2. Detection of trends and change periods

2.1 Introduction to fuzzy logic

Due to the complicated human mind and discrepant cognition in language, most problems concerning people often reveal its vague, obscure and uncertain property. However, most of our traditional tools for formal modeling, reasoning, and computing are, in the converse way, crisp, deterministic and precise in character. By crisp we mean dichotomous, that is, yes-or-no-type rather than more-or-less-type. In conventional dual logic, for instance, a statement can be true or false, and nothing in between. In set theory, an element can either belong to a set or not. Generally, precision also implies certainty, that is, no ambiguities.

In many cases, we receive information that shows certain sense of vague concepts. Consider these statements, for example, " It's very hot outside.", "Raise it higher a little bit.", "She is young". These statements are not, in comprehension, precise at all. Thirty degree Celsius high may be very hot for someone but not too hot for another. The age of thirty may means young to some people but not at all to some others. Under these situations, it is inappropriate to apply traditional mathematical tools to analyze since the conditions to the mathematical tools fail to hold.

The usefulness of mathematical language for modeling purpose is undisputed. However, there are limits to the usefulness and the possibility of using classical mathematical language, based on the dichotomous character of set theory, to model particular systems and phenomena in social sciences. Real situations are very often uncertain or vague in a number of ways.

Due to lack of information, the future state of the system might not be known completely. This type of uncertainty has long been handled appropriately by probability theory and statistics. This Kolmogoroff-type probability is essentially frequentistic and bases on set-theoretic considerations. Koopman's probability refers to the truth of statements and therefore bases on logic. On both types of probabilistic approaches it is assumed, however, that the events or statements, respectively, are well defined. We shall call this type of uncertainty or vagueness stochastic uncertainty by contrast to the vagueness concerning statements themselves, which we shall call fuzziness.

For this, Zadeh [1965] proposed fuzzy set theory, a new tool to generalize the classical notation of a set and accommodate semantic and conceptual fuzziness in statements. Fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. There is nowhere fuzzy in the analysis of fuzzy set theory but the name "fuzzy set theory" itself. It can also be considered as a modeling language well suited for situations in which fuzzy relations, criteria, and phenomena exits.

Fuzzy set theory can be found be applied in many areas, such as in engineering, in medicine, in meteorology, in manufacturing, in bioinformatics, and others. It is particular frequent in all areas in which human judgment, evaluation, and decisions are important. These are the areas of decision making, reasoning, learning, and so on. Before studying fuzzy set theory, we need to know some definitions.

Definition 2.1

If X is a collection of objects denoted generically by x (the universe of discourse) then a fuzzy set A in X is a set of ordered pairs: $A=\{(x, m_A(x)) | x \in X\}$, $m_A(x)$ *is called the membership function or grade of membership (also degree of truth) of x in A which maps X to the membership space M. (When M contains only the two points 0 and 1, A is nonfuzzy and* $m_A(x)$ *is identical to the characteristic function of a nonfuzzy set.) The range of the membership function is [0, 1]. Elements with a zero degree of membership are normally not listed.*

Example 2.1

Let the fuzzy set *A* denoted the numbers of hours you sleep a day. Suppose that *X* is the universe of discourse with integers. That is, $X=\{0, 1, \cdots, 24\}$. Then the fuzzy set *A* may be described as: $A=\{(6, 0.1), (7, 0.2), (8, 0.4), (9, 0.2), (10, 0.1)\}$.

Example 2.2 [Zimmermann 1991]

A = " real numbers close to 10"

$$A=\left\{(x, m_A(x)) \middle| m_A(x)=\left(1+(x-10)^2\right)^{-1}\right\}.$$

There is another way of denoting fuzzy sets. When the universe of discourse *X* is discrete, we write

$$A=\frac{m_A(x_1)}{x_1}+\frac{m_A(x_2)}{x_2}+\cdots \quad =\sum_{i=1}\frac{m_A(x_i)}{x_i}$$

When the universe of discourse *X* is continuous, we write

$$A=\int_X \frac{m_A(x)}{x}$$

Example 2.3

In example 2.1 we can write:

$$A = \frac{0.1}{6} + \frac{0.2}{7} + \frac{0.4}{8} + \frac{0.2}{9} + \frac{0.1}{10}$$

Example 2.4 [Zimmermann 1991]

A = “ real numbers close to 10”

$$A = \int_R \frac{\left(1 + (x-10)^2\right)^{-1}}{x}$$

In the previous examples we found that the supreme of a membership function of a fuzzy set may not equal 1. This type of the fuzzy sets whose supreme of membership function equals 1 is of more interests. We call this type of fuzzy set normal. Each nonempty fuzzy set can always be normalize by dividing $m_A(x)$ by $\sup_x m_A(x)$.

