Chapter 2

The Model

2.1 Interarrival and Service Times

We assume that both interarrival and service times are of Coxian distribution with k and m stages respectively. It means that an arrival may go through at most up to k phases, and the length of phase j is exponentially distributed with a given rate λ_j for $j = 1, \ldots, k$. After phase j, $j = 1, 2, \ldots, k$, the interarrival time comes to an end with probability p_j, and it enters the next phase with probability $1 - p_j$. Obviously, $p_k = 1$. A similar notation for μ_j and q_j, $j = 1, 2, \ldots, m$, in the service distribution is assumed.

Let $F_a(t)$ be the interarrival time distribution with phase k and its mean by $\frac{1}{\lambda}$. Then, it is known that

$$F_a(t) = 1 - \boldsymbol{\tau}_1 exp(\mathbf{T}_1 t)\mathbf{e}^{'} = -\sum_{n=1}^{\infty} \boldsymbol{\tau}_1 \mathbf{T}_1^n \mathbf{e}^{'} \frac{t^n}{n!},$$

where the $\boldsymbol{\tau}_1$ is the initial probability of $1 \times k$ vector

$$\boldsymbol{\tau}_1 = (1, 0, \ldots, 0),$$

$$\mathbf{T}_1 = \begin{bmatrix} -\lambda_1 & (1-p_1)\lambda_1 & & & 0 \\ & -\lambda_2 & (1-p_2)\lambda_2 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{k-1} & (1-p_{k-1})\lambda_{k-1} \\ 0 & & & & -\lambda_k \end{bmatrix}$$

is a squared matrix of order k, and $\mathbf{e}'$ is a column vector of all entries equal to 1 in a proper dimension depending on its multiplier.

Similarly, the service time distribution $F_s(\cdot)$ has mean $\frac{1}{\mu}$ and representation $(\boldsymbol{\tau}_2, \mathbf{T}_2)$ of dimension m, where

$$\boldsymbol{\tau}_2 = (1, 0, \ldots, 0)$$

is a $1 \times m$ vector and

$$\mathbf{T}_2 = \begin{bmatrix} -\mu_1 & (1-q_1)\mu_1 & & & 0 \\ & -\mu_2 & (1-q_2)\mu_2 & & \\ & & \ddots & \ddots & \\ & & & -\mu_{m-1} & (1-q_{m-1})\mu_{m-1} \\ 0 & & & & -\mu_m \end{bmatrix}$$

is the squared matrix of order m. The distribution is given by

$$F_s(t) = 1 - \boldsymbol{\tau}_2 exp(\mathbf{T}_2 t)\mathbf{e}' = -\sum_{n=1}^{\infty} \boldsymbol{\tau}_2 \mathbf{T}_2^n \mathbf{e}' \frac{t^n}{n!}.$$

The Laplace transform of the interarrival time has the form

$$\begin{aligned} F_a^*(x) &= \int_0^\infty e^{-xt} dF_a(t) \\ &= \int_0^\infty e^{-xt}\{-\boldsymbol{\tau}_1 exp(\mathbf{T}_1 t)\mathbf{T}_1\mathbf{e}'\}dt \\ &= \int_0^\infty -\boldsymbol{\tau}_1 exp\{(\mathbf{T}_1 - x\mathbf{I}_1)t\}\mathbf{T}_1\mathbf{e}' dt \\ &= -\boldsymbol{\tau}_1(\mathbf{T}_1 - x\mathbf{I}_1)^{-1} exp(\mathbf{T}_1 - x\mathbf{I}_1)|_0^\infty \mathbf{T}_1\mathbf{e}' \\ &= \boldsymbol{\tau}_1(x\mathbf{I}_1 - \mathbf{T}_1)^{-1}\boldsymbol{\gamma}_1, \end{aligned}$$

where $\boldsymbol{\gamma}_1 = -\mathbf{T}_1\mathbf{e}'$, and $\mathbf{I}_1$ is an identity matrix with proper dimension in equation. Similarly, the Laplace transform of the service time has the form

$$F_s^*(x) = \boldsymbol{\tau}_2(x\mathbf{I}_2 - \mathbf{T}_2)^{-1}\boldsymbol{\gamma}_2,$$

where $\boldsymbol{\gamma}_2 = -\mathbf{T}_2\mathbf{e}^{'}$. The utilization factor is defined as

