outcomes. The inputs might include multiple barometers of the state of the economy, in addition to the results of opinion surveys and approval ratings. It is possible, of course, to complement every parametric electoral outcome model with an equivalent DEA solution, and there is no reason, therefore, why DEA outputs should not provide insightful materials that might extend and enrich current practice not only in modeling of US elections, but also in comparative politics. Are electoral outcomes in the USA, Canada and Britain equally efficient/inefficient? Are stronger party systems and greater ideological commitment associated with greater or lesser degrees of campaign efficiency? Do variations in length of the campaign season play a role in separating efficiency and outcome?

The nature of any advance in scientific instrumentation is to introduce new observations and to extend the frontiers of inquiry by raising new questions, as well as addressing older questions in new ways. DEA affords that opportunity.

Appendix A

Efficiency measurement using data envelopment analysis

Data envelopment analysis (DEA) is a mathematical programming procedure that can be applied to observed data to provide empirical estimates of extremal relationships such as the efficient production possibility surfaces that are the cornerstones of modern economics. The CCR method was introduced by Charnes et al. (1978), assuming constant returns to scale. An extension to variable returns to scale, the BCC method, was made by Banker et al. (1984). Both cases are described below. In a recent comprehensive bibliography, Seiford (1996) lists over 700 articles that have employed the technique.

In DEA, the focus is upon production units responsible for converting inputs into outputs, called by CCR ‘decision making units’ (DMUs). To extend the analysis to presidential elections, we substitute the notion of an incumbent party campaign (IC) for the DMU. DEA comes in two varieties. Input-oriented DEA focuses on the minimum inputs required to produce specified outputs. Output-oriented DEA focuses on the maximum outputs that are achievable with given inputs. The presidential elections case examined in this paper is clearly of the latter kind. The CCR formulation thus becomes one of considering *n* ICs, each of which uses *m* inputs and generates *s* outputs. Inputs are denoted by x and outputs by *y*:

$$
\begin{array}{cccccccccc}
 & & & IC_1 & IC_2 & IC_3 & \cdots\cdots & IC_n & & \\
 & & & \lambda_1 & \lambda_2 & \lambda_3 & \cdots\cdots & \lambda_n & & \\
v_1 & 1 & \rightarrow & x_{11} & x_{12} & x_{13} & \cdots\cdots & x_{1n} & & \\
v_2 & 2 & \rightarrow & x_{21} & x_{22} & x_{23} & \cdots\cdots & x_{2n} & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \\
v_m & m & \rightarrow & x_{m1} & x_{m2} & x_{m3} & \cdots\cdots & x_{mn} & & \\
 & & & y_{11} & y_{12} & y_{13} & \cdots\cdots & y_{1n} & \rightarrow 1 & u_1 \\
 & & & y_{21} & y_{22} & y_{23} & \cdots\cdots & y_{2n} & \rightarrow 2 & u_2 \\
 & & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & & & y_{s1} & y_{s2} & y_{s3} & \cdots\cdots & y_{sn} & \rightarrow s & u_s
\end{array}
$$

where

v_i is a weight attached to the *i*th input, $v_i \geq 0$, i = 1,2,…,*m*

and

u_r is a weight attached to the *r*th output, $u_r \geq 0$, r = 1,2,…,*s*

Now write

$$x_j = [x_{1j}, x_{2j} \ldots\ldots x_{mj}]^T \quad j = 1,2,\ldots\ldots,n$$

$$y_j = [y_{1j}, y_{2j} \ldots\ldots y_{sj}]^T \quad j = 1,2,\ldots\ldots,n$$

$$v = [v_1, v_2, \ldots\ldots v_m]^T$$

$$u = [u_1, u_2, \ldots\ldots u_s]^T$$

The efficiency of any ICj is obtained by minimizing the ratio of weighted input to weighted output:

$$\underset{(u,v)}{\text{minimize}} \quad q_j = v^T x_j / u^T y_j \quad j = 1,2,\ldots\ldots,n$$

The vectors u and v must be chosen such that for all ICs

$$q_j \geqq 1 \quad j = 1,2,\ldots\ldots,n$$

Writing the vectors of efficiency scores, inputs and outputs as q_o, x_o and y_o, the relative efficiency of the ICs may thus be obtained as follows:

$$\underset{(u,v)}{\text{minimize}} \quad q_o = v^T x_o / u^T y_o$$

$$\text{subject to} \quad v^T x_j / u^T y_j \geqq 1 \quad j = 1,2,\dots,n$$

$$u \geq 0,\ v \geq 0$$

The notation '$\geqq$' means that $x,y \in \Re^N$, $x \geqq y$ if and only if $x_n \geqq y_n$, n = 1,2,…,N; '$\geq$' means that $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

This is an extended nonlinear programming formulation of an ordinary fractional programming problem. Charnes and Cooper (1962) showed that it may be replaced with linear programming equivalents, viz.:

Let

$$t = 1/u^T y_o,\ \omega = tv,\ \mu = tu$$

Hence

$$t > 0,\ \omega \geq 0,\ \mu \geq 0$$

The objective function in the new linear programming model is

$$q_o = v^T x_o / u^T y_o = t v^T x_o = \omega^T x_o$$

The restrictions

$$v^T x_j / u^T y_j \geqq 1 \quad j = 1,2,\dots\dots,n$$

are equivalent to

$$v^T x_j \geqq u^T y_j \quad j = 1,2,\dots\dots,n$$

which may be rewritten

$$t v^T x_j - t u^T y_j \geqq 0, \quad j = 1,2,\dots\dots,n$$

Thus

$$\omega^T x_j - \mu^T y_j \geqq 0, \quad j = 1,2,\dots\dots,n$$

Since t = 1/uTyo, we have

$$t u^T y_o = 1$$

$$\mu^T y_o = 1$$

The new equivalent linear programming model is thus:

$$\underset{(\mu,\omega)}{\text{minimize}} \quad q_o = \omega^T x_o$$

$$\text{subject to} \quad \omega^T x_j - \mu^T y_j \geqq 0 \quad j = 1,2,\ldots\ldots,n$$

$$\mu^T y_o = 1$$

$$\mu \geqq 0 \text{ and } \omega \geqq 0$$

$\because \mu^T y_o = 1$, $\because \mu \geqq 0$, so then $\mu \geqq 0$ and $\omega \geqq 0$ which are equivalent to $\mu \geq 0$ and $\omega \geq 0$, respectively.

The dual formulation is

$$\underset{(\lambda)}{\text{maximize}} \quad \phi$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_{rj}\lambda_j \geq \phi y_{ro} \quad r = 1,2,\ldots\ldots,s$$

$$\sum_{j=1}^{n} x_{ij}\lambda_j \leq x_{io} \quad i = 1,2,\ldots\ldots,m$$

$$\lambda_j \geqq 0 \quad j = 1,2,\ldots\ldots,n$$

$$\phi \text{ is unrestricted.}$$

λ is a column vector used to construct a convex hull connecting all the data points. ϕ measures the relative efficiency of an IC, taking on a value of 1 when the IC is maximally efficient. Via the duality theorem of linear programming, min q_o = max ϕ. In computation, this dual program is more tractable than the primal. In the primal program, the constraints are indexed on all ICs. By contrast, in the dual the constraints are indexed on inputs and outputs and sum over ICs. Because the number of inputs and outputs is never likely to exceed the number of ICs, the dual program, with only (m + s) constraints on inputs and outputs, is computed in preference to its (equivalent) primal with n constraints.

The production possibility set (T) in the CCR model satisfies the following postulates:

1. Concavity: If $(x_1,y_1) \in T$, and $(x_2,y_2) \in T$, and $\lambda \in [0,1]$, then $\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2) \in T$.
2. Conity: If $(x_1,y_1) \in T$ and $\alpha \geq 0$, then $\alpha(x_1,y_1) \in T$
3. Inefficiency: If $(x_1,y_1) \in T$ and $x_2 \geqq x_1$, $y_2 \leqq y_1$, then $(x_2,y_2) \in T$
4. Minimum extrapolation: T is the intersection of all sets satisfying postulates 1, 2, and 3.

Since (xj,yj), $j = 1,2,\ldots,n$, are the observed points, then $(xj,yj) \in T$, $j = 1,2,\ldots\ldots,n$, and on the basis of postulate 1 and 2, we know that

$$\left[\sum_{j=1}^{n} (x_j,y_j)\lambda_j \mid \lambda_j \geqq 0,\ j = 1,2,\ldots,n\right] \subset T$$

From postulate 3, we also know that

$$[(x,y)| \sum_{j=1}^{n} x_j\lambda_j \leqq x, \sum_{j=1}^{n} y_j\lambda_j \geqq y, \lambda_j \geqq 0,\ j = 1,2,\ldots\ldots n] \subset T$$

Since the set

$$[(x,y)| \sum_{j=1}^{n} x_j\lambda_j \leqq x, \sum_{j=1}^{n} y_j\lambda_j \geqq y, \lambda_j \geqq 0,\ j = 1,2,\ldots\ldots n]$$

has been proved to satisfy postulates 1, 2, and 3, and from minimum extrapolation (postulate 4), we know

$$[(x,y)| \sum_{j=1}^{n} x_{j,}\lambda_j \leqq x, \sum_{j=1}^{n} y_j\lambda_j \geqq y, \lambda_j \geqq 0,\ j = 1,2,\ldots\ldots n] \supset T$$

we can get the production possibility set

$$T = [(x,y)| \sum_{j=1}^{n} x_j\lambda_j \leqq x, \sum_{j=1}^{n} y_j\lambda_j \geqq y, \lambda_j \geqq 0,\ j = 1,2,\ldots\ldots n]$$

from which an 'efficient' subset can be obtained.