One special type of variables whose values are not numbers but words or sentence in a natural or artificial language also plays an important role in fuzzy set theory. This type of variable is defined to be linguistic variables. We now present its definition.

Definition 2.2 [Zadeh 1973]

A linguistic variable is characterized by a quintuple $\left(x, T(x), U, G, \tilde{M}\right)$ *in which x is the name of the variable; T(x) (or simply T) denotes the term set of x, that is, the set of names of linguistic values of x, with each value being a fuzzy variable denoted generically by x and* ranging *over a universe of discourse U which is associated with the base variable u; G is a syntactic rule (which usually has the form of a grammar) for generating the name, X, of values x; and M is a semantic rule for associating with each X its meaning,* $\tilde{M}(x)$ *which is a fuzzy subset of U. A particular X, that is a name generated by G, is called a term. It should be noted that the base variable u can also be vector-valued.*

In order to facilitate the symbolism in what follows, some symbols will have two meanings wherever clarity allows this: *x* will denote the name of the variable (“the lable”) and the generic name of its values. The same will be true for *X*, and $\tilde{M}(x)$.

Example 2.5 [Zadeh 1973]

Let X be a linguistic variable with the label "Age" (i.e., the label of this variable is "Age" and the values of it will also be called "Age") with U = [0, 100]. Terms of this linguistic variable, which are again fuzzy sets, could be called "old", "young", "very old", and so on. The base-variable u is the age in years of life. $\tilde{M}(x)$ is the rule that assigns a meaning, that is, a fuzzy set, to the terms.

$$\tilde{M}(old) = \{(u, m_{old}(u)) \mid u \in [0, 100]\}$$

where

$$m_{old}(u) = \begin{cases} \left(1 + (\frac{u-50}{5})^{-2}\right)^{-1} & u \in [50, 100] \\ 0 & u \in [0, 50] \end{cases}$$

T(x) will define the term set of *x*, for instance, in this case,

$T(\text{Age}) = \{old, very\ old, not\ so\ old, more\ or\ less\ young, quite\ young, very\ young\}$

where *G(x)* is a rule which generates the (label of) terms in the term set.

Forty years after the proposition, fuzzy set theory has been successfully applied in more and more areas and has caught more and more attention. Theoretical advances have been made in many directions. Therefore, the fundamentals of the theory have also been well developed. Interested readers may refer to Klir and Folger [1988] or Zimmermann [1991] for more basic theory and details.

2.2 Concept of fuzzy time series

A time series is a set of observed values recorded with time. These observed values could be either continuous, which is called continuous time series, or discrete, which is called discrete time series. A time series is usually denoted by $\{X_t\}$, $X_{t_1}, X_{t_2}, \cdots\cdots, X_{t_n}$ refer to the observed values at time $t_1, t_2, \cdots\cdots, t_n$ respectively.

Time series analysis plays a very important role in forecasting and is very successful in many applications. Each observed value seems to be a single and precise one in traditional time series analysis. However, the measurement error of collecting data, the time lag or the interaction between variables may turn the single value into a range of possible values. For example, when we talk about the stock index of a day, which value do we indeed specify, the index at beginning of the day, the index at the end of the day, the highest index of the day or the lowest of the day?

Conventional study on the time series analysis is based on the concept that the observed data are random with certain measurement errors or noise. However, in the empirical study we often encounter the situation that those data reveals not only the property of randomization but also the perception of fuzziness. In this case, the application of fuzzy time series leads to a better result. Thus, we firstable define the fuzzy time series.

