Chapter 1

Introduction

A tree is a connected acyclic graph. A graceful labelling of an undirected graph G with n edges is a one-to-one function from the set of vertices of G to the set $\{0, 1, 2, ..., n\}$ such that the induced edge labels are all distinct, where the edge label is the absolute value of the difference between the two endvertex labels. If a tree can be gracefully labelled, it can be called a 'graceful tree' (Rosa 1967).

The famous Graceful Tree Conjecture (GTC) or Ringel-Kotzig Conjecture, states that all trees are graceful. This labelling was originally introduced in 1967 by Rosa who also showed that the existence of a graceful labelling of a given graph G with n edges is a sufficient condition for the existence of a cyclic decomposition of a complete graph of order $2n + 1$ into subgraphs isomorphic to G. So far, no proof of the truth or falsity of the conjecture has been found. In the absence of generic proof, two approaches have been used in studying the GTC: One is to provide the gracefulness of special classes of trees and the other is to exhaustively test up to a specified size for graceful labellings.

Some types of trees have been proved to be graceful. Many of the proofs for graceful labellings of some classes of a pattern are constructive. The following are some classes of graceful trees.

Paths

A path a single line of vertices. It is well known that paths can always be labelled gracefully [19]. However, the number of graceful labellings of a path of length n is not known.

Caterpillars

A Caterpillar is a tree with a single path with vertices either a single vertex or the root of a star. Caterpillars were shown to be graceful early on by Rosa [19]. They can be labelled using a similar strategy as for paths, as shown in Figure 1.1.

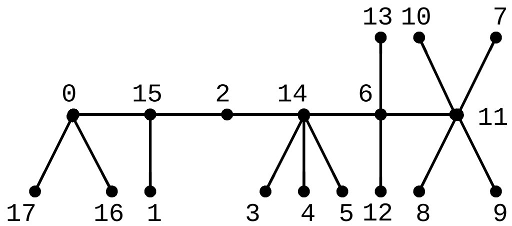

Figure 1.1: An example of graceful labelling for a caterpillar.

***n*-Stars**

An $n-$star has a single root with any number of paths of length n attached to it. Cahit and Cahit [11] proved that all n-stars have a graceful labelling as shown in Figure 1.2.

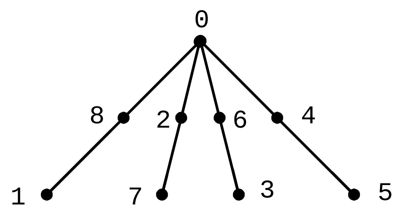

Figure 1.2: An example of graceful labelling for a 2-star.

Trees with diameter five

The diameter of a graph is the maximum of the shortest paths between its vertices.

Hrnciar and Havier extended the proof of caterpillars to prove that all trees of diameter five are graceful [14].

Firecrackers

A lobster is a tree with a single path from which each vertex not in the path has distance at most two. Bermond [3] conjectured that all lobsters are graceful. This conjecture is still open, with only a few limited cases being solved. A firecracker is a special class of lobster and can be seen as a collection of stars where one end vertex from each star is chosen and they are all connected in a path. Chen et al. [13] proved in 1997 that all firecrackers are graceful. See the Figure 1.3 below for an example labelling.

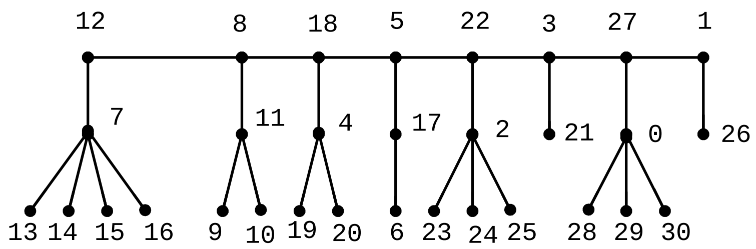

Figure 1.3: An example of graceful labelling for a firecracker.

Banana trees

A banana tree is constructed by bringing multiple stars together at a single vertex (see Figure 1.4). Chen,Lu, and Yeh [13] have conjectured that all banana trees are graceful, and some classes of banana trees been shown to be graceful [5].

Olive trees

An olive tree have a root joining i paths, where path i has length i. Olive trees were shown to be graceful in 1978 by Pastel and Raynaud [18] (see Figure 1.5).

Symmetrical trees

A symmetrical tree is a rooted tree where all the vertices at the same distance from the root have the same degree. All symmetrical trees are graceful, a result by Bermond and Sotteau in 1975 [4], (see Figure 1.6).

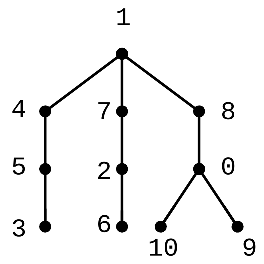

Figure 1.4: An example of graceful labelling for a banana tree.

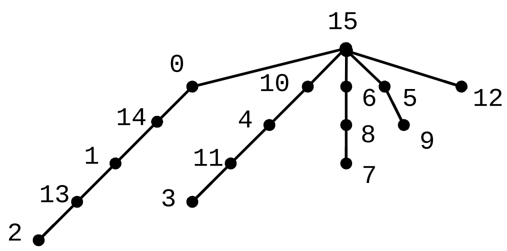

Figure 1.5: An olive tree with a graceful labelling.

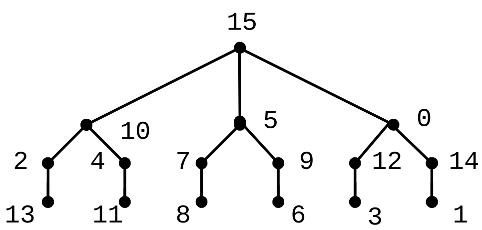

Figure 1.6: An example of a graceful labelling for a symmetrical tree.