The CCR model assumes constant returns to scale of inputs. To permit variable returns to scale, Banker et al. (1984) reformulated the BCC model by relaxing the conity postulate, viz.:

$$\underset{(\lambda)}{\text{maximize}} \quad \phi$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_{rj}\lambda_j \geqq \phi y_{ro} \quad r = 1,2,\ldots\ldots,s$$

$$\sum_{j=1}^{n} x_{ij}\lambda_j \leqq x_{io} \quad i = 1,2,\ldots\ldots,m$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geqq 0 \quad j = 1,2,\ldots\ldots,n$$

$$\phi \text{ is unrestricted.}$$

It is inclusion of the constraint $\sum_{j=1}^{n} \lambda_j = 1$ that relaxes the conity postulate and permits variable returns to scale. The corresponding production possibility set in the BCC model is changed as follows:

$$[(x,y)| \sum_{j=1}^{n} x_{j,}\lambda_j \leqq x, \sum_{j=1}^{n} y_j\lambda_j \geqq y, \sum_{j=1}^{n} \lambda_j = 1]$$

For ease in computation, a non-Archimedean (infinitesimal) constraint, e, is introduced as an artifact to ensure that all of the observed inputs and outputs will have 'some' positive value assigned to them. This value, which need not be prescribed explicitly, serves as a lower limit for the values that can be assigned to the variables μ_r and ω_i. The model thus becomes in its primal form

$$\underset{(\mu,\omega)}{\text{minimize}} \quad q_o = \omega^T x_o$$

$$\text{subject to} \quad \omega^T x_j - \mu^T y_j \geqq 0 \quad j = 1,2,\ldots\ldots,n,$$

$$\mu^T y_o = 1$$

$$\mu_r \geqq \epsilon \text{ and } \omega_i \geqq \epsilon$$

The dual form is

$$\underset{(\lambda,s^-,s^+)}{\text{maximize}} \quad \phi + \epsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} S_r^+\right) = \text{omicron}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \quad i = 1,2,\ldots\ldots,m$$

$$\phi y_{ro} - \sum_{j=1}^{n} y_{rj}\lambda_j + s_r^+ = 0 \quad r = 1,2,\ldots\ldots,s$$

$$\lambda_j \geqq 0 \; j = 1,2,\ldots\ldots,n$$

$$s_i^- \geqq 0, \quad i = 1,2,\ldots\ldots,m$$

$$s_r^+ \geqq 0, \quad r = 1,2,\ldots\ldots,s$$

$$\phi \text{ is unrestricted.}$$

where s_i^- and s_r^+ are slack variables which can be defined as

$$s_i^- = x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j, \quad s_r^+ = \sum_{j=1}^{n} y_{rj}\lambda_j - \phi y_{ro}$$

The value of ϵ is defined to be so small that ϵ will not affect the maximizing value of ϕ. From the duality theory of linear programming, we have

$$\text{minimum } q_o = \text{maximum } \phi + \epsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$

The y_{ro} and x_{io} values represent observed output and input values for ICo, the IC being evaluated. That is, the y_{ro} and x_{io}, like the y_{ij} and x_{ij}, are all known constants. The values that can be assigned to the slack variables are constrained to be nonnegative. Hence, $x_{io} \leqq \sum_{j=1}^{n} x_{ij}\lambda_j$, and efficiency comparisons will be effected only from solutions with input values at least as great as the inputs utilized by ICo in every case. Similarly, S_r^+ means the solutions will satisfy $\phi y_{ro} \geqq \sum_{j=1}^{n} y_{rj}\lambda_j$ for each of r = 1,…,s outputs achieved by ICo.

In the above, an efficient ICo is indicated by a value of ϕ* equal to unity and s^- (input slacks) and s^+ (output slacks) equal to zero.

This BCC model also can be restated in primal form:

$$\begin{aligned}
\underset{(\mu,\omega)}{\text{minimize}} \quad & q_o + \omega_o^T = \text{I} \\
\text{subject to} \quad & \omega^T x_j - \mu^T y_j + \omega_o \geqq 0 \quad j = 1,2,\dots\dots,n, \\
& \mu^T y_o = 1 \\
& \mu_r \geqq \epsilon \text{ and } \omega_i \geqq \epsilon \\
& \omega_o^T \text{ is unrestricted.}
\end{aligned}$$

The dual form is

$$\begin{aligned}
\underset{(\lambda,s^-,s^+)}{\text{maximize}} \quad & \phi + \epsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} S_r^+\right) = \text{omicron} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \quad i = 1,2,\dots\dots,m \\
& \phi y_{ro} - \sum_{j=1}^{n} y_{rj}\lambda_j + s_r^+ = 0 \quad r = 1,2,\dots\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geqq 0 \; j = 1,2,\dots\dots,n \\
& s_i^- \geqq 0, \quad i = 1,2,\dots\dots,m \\
& s_r^+ \geqq 0, \quad r = 1,2,\dots\dots,s \\
& \phi \text{ is unrestricted.}
\end{aligned}$$

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