Definition 2.3 A fuzzy time series

Let $\{X_t \in R,\ t = 1, 2, 3, \cdots\}$ *be a time series and U be the universe of discourse. Let* $\left\{P_i,\ i=1, 2, 3, \cdots, m.\ \ \bigcup_{i=1}^{m} P_i = U\right\}$ *be an ordered partition of U on which linguistic variables* $\{L_i,\ i=1, 2, 3, \cdots, m\}$ *are given. For each* X_t, *if* $F(X_t)$ *is its correspondent fuzzy set on U consists of membership* $\{u_{t1}, u_{t2}, \cdots, u_{tm}\}$ *for* $\{L_1, L_2, \cdots, L_m\}$. *Then we say* $\{F(X_t)\}$ *is a fuzzy time series corresponding to* $\{X_t\}$, *and denoted by*

$$F(X_t) = \frac{m_{t1}(X_t)}{L_1} + \frac{m_{t2}(X_t)}{L_2} + \cdots + \frac{m_{tm}(X_t)}{L_m} \tag{2.1}$$

where "+" denotes the connection, " $\frac{m_{ti}(X_t)}{L_i}$ *" specifies the corresponding relation of the membership* $m_{ti}(X_t)$ *of* X_t *with respect to the linguistic variable* L_i,

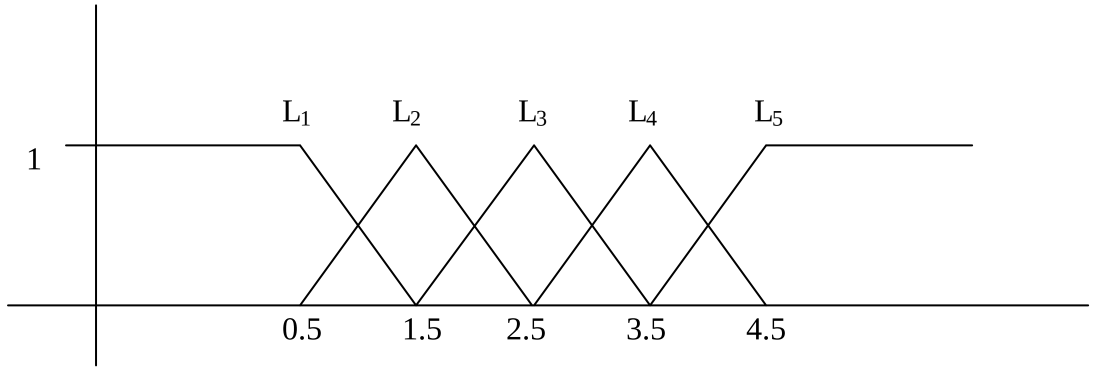
$m_{ti} : R \to [0,1]$, *and* $\sum_{i=1}^{m} m_{ti} = 1$

For simplicity, we write $F(X_t) = (m_{t1}, m_{t2}, \cdots, m_{tm})$ instead of (2.1), and take the triangle-shaped and trapedzoid-shaped membership function as our mainly interested membership function in this paper. The set $\{L_i,\ i=1, 2, \cdots, m\}$ is regarded as a sequence of linguistic variable, and the element of fuzzy time series $\{F(X_t);\ t=1, 2, \cdots, n\}$ is consisted by the memberships of linguistic variables. That is, for any $F(X_t)(t=1, 2, \cdots, n)$, it contains the membership of linguistic hedge corresponding to each L_i.

Example 2.6

Let $\{X_t(t)\} = \{0.8,\ 1.7,\ 2.9,\ 4.1,\ 3.5,\ 3.2,\ 4.3,\ 3.6\}$. Choose U = $\{[0,1], [1,2], [2,3], [3,4], [4,5]\}$, and define the linguistic variable to be very low $= L_1$, low = L_2, middle $= L_3$, high $= L_4$, very high $= L_5$. Moreover, we take the average number of the intervals as our typical values. The typical values corresponding to

$L_1, \ldots, L_5$ are now defined to be 0.5, 1.5, 2.5, 3.5, 4.5 respectively. Figure 2.1 shows the membership function of those linguistic variables.