$$\rho = \frac{\lambda}{\mu}$$

Since

$$\frac{1}{\lambda} = \int_0^\infty t dF_a(t) = -F_a^{*'}(0)$$

and

$$\frac{1}{\mu} = \int_0^\infty t dF_s(t) = -F_s^{*'}(0)$$

we have

$$\rho = \frac{F_s^{*'}(0)}{F_a^{*'}(0)}.$$

2.2 Matrix of Transition Rates

The $C_k/C_m/1/N$ queueing system may be studied as a Quasi-Birth-Death process. A state of system is denoted by (n, i, j), where n is the number of customers in the system, $n \geq 0$, and i (resp. j) is the phase of customer presents in the interarrival fictitious center (resp. the service center), $1 \leq i \leq k$, $1 \leq j \leq m$. We arrange the states (n, i, j) in lexicographic order and partition of the state space according to the number of customers, n, i.e.

$$\mathcal{S}_n = \{(n, i, j) | 1 \leq i \leq k, \;\; 1 \leq j \leq m\}, \;\; n = 0, 1, 2, \cdots, N.$$

For fixed n the state can be lexicographically in according with phase i and j. The state space can be organized into three groups:

$$\mathcal{S}_0 = \{(0, 1, 0), (0, 2, 0), \cdots, (0, k, 0)\},$$

$$\begin{aligned}
\mathcal{S}_n = \{&(n, 1, 1), & (n, 1, 2), & \;\; \cdots, & (n, 1, m); \\
&(n, 2, 1), & (n, 2, 2), & \;\; \cdots, & (n, 2, m); \\
&\;\;\vdots & \vdots & \;\; \cdots & \vdots \\
&(n, k, 1), & (n, k, 2), & \;\; \cdots, & (n, k, m)\},
\end{aligned}$$

where $1 \le n \le N-1$,

$$\begin{aligned}
\mathcal{S}_N = \{&(N,1,1), & &(N,1,2), & &\cdots, & &(N,1,m); \\
&(N,2,1), & &(N,2,2), & &\cdots, & &(N,2,m); \\
&\ \vdots & &\ \vdots & &\cdots & &\ \vdots \\
&(N,k,1), & &(N,k,2), & &\cdots, & &(N,k,m)\}.
\end{aligned}$$

$\mathcal{S}_0$ and $\mathcal{S}_N$ are defined for boundary states. Likewise, $\mathcal{S}_n$, $1 \le n \le N-1$ is defined for unboundary state. Denoted by $\mathbf{P}$ the stationary probability row-vector partitioned corresponding to $\mathcal{S}_n$ as:

$$\mathbf{P} = (\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_N),$$

where $\mathbf{P}_n$ is a stationary probability row-vector when n customer in system. Define $\mathbf{Q}$ the transition rate matrix of the chain according to the arrangement of $\mathcal{S}_n$. Then $\mathbf{Q}$ is of the block-tridiagonal form and written as

$$\mathbf{Q} = \begin{array}{c} \\ \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \vdots \\ \mathcal{S}_{N-2} \\ \mathcal{S}_{N-1} \\ \mathcal{S}_N \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_2 & \cdots & \mathcal{S}_{N-2} & \mathcal{S}_{N-1} & \mathcal{S}_N \end{array} \\
\begin{bmatrix}
\mathbf{B}_0 & \mathbf{A}_0 & & & & & \\
\mathbf{C}_0 & \mathbf{B} & \mathbf{A} & & & & \\
 & \mathbf{C} & \mathbf{B} & \mathbf{A} & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\
 & & & & \mathbf{C} & \mathbf{B} & \mathbf{A} \\
 & & & & & \mathbf{C} & \mathbf{B}_1
\end{bmatrix}
\end{array}.$$

$\mathbf{A}_0$ is a $k \times km$ transition rate matrix from states of $\mathcal{S}_0$ to states of $\mathcal{S}_1$. $\mathbf{B}_0$ is a $k \times k$ transition rate matrix among states of $\mathcal{S}_0$. $\mathbf{C}_0$ is a $km \times k$ transition rate matrix from states of $\mathcal{S}_1$ to states of $\mathcal{S}_0$. $\mathbf{A}$ is a $km \times km$ transition rate matrix from states of $\mathcal{S}_n$ to states of $\mathcal{S}_{n+1}$, $1 \le n \le N-1$. $\mathbf{B}$ is a $km \times km$ transition rate matrix among states of $\mathcal{S}_n$, $1 \le n \le N-1$. $\mathbf{C}$ is a $km \times km$ transition rate matrix from states of $\mathcal{S}_n$ to states of $\mathcal{S}_{n-1}$, $2 \le n \le N-1$. $\mathbf{B}_1$ is a $km \times km$ transition rate matrix among states of $\mathcal{S}_N$.

