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Economic production quantity models for deteriorating items with up-stream full trade credit and down-stream partial trade credit

Sheng-Chih Chen [a], Jinn-Tsair Teng [b,*], Konstantina Skouri [c]

[a] Master's Program of Digital Content and Technologies, College of Communication, and Center for Creativity and Innovation Studies, National ChengChi University, Taipei 11605, Taiwan, ROC
[b] Department of Marketing and Management Sciences, Cotsakos College of Business, The William Paterson University of New Jersey, Wayne, NJ 07470, USA
[c] Department of Mathematics, University of Ioannina, Ipiros, Greece

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ABSTRACT

In practice, in order to reduce default risks with credit-risk customers, a seller (e.g., a manufacturer or a retailer) frequently requests its credit-risk customers to pay a fraction of the purchase amount at the time of placing an order as collateral deposit, and then grants a permissible delay on the outstanding balance (i.e., a down-stream partial trade credit). By contrast, the seller usually receives a permissible delay on the entire purchase amount from the supplier (i.e., an up-stream full trade credit). In this paper, we propose an economic production quantity (EPQ) model for deteriorating items in a supply chain with both up-stream and down-stream trade credit financing. By using fractional programming results, we can prove that the optimal solution not only exists but also is unique. Moreover, we propose three discrimination terms to identify the optimal solution among possible alternatives. Finally, some numerical examples are presented to highlight the theoretical results and managerial insights.

1. Introduction

Financial supply chain management and working capital management are increasingly recognized as important means to increase profitability in a supply chain. The physical product flow has long been addressed by researchers and practitioners. However, now companies have identified the financial side of the supply chain as a promising area for improvements. By actively managing payment terms and working capital requirements, managers can influence financial performance and achieve significant cost savings. The permissible delay in payments (i.e., trade credit) allows a buyer to accumulate revenue and earn interest during the credit period. However, beyond this credit period the seller charges the buyer interest on the unpaid balance. Hence, from the buyer's perspective, a permissible delay in payments reduces its holding cost, and thus is a powerful promotional tool to attract new customers, who consider it as an alternative incentive policy to price discounts. On the other hand, from the seller's perspective, although offering trade credit increases its opportunity cost due to interest loss during the credit period, it reduces its buyer's holding cost, attracts new customers, and in turn increases its profit.

In 1913, the economic order quantity (EOQ) was first proposed by Harris (1913). Since then prolific extensions of his EOQ model have been developed by researchers. Grubbstrom (1980) built an inventory model with two trade credit periods with no optimization. Goyal (1985) obtained the retailer's optimal order quantity in an EOQ model when the supplier offers a permissible delay in payments. Aggarwal and Jaggi (1995) extended the EOQ model with trade credit financing from non-deteriorating items to deteriorating items. Jamal et al. (1997) further generalized the EOQ model to allow for shortages. Chang et al. (2003) developed an EOQ model for deteriorating items under supplier credits linked to ordering quantity. Huang (2003) proposed an inventory model by assuming that the supplier offers the retailer a permissible delay and the retailer also provides its customers another permissible delay to stimulate demand. Ouyang et al. (2006) established an optimal ordering policy for deteriorating items under trade credits. Liao (2007) presented an economic production quantity (EPQ) model for deteriorating items under permissible delay in payments. Teng et al. (2009) proposed an EOQ model with two warehouses and solved the problem by an arithmetic–geometric inequality method. Hu and Liu (2010) presented an EPQ model with permissible delay in payments and allowable shortages. Teng et al. (2011) extended an EOQ model for stock-dependent demand to supplier's trade credit with a progressive payment scheme. Skouri et al. (2011) studied supply chain models for deteriorating items with ramp-type demand rate under permissible delay in payments. Teng et al. (2012a) discussed vendor-buyer inventory models with trade credit financing under a non-cooperative and an integrated environments. Concurrently, Teng et al. (2012b) established an EOQ model with trade credit financing for increasing demand. Min et al. (2012) developed an EPQ model with

* Corresponding author. Tel.: +1 9737202651; fax: +1 9737202809.
E-mail addresses: tengj@wpunj.edu, tengjt@gmail.com (J.-T. Teng).

