Chapter 3

A Summary of the Algorithm

In this chapter, we will solve the stationary probabilities by product-form method. Note the number of the unknowns is independent of N. In Section 3.1, we will introduce the algorithm which suits the case of simple roots. In Section 3.2, we will present two illustrative examples.

3.1 The Algorithm

We describe the algorithm for solving stationary probabilities of a $C_k/C_m/1/N$ system in the following steps.

Step 1 Solve equation (2.11), $F_a^*(x)F_s^*(-x) = 1$. Let x_α be a solution of (2.11), $\alpha = 1, \cdots, t$.

Step 2 Compute w_α, $\mathbf{u}_\alpha$, $\mathbf{v}_\alpha$.

1. Compute w_α defined in $w_\alpha = F_a^*(x_\alpha)$.
2. Compute $\mathbf{u}_\alpha$ defined in (2.12), $\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}$.
3. Compute $\mathbf{v}_\alpha$ defined in (2.13), $\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_\alpha\mathbf{I}_2)^{-1}$.

Step 3 Compute $\mathbf{w}_{\alpha,n}$ defined in (3.16), $\mathbf{w}_{\alpha,n} = w_\alpha^{n-1}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)$, $1 \leq n \leq N-1$.

Step 4 Let $\mathbf{P}_n$ be a linear combination of $\mathbf{w}_{\alpha,n}$ that is $\mathbf{P}_n = \sum_{\alpha=1}^{t} b_\alpha \mathbf{w}_{\alpha,n}$, $b_\alpha \in \mathbb{C}$.

Step 5 Set a linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25).

Step 6 Use the Cholesky factorization to solve the linear nonhomogeneous system and obtain coefficients b_α, $\alpha = 1, \ldots, t$ and boundary stationary probabilities $\mathbf{P}_0$ and $\mathbf{P}_N$.

Step 7 Substituting coefficients b_α, $\alpha = 1, \ldots, t$, to (2.15) and obtain unboundary stationary probabilities $\mathbf{P}_n$, $1 \leq n \leq N-1$.

Step 8 Compute the system-size probability $\boldsymbol{\pi}_n$, $n = 1, \ldots, N$.

It is important to note that no matter how large the system size N is, we only need to solve coefficients $b_\alpha, \alpha = 1, \ldots, t$. Hence the computational complexity is greatly reduced.

3.2 Examples of $C_2/C_2/1/7$ System

3.2.1 The Example of Case 1 of $\rho < 1$

The system has the following features:

$$N = 7, \quad \tau_1 = \tau_2 = (1, 0),$$

$$\lambda_1 = \lambda_2 = 4, \quad p_1 = 0.5, \quad p_2 = 1,$$

$$\mu_1 = \mu_2 = 5, \quad q_1 = 0.5, \quad q_2 = 1.$$

Step 1 Solve equation (2.11), $F_a^*(x)F_s^*(-x) = 1$. Let x_α be a solution with positive real parts of (2.11), $\alpha = 1,\ 2$. We have

$$F_a^*(x) = \frac{2}{(x+4)} + \frac{8}{(x+4)^2},$$

$$F_s^*(x) = \frac{5}{2(x+5)} + \frac{25}{2(x+5)^2},$$

and the solutions of $F_a^*(x)F_s^*(-x) = 1$ are

$$x_1 = 6.5131, \quad x_2 = 0.8576.$$

Step 2 Compute w_α, $\mathbf{u}_\alpha$, $\mathbf{v}_\alpha$.

1. Compute w_α defined in $w_\alpha = F_a^*(x_\alpha)$.

$$w_1 = F_a^*(x_1) = 0.2626,$$

$$w_2 = F_a^*(x_2) = 0.7508.$$

2. Compute $\mathbf{u}_\alpha$ defined in (2.12), $\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}$.

$$\mathbf{u}_1 = (0.8402, 0.1598),$$

$$\mathbf{u}_2 = (0.7084, 0.2916).$$

3. Compute $\mathbf{v}_\alpha$ defined in (2.13), $\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_\alpha\mathbf{I}_2)^{-1}$.