The submatrices could be written as Kronecker product and Kronecker sum, defined in Bellman [1], which were denoted by $\otimes$ and $\oplus$, respectively. Kronecker product and Kronecker sum were used to simplify the representation of the system of balance equations for queues by many researchers, for example [8] and [9]. Here, the submatrices $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{B}_1$ are given below:

$$\begin{array}{lll} \mathbf{A}_0 = \boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2 & \mathbf{B}_0 = \mathbf{T}_1 & \mathbf{C}_0 = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \\ \mathbf{A} = \boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2 & \mathbf{B} = \mathbf{T}_1 \oplus \mathbf{T}_2 & \mathbf{C} = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2 \\ & \mathbf{B}_1 = (\mathbf{T}_1 + \mathbf{R}_1) \oplus \mathbf{T}_2 & \end{array} \tag{2.1}$$

where $\mathbf{R}_1$ is $\mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_k)$.

2.3 Balance Equations

For the balance equations $\mathbf{PQ} = \mathbf{0}$ and the normalization condition $\mathbf{Pe}' = 1$, we obtain the following equations:

$$\begin{cases} \mathbf{P}_0\mathbf{B}_0 + \mathbf{P}_1\mathbf{C}_0 = \mathbf{0} & & (2.2) \\ \mathbf{P}_0\mathbf{A}_0 + \mathbf{P}_1\mathbf{B} + \mathbf{P}_2\mathbf{C} = \mathbf{0} & & (2.3) \\ \mathbf{P}_{n-1}\mathbf{A} + \mathbf{P}_n\mathbf{B} + \mathbf{P}_{n+1}\mathbf{C} = \mathbf{0} & 2 \le n \le N-1 & (2.4) \\ \mathbf{P}_{N-1}\mathbf{A} + \mathbf{P}_N\mathbf{B}_1 = \mathbf{0} & & (2.5) \\ \mathbf{Pe}' = 1 & & (2.6) \end{cases}$$

It is easy to rewrite the balance equations by substitute (2.1) into equations (2.2)$\sim$(2.5):

$$\mathbf{P}_0\mathbf{T}_1 + \mathbf{P}_1(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2) = 0, \tag{2.7}$$

$$\mathbf{P}_0(\boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2) + \mathbf{P}_1(\mathbf{T}_1 \oplus \mathbf{T}_2) + \mathbf{P}_2(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2) = 0, \tag{2.8}$$

$$\mathbf{P}_{n-1}(\boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + \mathbf{P}_n(\mathbf{T}_1 \oplus \mathbf{T}_2) + \mathbf{P}_{n+1}(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2) = 0, \tag{2.9}$$

$$2 \le n \le N-1,$$

$$\mathbf{P}_{N-1}(\boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + \mathbf{P}_N((\mathbf{T}_1 + \mathbf{R}_1) \oplus \mathbf{T}_2) = 0. \tag{2.10}$$

2.4 Vector Product-Form Solutions

2.4.1 Case of Simple Roots

In [10], Wang expressed the unboundary stationary probabilities ($\mathbf{P}_n, n = 1, \ldots, N-1$) of $C_k/C_m/1/N$ system can be written as a linear combination of product-forms. We review an important results from [10].

Proposition 1 *(p.12 in [10]) The equation: $F_a^*(x)F_s^*(-x) = 1$ has t solutions which we need. If $\rho < 1$, t equals m and the equation has m solutions with positive real parts. If $\rho > 1$, t equals k and the equation has k solutions with negative real parts.*

The proof is referred to [10].

According to Proposition 1, we assume the equation:

$$F_a^*(x)F_s^*(-x) = 1 \tag{2.11}$$

has t solutions. For simplicity, we first assume all roots of (2.11) are simple. When the utilization factor, ρ, equals to one, we need to adjust equation (2.11) to avoid $x = 0$. For example, we can rewrite (2.11) as

$$F_a^*(x)F_s^*(-x) = 1.000001.$$

Therefore, if $\rho < 1$, we need the roots of (2.11) with positive real parts. If $\rho > 1$, we need the roots of (2.11) with negative real part. If $\rho = 1$, we need all roots of (2.11) with small adjustment to avoid $x = 0$.