Thus, we have the fuzzy time series $\{F(t)\}$ corresponding to $\{X_t\}$ as follows:

	Very low	low	middle	high	very high
$F(X_1) = ($	0.7	0.3	0	0	0)
$F(X_2) = ($	0	0.8	0.2	0	0)
$F(X_3) = ($	0	0	0.6	0.4	0)
$F(X_4) = ($	0	0	0	0.4	0.6)
$F(X_5) = ($	0	0	0	1	0)
$F(X_6) = ($	0	0	0.2	0.8	0)
$F(X_7) = ($	0	0	0	0.2	0.8)
$F(X_8) = ($	0	0	0	0.9	0.1)

2.3 Detection of change period

Because the structural change of a time series from one pattern to another may not switch at once but rather experience a period of adjustment time, it is natural for us to apply the concept of change period instead of change points when analyzing a structural change process. Taking different view from the change points, the concept of change period provides us with a more reasonable, more comprehensible and more flexible way to analyze the real world problems.

In the paper, we present an approach to find the change periods in a time series with fuzzy statistics. The following definitions are required.

Definition 2.4 Fuzzy trend indicator series

Let $F(X_t)=\frac{m_{t1}(X_t)}{L_1}+\frac{m_{t2}(X_t)}{L_2}+\cdots+\frac{m_{tm}(X_t)}{L_m}$ $t=1,2,3,\cdots$ *be a fuzzy time series, then the series* $FI(t)=a_1m_{t1}(X_t)+a_2m_{t2}(X_t)+\cdots+a_m m_{tm}(X_t)$, $a_i\in R$ $\forall i=1,2,\cdots,m$ *is defined to be the fuzzy trend indicator series, and* $a_1,a_2,\cdots,a_m$ *are called the fuzzy weights of this fuzzy time series.*

Example 2.7

In example 2.1, we have transformed the time series $\{X_t\}$ into a fuzzy time series $\{F(X_t)\}$. Now let the fuzzy weights of $\{F(X_t)\}$ $a_1=-2$, $a_2=-1$, $a_3=0$, $a_4=1$, $a_5=2$, then we have

$$FI(1)=(-2)\times 0.7+(-1)\times 0.3+0\times 0+1\times 0+2\times 0=-1.7$$
$$FI(2)=(-2)\times 0+(-1)\times 0.8+0\times 0.2+1\times 0+2\times 0=-0.8$$
$$FI(3)=(-2)\times 0+(-1)\times 0+0\times 0.6+1\times 0.4+2\times 0=0.4$$
$$FI(4)=(-2)\times 0+(-1)\times 0+0\times 0+1\times 0.4+2\times 0.6=1.6$$
$$FI(5)=(-2)\times 0+(-1)\times 0+0\times 0+1\times 1+2\times 0=1$$
$$FI(6)=(-2)\times 0+(-1)\times 0+0\times 0.2+1\times 0.8+2\times 0=0.8$$
$$FI(7)=(-2)\times 0+(-1)\times 0+0\times 0+1\times 0.2+2\times 0.8=1.8$$
$$FI(8)=(-2)\times 0+(-1)\times 0+0\times 0+1\times 0.9+2\times 0.1=1.1$$

Definition 2.5 Change period detection sequence

If $\{FI(t)\}_{t=1}^{\infty}$ *is a fuzzy trend indicator series, let* $d(t)=\sum_{t'=t-n+1}^{t}FI(t')$ *, then* $\{d(t)\}_{t=n}^{\infty}$ *forms a sequence and is called a change period detection sequence of degree n.*

Example 2.8

Let $\{FI(t)\quad t=1,2,\cdots,8\}$ be the same as that in example 2.2. The change period detection sequence of degree 3 is constructed as follows:

$$d(3)=(-1.7)+(-0.8)+0.4=-2.1$$
$$d(4)=(-0.8)+0.4+1.6=1.2$$
$$d(5)=0.4+1.6+1=2.1$$
$$d(6)=1.6+1+0.8=3.4$$
$$d(7)=1+0.8+1.8=3.6$$
$$d(8)=0.8+1.8+1.1=3.7$$

Note that the change period detection sequence of degree n is the sequence of the partial cumulative sum of consecutive n elements in fuzzy trend indicator series. The decision of the degree of a change period detection sequence and the fuzzy weight of a fuzzy time series determines what kind of change period you can find. With the definitions defined above, we can now introduce our method to find the change periods in a time series.