$$\mathbf{v}_1 = (-1.5331, 2.5331),$$

$$\mathbf{v}_2 = (0.6236, 0.3764).$$

Step 3 Compute $\mathbf{w}_{\alpha,n}$ defined in (2.14), $\mathbf{w}_{\alpha,n} = w_\alpha^{n-1}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)$, $1 \le n \le 6$.

$$\mathbf{w}_{1,1} = (-1.2881, 2.1282, -0.2450, 0.4049),$$

$$\mathbf{w}_{1,2} = (-0.3383, 0.5589, -0.0644, 0.1063),$$

$$\mathbf{w}_{1,3} = (-0.0888, 0.1468, -0.0169, 0.0279),$$

$$\mathbf{w}_{1,4} = (-0.0233, 0.0385, -0.0044, 0.0073),$$

$$\mathbf{w}_{1,5} = (-0.0061, 0.0101, -0.0012, 0.0019),$$

$$\mathbf{w}_{1,6} = (-0.0016, 0.0027, -0.0003, 0.0005),$$

$$\mathbf{w}_{2,1} = (0.4417, 0.2666, 0.1819, 0.1098),$$

$$\mathbf{w}_{2,2} = (0.3316, 0.2002, 0.1365, 0.0824),$$

$$\mathbf{w}_{2,3} = (0.2490, 0.1503, 0.1025, 0.0619),$$

$$\mathbf{w}_{2,4} = (0.1869, 0.1128, 0.0770, 0.0464),$$

$$\mathbf{w}_{2,5} = (0.1403, 0.0847, 0.0578, 0.0349),$$

$$\mathbf{w}_{2,6} = (0.1054, 0.0636, 0.0434, 0.0262),$$

Step 4 Let $\mathbf{P}_n$ be a linear combination of $\mathbf{w}_{\alpha,n}$ that is $\mathbf{P}_n = \sum_{\alpha=1}^{2} b_\alpha \mathbf{w}_{\alpha,n}$, $b_\alpha \in \mathbb{C}$.

Step 5 Set a linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25).

$$\mathbf{z}\begin{bmatrix}
-4 & 0 & 7.4210 & 2.4374 & 0 & 0 & 0 & 0 \\
2 & -4 & 1.4118 & 1.0035 & 0 & 0 & 0 & 0 \\
2 & 4 & 13.5414 & -2.1459 & 0 & 0 & 0 & 0 \\
0 & 0 & -22.3741 & -1.2951 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.0024 & -0.4364 & 2.5 & 5 & 0 & 0 \\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.0005 & -0.1797 & 0 & 0 & 2.5 & 5 \\
0 & 0 & 0.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.0044 & 0.3842 & -7.0000 & 0 & 0 & 0 \\
0 & 0 & 0.0073 & 0.2319 & 2.5 & -7 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2.5 & -5 \\
1 & 1 & 1.3557 & 3.2937 & 1 & 1 & 1 & 1
\end{bmatrix}^{t} = \mathbf{b}.$$

where $\mathbf{z} = (P_{0,1,0}, \cdots, P_{0,k,0}, b_1, \cdots, b_t, P_{N,1,1}, \cdots, P_{N,i,j}, \cdots, P_{N,k,m})$, and $\mathbf{b} = (0, 0, \cdots, 0, 1)$.

Step 6 Use the Cholesky factorization to solve the linear nonhomogeneous system and obtain coefficients $b_\alpha,\ \alpha = 1,\ 2$, and boundary stationary probabilities $\mathbf{P}_0$ and $\mathbf{P}_7$.

$$\mathbf{P}_0 = (0.1152, 0.1105),$$

$$\mathbf{P}_7 = (0.0129, 0.0126, 0.0045, 0.0066),$$

$$b_1 = -0.0133, \quad b_2 = 0.2294.$$

Step 7 Substituting coefficients b_α, $\alpha = 1, 2$, to (2.15) and obtain unboundary stationary probabilities $\mathbf{P}_n$, $1 \le n \le 6$.