Let x_α be a solution of (2.11) which we need, $\alpha = 1, \cdots, t$ and set $w_\alpha = F_a^*(x_\alpha)$, for $w_\alpha \neq 0$. Given x_α, we define $\mathbf{u}_\alpha$ and $\mathbf{v}_\alpha$ as follows,

$$\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 (\mathbf{T}_1 - x_\alpha \mathbf{I}_1)^{-1}, \tag{2.12}$$

$$\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha} \boldsymbol{\tau}_2 (\mathbf{T}_2 + x_\alpha \mathbf{I}_2)^{-1}, \tag{2.13}$$

where $a_{\mathbf{u}_\alpha}$, $a_{\mathbf{v}_\alpha}$ are constants such that $\mathbf{u}_\alpha \mathbf{e}' = \mathbf{v}_\alpha \mathbf{e}' = 1$. Simply, set

$$a_{\mathbf{u}_\alpha} = \frac{x_\alpha}{\omega_\alpha - 1}, \qquad a_{\mathbf{v}_\alpha} = \frac{x_\alpha \omega_\alpha}{\omega_\alpha - 1}, \qquad for \quad \omega_\alpha \neq 1.$$

Since there are t solution, we define $\alpha = 1, 2, \ldots, t$,

$$\mathbf{w}_{\alpha,n} = w_\alpha^{n-1}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha), \quad 1 \leq n \leq N-1. \tag{2.14}$$

Therefore, by [10], we define the unboundary state probabilities are of the form,

$$\mathbf{P}_n = \sum_{\alpha=1}^{t} b_\alpha \mathbf{w}_{\alpha,n}, \quad b_\alpha \in \mathbb{C}, \quad 1 \leq n \leq N-1, \tag{2.15}$$

where b_α is the coefficients respect to $\mathbf{w}_{\alpha,n}$.

Proposition 2 *([6]) Given x_α, $\mathbf{w}_{\alpha,n}, 2 \leq n \leq N-2$ satisfies equation (2.9)*

Proof :

(1) For any given α, rewriting (2.12) and multiplying it by $(\mathbf{T}_1 - x_\alpha \mathbf{I}_1)$, we have

$$\mathbf{u}_\alpha(\mathbf{T}_1 - x_\alpha \mathbf{I}_1) = a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 (\mathbf{T}_1 - x_\alpha \mathbf{I}_1)^{-1} (\mathbf{T}_1 - x_\alpha \mathbf{I}_1)$$

$$\Longrightarrow \mathbf{u}_\alpha \mathbf{T}_1 = a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 + x_\alpha \mathbf{u}_\alpha.$$

Similarly, rewriting (2.13), we have

$$\mathbf{v}_\alpha \mathbf{T}_2 = a_{\mathbf{v}_\alpha} \boldsymbol{\tau}_2 - x_\alpha \mathbf{v}_\alpha.$$

Therefore, it is derived

$$\mathbf{u}_\alpha \mathbf{T}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes \mathbf{v}_\alpha \mathbf{T}_2 = a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes a_{\mathbf{v}_\alpha} \boldsymbol{\tau}_2.$$

(2) For any given α, rewriting (2.12) and multiplying it by $\mathbf{T}_1 \mathbf{e}' (= -\boldsymbol{\gamma}_1)$, we have

$$\begin{aligned}
\mathbf{u}_\alpha \mathbf{T}_1 \mathbf{e}' &= a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 (\mathbf{T}_1 - x_\alpha \mathbf{I}_1)^{-1} \mathbf{T}_1 \mathbf{e}' \\
&= a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 (\mathbf{T}_1 - x_\alpha \mathbf{I}_1)^{-1} (-\boldsymbol{\gamma}_1) \\
&= a_{\mathbf{u}_\alpha} \boldsymbol{\tau}_1 (x_\alpha \mathbf{I}_1 - \mathbf{T}_1)^{-1} \boldsymbol{\gamma}_1 \\
&= a_{\mathbf{u}_\alpha} F_a^*(x_\alpha) \\
&= a_{\mathbf{u}_\alpha} w_\alpha.
\end{aligned}$$