The first step to do when we have a time series is to make the first difference, that is, suppose that $\{X_t\}$ is the time series then we find its first difference time series $\{Y_t\}$ where $Y_t = X_t - X_{t-1}$. Intuitively speaking, Y_t is the amplitude of rising or falling at time t comparison to time *t-1*. That is to say, $Y_t > 0$ means that X_{t-1} rises to X_t at time t with the rising amplitude Y_t. Similarly, $Y_t < 0$ means that X_{t-1} falls to X_t at time t with the falling amplitude Y_t; $Y_t = 0$ means that there's no difference between X_{t-1} and X_t.

After forming the time series $\{Y_t\}$, the first difference time series of $\{X_t\}$, the second step is to transform $\{Y_t\}$ into the corresponding fuzzy time series $F(Y_t)$ with L_i as its linguistic level, $i = 1, 2, \cdots, m$. If a time series $\{X_t\}$ illustrates the senses of fuzziness, then so is its first difference $\{Y_t\}$. Therefore, that is the reason why we transform the first difference time series $\{Y_t\}$ into the corresponding fuzzy time series $F(Y_t)$.

The third step is to construct the fuzzy trend indicator series $FI(t) = a_1 m_{t1}(Y_t) + a_2 m_{t2}(Y_t) + \cdots + a_m m_{tm}(Y_t)$ of the fuzzy time series $F(Y_t)$. The rule to decide the fuzzy weights of the fuzzy time series is that if a_i is the fuzzy weight of some linguistic variable L_i where the negative Y_t takes the membership function $m_{ti}(Y_t)$ greater than zero, then we assign a_i a nonpositve value; if a_i is the fuzzy weight of some linguistic variable L_i where the positive Y_t takes the membership function $m_{ti}(Y_t)$ greater than zero, then we assign a_i a nonnegative value.

The next step is to compute the change period detection sequence from the fuzzy trend indicator series we have constructed. We are going to detect trends and change periods in a time series through change period detection sequence. The degree n of the sequence plays a very important role in deciding what types of change periods you could find.

Sometimes an obvious change in trend or a change period in a small observation range may seem just a small noise caused by randomness and fuzziness in a larger observation range. The observation range does dominate our cognition of what a change period will be. More than that, a noise recognized by a person in a certain observation range under some certain situation may be recognized as a change period by the same person in the same observation range but just under a different situation.

This shows that the standard of a change periods depends on factors like the range of observation, different situations, and so on. Therefore, an essential question arise: can we always find the appropriate type of change periods that fits our demand?

In this paper, we propose a method to detect the suitable type of change period by controlling the degree n of a change period detection sequence and the fuzzy weights. Intuitively, the greater the n is, the greater change period you are able to find. In the following definition, we define what we called a change period.

Definition 2.6 Change period

Suppose that $\{d(t)\}_{t=n}^{\infty}$ *is a change period detection sequence with degree n, for a given h > 0, if there is a time interval* $T_1 = \{t_l, t_{l+1}, \cdots, t_{l+m}\}$ *such that* $-h < d(t) < h$ $\forall\, t \in T_1$ *, and if there exist two time intervals of its direct consecutive predecessors and successors* $T_2 = \{t_{l-j}, t_{l-(j-1),} \cdots, t_{l-1}\}$ *and* $T_3 = \{t_{l+m+1,}\ t_{l+m+2}, \cdots, t_{l+m+k}\}$ *such that the signs of d(t) are all the same* $\forall\, t \in T_2$ (T_3)*, but opposite to the signs of d(t)* $\forall\, t \in T_3$ (T_2)*, then we call* $T = \left\{t_{[l-\frac{n+1}{2}]}, t_{[l-\frac{n+1}{2}]+1}, t_{[l-\frac{n+1}{2}]+2,} \cdots, t_{[l-\frac{n+1}{2}]+m}\right\}$ *a change period, where* $[x]$ *denotes the greatest integer function.*

Example 2.9

Suppose that d(t) is a change period detection sequence of degree 10 with details in the following table.