$$\mathbf{P}_1 = (0.1185, 0.0329, 0.0450, 0.0198),$$

$$\mathbf{P}_2 = (0.0806, 0.0385, 0.0322, 0.0175),$$

$$\mathbf{P}_3 = (0.0583, 0.0325, 0.0237, 0.0138).$$

$$\mathbf{P}_4 = (0.0432, 0.0254, 0.0177, 0.0106).$$

$$\mathbf{P}_5 = (0.0323, 0.0193, 0.0133, 0.0080).$$

$$\mathbf{P}_6 = (0.0242, 0.0146, 0.0100, 0.0060).$$

Step 8 Compute the system-size probability $\boldsymbol{\pi}_n$, $n = 0, \ldots, 7$.

$$\boldsymbol{\pi}_0 = 0.2256,$$

$$\boldsymbol{\pi}_1 = 0.2162,$$

$$\boldsymbol{\pi}_2 = 0.1688,$$

$$\boldsymbol{\pi}_3 = 0.1284,$$

$$\boldsymbol{\pi}_4 = 0.0968,$$

$$\boldsymbol{\pi}_5 = 0.0728,$$

$$\boldsymbol{\pi}_6 = 0.0547,$$

$$\boldsymbol{\pi}_7 = 0.0366,$$

We know that the idle probability of the system is $\boldsymbol{\pi}_0 = 0.2256$, and the blocking probability of the $C_2/C_2/1/7$ queueing system is $\boldsymbol{\pi}_7 = 0.0366$.

3.2.2 The Example of Case 2 of $\rho > 1$

The system has the following features:

$$N = 7, \quad \tau_1 = \tau_2 = (1, 0),$$

$$\lambda_1 = \lambda_2 = 5, \quad p_1 = 0.5, \quad p_2 = 1,$$

$$\mu_1 = \mu_2 = 4, \quad q_1 = 0.5, \quad q_2 = 1.$$

Step 1 Solve equation (2.11), $F_a^*(x)F_s^*(-x) = 1$. Let x_α be a solution with negative real parts of (2.11), $\alpha = 1,\ 2$. We have

$$F_a^*(x) = \frac{5}{2(x+5)} + \frac{25}{2(x+5)^2},$$

$$F_s^*(x) = \frac{2}{(x+4)} + \frac{8}{(x+4)^2},$$

and the solutions of $F_a^*(x)F_s^*(-x) = 1$ are

$$x_1 = -6.5131, \quad x_2 = -0.8576.$$

Step 2 Compute w_α, $\mathbf{u}_\alpha$, $\mathbf{v}_\alpha$.

1. Compute w_α defined in $w_\alpha = F_a^*(x_\alpha)$.

$$w_1 = F_a^*(x_1) = 3.8078,$$

$$w_2 = F_a^*(x_2) = 1.3320.$$

2. Compute $\mathbf{u}_\alpha$ defined in (2.12), $\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}$.

$$\mathbf{u}_1 = (-1.5331, 2.5331),$$

$$\mathbf{u}_2 = (0.6236, 0.3764).$$

3. Compute $\mathbf{v}_\alpha$ defined in (2.13), $\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_\alpha\mathbf{I}_2)^{-1}$.

$$\mathbf{v}_1 = (0.8402, 0.1598),$$

$$\mathbf{v}_2 = (0.7084, 0.2916).$$

Step 3 Compute $\mathbf{w}_{\alpha,n}$ defined in (2.14), $\mathbf{w}_{\alpha,n} = w_\alpha^{n-1}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)$, $1 \le n \le 6$.