Similarly, rewriting (2.13), we have

$$\mathbf{v}_\alpha \mathbf{T}_2 \mathbf{e}' = a_{\mathbf{v}_\alpha} F_s^*(-x_\alpha) = \frac{a_{\mathbf{v}_\alpha}}{w_\alpha}.$$

Therefore, we can derive

$$\begin{aligned} & \tfrac{1}{w_\alpha}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1\mathbf{e}'\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + w_\alpha(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \mathbf{T}_2\mathbf{e}'\boldsymbol{\tau}_2) \\ = \; & \tfrac{1}{w_\alpha}(a_{\mathbf{u}_\alpha} w_\alpha \boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha) + w_\alpha(\mathbf{u}_\alpha \otimes (\tfrac{a_{\mathbf{v}_\alpha}}{w_\alpha})\boldsymbol{\tau}_2). \end{aligned}$$

(3) Inserting (2.14) into (2.9) divided by w_α^{n-1}, it becomes

$$\begin{aligned} & \tfrac{w_\alpha^{n-2}}{w_\alpha^{n-1}}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + \tfrac{w_\alpha^{n-1}}{w_\alpha^{n-1}}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1 \oplus \mathbf{T}_2) + \tfrac{w_\alpha^{n}}{w_\alpha^{n-1}}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2) \\ = \; & (\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1 \oplus \mathbf{T}_2) - \{\tfrac{1}{w_\alpha}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1\mathbf{e}'\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + w_\alpha(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \mathbf{T}_2\mathbf{e}'\boldsymbol{\tau}_2)\} \\ = \; & \mathbf{u}_\alpha\mathbf{T}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes \mathbf{v}_\alpha\mathbf{T}_2 - \{\tfrac{1}{w_\alpha}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1\mathbf{e}'\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + w_\alpha(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \mathbf{T}_2\mathbf{e}'\boldsymbol{\tau}_2)\} \\ = \; & a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2 - \{\tfrac{1}{w_\alpha}(a_{\mathbf{u}_\alpha} w_\alpha\boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha) + w_\alpha(\mathbf{u}_\alpha \otimes (\tfrac{a_{\mathbf{v}_\alpha}}{w_\alpha})\boldsymbol{\tau}_2)\} \end{aligned}$$

Hence it balances the equation (2.9). □

We can rewrite the balance equations by substitute(2.15) into equations (2.2) $\sim$ (2.6). According to Proposition 2, any linear combination of $\mathbf{w}_{\alpha,n}$, $1 \leq n \leq N$ satisfies the balance equations (2.4). When we substitute (2.15) into balance equations, we can ignore the equations (2.4) excepted $n = N-1$.

$$\begin{cases} \mathbf{P}_0\mathbf{B}_0 + (\sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,1})\mathbf{C}_0 = \mathbf{0} & (2.16) \\ \mathbf{P}_0\mathbf{A}_0 + (\sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,1})\mathbf{B} + (\sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,2})\mathbf{C} = \mathbf{0} & (2.17) \\ (\sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,N-2})\mathbf{A} + (\sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,N-1})\mathbf{B} + \mathbf{P}_N\mathbf{C} = \mathbf{0} & (2.18) \\ (\sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,N-1})\mathbf{A} + \mathbf{P}_N\mathbf{B}_1 = \mathbf{0} & (2.19) \\ \mathbf{P}_0\mathbf{e}' + \sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,1}\mathbf{e}' + \sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,2}\mathbf{e}' + \cdots + \sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,N-1}\mathbf{e}' + \mathbf{P}_N\mathbf{e}' = 1 & (2.20) \end{cases}$$

After rewriting above equations:

$$\begin{cases} \mathbf{P}_0\mathbf{B}_0 + b_1(\mathbf{w}_{1,1}\mathbf{C}_0) + b_2(\mathbf{w}_{2,1}\mathbf{C}_0) + \cdots + b_t(\mathbf{w}_{t,1}\mathbf{C}_0) = \mathbf{0} & (2.21) \\ \mathbf{P}_0\mathbf{A}_0 + b_1(\mathbf{w}_{1,1}\mathbf{B} + \mathbf{w}_{1,2}\mathbf{C}) + b_2(\mathbf{w}_{2,1}\mathbf{B} + \mathbf{w}_{2,2}\mathbf{C}) + \cdots + b_t(\mathbf{w}_{t,1}\mathbf{B} + \mathbf{w}_{t,2}\mathbf{C}) = \mathbf{0} & (2.22) \\ b_1(\mathbf{w}_{1,N-2}\mathbf{A} + \mathbf{w}_{1,N-1}\mathbf{B}) + \cdots + b_t(\mathbf{w}_{t,N-2}\mathbf{A} + \mathbf{w}_{t,N-1}\mathbf{B}) + \mathbf{P}_N\mathbf{C} = \mathbf{0} & (2.23) \\ b_1(\mathbf{w}_{l,N-1}\mathbf{A}) + b_2(\mathbf{w}_{2,N-1}\mathbf{A}) + \cdots + b_t(\mathbf{w}_{t,N-1}\mathbf{A}) + \mathbf{P}_N\mathbf{B}_1 = \mathbf{0} & (2.24) \\ \mathbf{P}_0\mathbf{e}' + b_1(\sum_{j=1}^{N-1}\mathbf{w}_{1,j}\mathbf{e}') + b_2(\sum_{j=1}^{N-1}\mathbf{w}_{2,j}\mathbf{e}') + \cdots + b_t(\sum_{j=1}^{N-1}\mathbf{w}_{t,j}\mathbf{e}') + \mathbf{P}_N\mathbf{e}' = 1 & (2.25) \end{cases}$$

Since the system is stable, at least one of the coefficient b_α must be nonull. Hence, for an appropriate choice of b_α, we can solve boundary probabilities.

2.4.2 A Simple Case of Multiple Roots

In this section, we discuss the situation when multiple roots occur in (2.11). If the equation (2.11) has multiple roots, the expression of the unboundary state probabilities will be very complicated.

Let x_1, x_2,..., x_s be the s distinct roots of (2.11) with multiplicity r_1, r_2,..., r_s. According to Proposition 1, we know (2.11) has t solutions. We assume $r_1 = 2$ and $r_2 = r_3 = \ldots = r_s = 1$, then we can get $s + 1 = t$. Since x_1 is a multiple root of (2.11), we can not define the unboundary state probabilities as (2.15). In [7], Liu provides the formula of vector product solution of unboundary stationary probabilities for it.

First, we set $w_\alpha = F_a^*(x_\alpha)$, for $\alpha = 1, 2, \ldots, s$, and define $\mathbf{u}_\alpha$ and $\mathbf{v}_\alpha$, $\alpha = 2, \ldots, s$, as (2.12), (2.13). Second, we define $\mathbf{u}_1^{(1)}$, $\mathbf{v}_1^{(1)}$, $\mathbf{u}_1^{(0)}$, $\mathbf{v}_1^{(0)}$, $\mathbf{u}_1^{(2)}$, $\mathbf{v}_1^{(2)}$, φ_{11}, φ_{10}, φ_{12} as follows,

$$\mathbf{u}_1^{(1)} = a_{\mathbf{u}_1}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1},$$

$$\mathbf{v}_1^{(1)} = a_{\mathbf{v}_1}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1},$$

$$\mathbf{u}_1^{(0)} = \frac{1}{w_1}\mathbf{u}_1^{(1)}(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1},$$

$$\mathbf{v}_1^{(0)} = \frac{1}{w_1}\mathbf{v}_1^{(1)}(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1},$$

$$\mathbf{u}_1^{(2)} = \frac{b_{\mathbf{u}_1}}{w_1}\mathbf{u}_1^{(1)}(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1},$$

$$\mathbf{v}_1^{(2)} = \frac{b_{\mathbf{v}_1}}{w_1}\mathbf{v}_1^{(1)}(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1},$$

$$\varphi_{11} = \mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(1)},$$

$$\varphi_{10} = \mathbf{u}_1^{(0)} \otimes \mathbf{v}_1^{(1)} - \mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(0)},$$

$$\varphi_{12} = \mathbf{u}_1^{(2)} \otimes \mathbf{v}_1^{(1)} - \mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(2)},$$

where

$$b_{\mathbf{u}_1} = \frac{-a_{\mathbf{u}_1}w_1}{\mathbf{u}_1^{(1)}(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1}$$

$$b_{\mathbf{v}_1} = \frac{-a_{\mathbf{v}_1}}{w_1\mathbf{v}_1^{(1)}(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2}$$