t	d(t)	t	d(t)	t	d(t)	t	d(t)	t	d(t)
1	6.768	6	8.822	11	2.084667	16	-4.23067	21	-6.01733
2	8.778667	7	5.718667	12	-0.46733	17	-4.272	22	-6.944
3	7.807333	8	6.76	13	0.902667	18	-9.52533	23	-9.65133
4	5.698667	9	6.998	14	0.557333	19	-8.63267	24	-10.074
5	9.156667	10	3.923333	15	-4.16667	20	-9.614	25	-7.81533

Let $h = 4$, we can see that at time 10, 11, 12, 13, 14, $-h <$ d(t) $< h$. We can find a time interval of the direct consecutive predecessors $T_2 = \{1, 2, \ldots, 9\}$ such that d(t) is positive $\forall\, t \in T_2$, and a time interval of the direct consecutive successors $T_3 = \{10, 11, \ldots, 25\}$ such that d(t) is negative $\forall\, t \in T_3$. Thus, the change period T is $\left\{[10 - \frac{10+1}{2}] = 4, 5, 6, 7, 8\right\}$.

In the end of this section, we sum up with the integrated process of change period detection for a time series:

Step 1. Obtaining the difference time series $\{Y_t\}$ *by make the first difference* $Y_t = X_t - X_{t-1}$ *of a time series* $\{X_t\}$.

Step 2. Transform the difference time series $\{Y_t\}$ *into a fuzzy time series* $\{F(Y_t)\}$ *with* L_i *as its linguistic level,* $i = 1, 2, \cdots, m$.

Step 3. Decide the weights of this fuzzy time series $\{F(Y_t)\}$, *and find the corresponding fuzzy trend indicator series FI(t).*

Step 4. Construct the change period detection sequence of degree n $\{d(t)\}$ *from the fuzzy trend indicator series FI(t) obtained in step 3.*

Step 5. Observe the change period detection sequence $\{d(t)\}$ *and check if we can find some time intervals T which satisfy the condition in definition 2.6. If such time intervals exist, then they are the change periods.*

2.4 Trends Detection

Trends detection is of much importance in many applications of both practical and theoretical areas. With the capability of knowing the beginning and the end of trends, we can make correct decisions and take appropriate actions. In the paper, we propose an approach to detect trends with change period detection sequence. Before introducing this method, we define a trend in a time series.

Definition 2.7 Trend

Suppose that $\{d(t)\}_{t=n}^{\infty}$ *is a change period detection sequence with degree n, if we can find an time interval* $T = \{t_m, t_{m+1}, \cdots, t_{m+n}\}$ *at which d(t) takes positive sign or zero for every* $t \in T$ *and some* $m,n \in N$, *then* $\{X(t_m), X(t_{m+1}), \cdots, X(t_{m+n})\}$ *(or X(T) for abbreviation) is called an upward trend; if we can find an time interval* $T = \{t_m, t_{m+1}, \cdots, t_{m+n}\}$ *at which d(t) takes negative sign or zero for every* $t \in T$ *and some* $m,n \in N$, *then X(T) is called a downward trend. Both the upward and downward trends are called trends.*

Property 2.1

A trend must occur between two change periods.

Proof. Suppose that $\{d(t)\}_{t=n}^{\infty}$ is a change period detection sequence with degree n and T_1 is a change period, then, by definition 2.6, there is a time interval T_2 of the successors of T_1 at which d(t) takes the same sign for every $t \in T_2$. Without lose of generality, we assume that d(t) take positive sign on T_2 and thus T_2 is an upward trend. The sighs of the successors of T_2 have only three possibilities: positive, negative or zero. If d(t) takes also positive sign or zero at the successors of T_2, then we can enlarge T_2 to include those successors of T_2 for a larger upward trend. If d(t) takes negative sign at the successors of T_2, the there must exist a h and a time interval T_3 such that $-h < d(t) < h$ on T_3. Clearly, T_3 is right behind T_2 and by definition 2.6 is a change period. Therefore an upward trend must occur between two change periods. A downward trend is similar. □

Property 2.1 is an important result. It tells us that when you detect a change period in a time series, it is the end of a trend.