$$\mathbf{w}_{1,1} = (-1.2881, -0.2450, 2.1282, 0.4049),$$

$$\mathbf{w}_{1,2} = (-4.9046, -0.9330, 8.1038, 1.5417),$$

$$\mathbf{w}_{1,3} = (-18.6755, -3.5528, 30.8571, 5.8702),$$

$$\mathbf{w}_{1,4} = (-71.112, -13.528, 117.5, 22.352),$$

$$\mathbf{w}_{1,5} = (-270.78, -51.512, 447.4, 85.113),$$

$$\mathbf{w}_{1,6} = (-1031.1, -196.15, 1703.6, 324.09),$$

$$\mathbf{w}_{2,1} = (0.4417, 0.1819, 0.2666, 0.1098),$$

$$\mathbf{w}_{2,2} = (0.5884, 0.2423, 0.3551, 0.1462),$$

$$\mathbf{w}_{2,3} = (0.7838, 0.3227, 0.4730, 0.1948),$$

$$\mathbf{w}_{2,4} = (1.044, 0.42982, 0.63005, 0.25941),$$

$$\mathbf{w}_{2,5} = (1.3905, 0.57252, 0.83923, 0.34553),$$

$$\mathbf{w}_{2,6} = (1.8522, 0.76259, 1.1178, 0.46024),$$

Step 4 Let $\mathbf{P}_n$ be a linear combination of $\mathbf{w}_{\alpha,n}$ that is $\mathbf{P}_n = \sum_{\alpha=1}^{2} b_\alpha \mathbf{w}_{\alpha,n}$, $b_\alpha \in \mathbb{C}$.

Step 5 Set a linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25).

$$\mathbf{z}\begin{bmatrix}
-5 & 0 & -3.5563 & 1.611 & 0 & 0 & 0 & 0 \\
2.5 & -5 & 5.8759 & 0.97228 & 0 & 0 & 0 & 0 \\
2.5 & 5 & -1.9489 & -1.8299 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.37076 & -0.75341 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 10840 & -8.9973 & 2 & 4 & 0 & 0 \\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & -17910 & -5.4301 & 0 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5940.3 & 10.22 & -6.5 & 0 & 0 & 0 \\
0 & 0 & 1130.1 & 4.2077 & 2 & -6.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 2.5 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.5 & 2 & -4 \\
1 & 1 & 1085.2 & 13.81 & 1 & 1 & 1 & 1
\end{bmatrix}^t = \mathbf{b},$$

where $\mathbf{z} = (P_{0,1,0}, \cdots, P_{0,k,0}, b_1, \cdots, b_t, P_{N,1,1}, \cdots, P_{N,i,j}, \cdots, P_{N,k,m})$, and $\mathbf{b} = (0, 0, \cdots, 0, 1)$.

Step 6 Use the Cholesky factorization to solve the linear nonhomogeneous system and obtain coefficients b_α, $\alpha = 1,\ 2$, and boundary stationary probabilities $\mathbf{P}_0$ and $\mathbf{P}_7$.

$$\mathbf{P}_0 = (0.0139, 0.0143),$$

$$\mathbf{P}_7 = (0.0845, 0.0598, 0.0529, 0.0639),$$

$$b_1 = 0.000005, \quad b_2 = 0.050952.$$

Step 7 Substituting coefficients b_α, $\alpha = 1,\ 2$, to (2.15) and obtain unboundary stationary probabilities $\mathbf{P}_n$, $1 \le n \le 6$.

$$\mathbf{P}_1 = (0.0225, 0.0093, 0.0136, 0.0056),$$

$$\mathbf{P}_2 = (0.0300, 0.0123, 0.0181, 0.0075),$$

$$\mathbf{P}_3 = (0.0398, 0.0164, 0.0242, 0.0100).$$

$$\mathbf{P}_4 = (0.0529, 0.0218, 0.0327, 0.0133).$$

$$\mathbf{P}_5 = (0.0696, 0.0289, 0.0449, 0.0180).$$

$$\mathbf{P}_6 = (0.0895, 0.0379, 0.0650, 0.0250).$$

Step 8 Compute the system-size probability $\boldsymbol{\pi}_n$, $n = 0, \ldots, 7$.

$$\boldsymbol{\pi}_0 = 0.0282,$$

$$\boldsymbol{\pi}_1 = 0.0510,$$

$$\boldsymbol{\pi}_2 = 0.0679,$$

$$\boldsymbol{\pi}_3 = 0.0905,$$

$$\boldsymbol{\pi}_4 = 0.1207,$$

$$\boldsymbol{\pi}_5 = 0.1614,$$

$$\boldsymbol{\pi}_6 = 0.2174,$$

$$\boldsymbol{\pi}_7 = 0.2610,$$

We know that the idle probability of the system is $\boldsymbol{\pi}_0 = 0.0282$, and the blocking probability of the $C_2/C_2/1/7$ queueing system is $\boldsymbol{\pi}_7 = 0.2610$.