Third, since there are s solutions, we define $\mathbf{w}_{\alpha,n}$, $\alpha = 2, \ldots, s$, as (2.14). By [7], we can define the unboundary state probailities in the following.
If $\boldsymbol{\tau}_1(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 = 0$, then

$$\mathbf{P}_n = b_{11}w_1^{n-1}\varphi_{11} + b_{12}w_1^{n-1}\varphi_{10} + \sum_{\alpha=2}^{s} b_\alpha \mathbf{w}_{\alpha,n}, \quad 1 \leq n \leq N-1. \tag{2.26}$$

If $\boldsymbol{\tau}_1(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_2 \neq 0$, then

$$\mathbf{P}_n = \psi_n(w_1) + \sum_{\alpha=2}^{s} b_\alpha \mathbf{w}_{\alpha,n}, \quad 1 \leq n \leq N-1, \tag{2.27}$$

where

$$\begin{aligned}
\psi_1(w_1) &= b_{11}\varphi_{11} + b_{12}\varphi_{12} \\
\psi_2(w_1) &= b_{11}w_1\varphi_{11} + b_{12}(w_1\varphi_{12} + \varphi_{11}) \\
&\vdots \\
\psi_n(w_1) &= b_{11}w_1^{n-1}\varphi_{11} + b_{12}(w_1^{n-1}\varphi_{12} + (n-1)w_1^{n-2}\varphi_{11}).
\end{aligned}$$

From (2.16), (2.17), we show the complexity of the case of multiple roots. In our numerical experience, we never find the multiple roots of (2.11). Therefore, In this thesis, we focus on the case of simple roots and the algorithm in chapter 3 only fits for the case of simple roots.

2.5 Boundary State Probabilities

In this section, we want to compare the product-form method with traditional method. Note that $\mathbf{P}_0$ and $\mathbf{P}_N$ both are row vectors. Let

$$\mathbf{P}_0 = (P_{0,1,0}, P_{0,2,0}, \cdots, P_{0,k,0}). \tag{2.28}$$

and

$$\mathbf{P}_N = (P_{N,1,1}, P_{N,1,2}, \cdots, P_{N,1,m}, P_{N,2,1}, \cdots, P_{N,i,j}, \cdots, P_{N,k,m}). \tag{2.29}$$

Therefore the total number of unknowns of product-form method are $k + t + km$. Observe the equations (2.21) $\sim$ (2.25). The total number of equations is $k + 3km +$

1. It is noted that the number of unknowns and equations are independent of system size N. However, in the traditional method, the total number of unknowns is $k + kmN$. Observe the equations (2.2) $\sim$ (2.6), the total number of equations are $k+kmN+1$. Obviously, the problem can be greatly reduced to a problem of solving a linear nonhomogenous system independent of N. Hence, the computing efficiency of the product-form method is much better than that of a traditional method when $N >> 3$.

According to proposition 1, t depends on the condition of ρ. Whatever $\rho > 1$ or $\rho < 1$, the number t is less or equal km. Therefore, the number of equations is greater than unknowns. Instead of checking the independent vector in (2.21)$\sim$(2.25) and solving by Gauss Elimination, the solution of this problem may be obtained by using some numerical methods.

2.6 Performance Measures

We denote the expected number of people waiting in the queue by L_q. Note that if 0 or 1 customer is present in the system, then nobody is waiting in line, but if j people are present($j \geq 1$), there will be $j-1$ people waiting in line. Thus, we have

$$L_q = \sum_{j=1}^{j=N} (j-1)\pi_j.$$

where

$$\pi_j = \mathbf{P}_j \mathbf{e}', \quad 0 \leq j \leq N.$$

Also of interest is L_s, the expected number of customers in system. We have

$$L_s = \sum_{j=1}^{j=N} (j)\pi_j.$$

Often we are interested in the amount of time that a customer spends in a queueing system. We define W_s as the expected time a customer spends in the queueing system, including time in line plus time in service, and W_q as the expected time a customer spends waiting in line. Both W_s and W_q are computed under the assumption that the steady state has been reached. By using a powerful result known as *Little's queuing formula*, W_s and W_q may be easily computed from L_s and L_q.

Proposition 3 *(Little queueing formula) For $C_k/C_m/1/N$ queueing system, the following relations hold:* $L_q = \lambda(1 - \pi_N)W_q$, $L_s = \lambda(1 - \pi_N)W_s$

Therefore, we can easily computed W_s and W_q.