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inventory-level-dependent demand and permissible delay in payments. Tsao (2012) considered manufacturer's production and warranty decisions for an imperfect production system under system maintenance and trade credit. Mahata (2012) proposed an EPQ model for deteriorating items by assuming that the retailer obtains a full trade credit offered by the supplier and offers a partial trade credit to his/her customers. Recently, Chung and Cárdenas-Barrón (2013) presented a simplified solution procedure to an EOQ model for deteriorating items by Min et al. (2010) with stock-dependent demand and two-level trade credit. Chern et al. (2013) established Stackelberg solution in a vendor-buyer supply chain model with permissible delay in payments. Ouyang and Chang (2013) proposed an optimal production lot with imperfect production process under permissible delay in payments and complete backlogging. Chen et al. (2013a) established the retailer's optimal EOQ when the supplier offers conditionally permissible delay in payments link to order quantity. Concurrently, Chen et al. (2013b) attempted to overcome some shortcomings of mathematical model and expressions in Liao et al. (2012).

In this paper we propose an EPQ model for deteriorating items in a supply chain in which a retailer receives a full trade credit from its supplier and simultaneously offers a partial trade credit to his/her customers. This model is closely related to that of Mahata (2012) but the interest earned and interest payable have been calculated in an different and, according to our opinion, more properly way. By applying convex fractional programming results, we obtain the necessary and sufficient conditions of an optimal solution and propose three discrimination terms to identify the global minimum solution among different alternatives. Finally, some numerical examples are used to illustrate the theoretical results and managerial insights.

2. Notation and assumptions

For simplicity, the notation and the assumptions used through the paper are presented below.Notation

D	demand rate in units per year
P	production rate in units per year, $P>D$
ρ	fraction of non-production time $=1-D/P$
A	set-up or ordering cost in dollars per order
h	holding cost in dollars per unit per year excluding interest charges
c	purchase cost per unit in dollars
p	selling price per unit in dollars, $p>c$
M	up-stream trade credit in years offered by the supplier to the retailer
N	down-stream trade credit in years offered by the retailer to its buyers
I_c	interest rate charged per dollar per year
I_e	interest rate earned per dollar per year
α	fraction of total purchase cost which the buyer must pay at the time of placing an order, $0\le\alpha\le1$
$1-\alpha$	fraction of total purchase cost which the buyer has a permissible delay of N years
t_1	time in years at which production stops
θ	constant deterioration rate, $0\le\theta<1$
T	replenishment cycle time in years
T^*	optimal replenishment cycle time
$TRC(T)$	seller's annual total relevant cost in dollars
$TRC^*(T^*)$	seller's optimal annual total relevant cost in dollars

2.1. Assumptions

(1) The demand rate is known and constant.
(2) Shortages are not allowed.
(3) Time horizon is infinite, and replenishments are instantaneous.
(4) A bank in general loans money only on the retailer's expected receivable revenue (i.e., the revenue received from future sales, which is not including deteriorated items). Therefore, the retailer's interest charged is based on the non-deteriorated items, which is not the entire on-hand inventory as that includes deteriorated items.
(5) The seller receives a full credit period of M years from its supplier, and in turn provides a partial trade credit to its credit-risk customers who must pay α portion of the total purchase cost at the time of placing an order as collateral deposit, and then receive a permissible delay of N years on the outstanding amount. Notice that to good-credit customers, the seller may provide a full trade credit in which we simply set $\alpha=0$. Hence, our proposed model includes the special case in which the seller offers a down-stream full trade credit to its customers.
(6) If $M\ge N$ then the seller deposits the sales revenue into an interest bearing account. If $M\ge T+N$ (i.e., the permissible delay period is longer than the time at which the retailer receives the last payment from its customers), then the seller receives all revenue and pays off the entire purchase cost at the end of the permissible delay M. Otherwise (if $M\le T+N$), the seller pays the supplier the sum of all units sold by $M-N$ and the collateral deposit received from N to M, keeps the profit for the use of the other activities, and starts paying for the interest charges on the items sold after $M-N$.
(7) If $N\ge M$, then the seller finances and pays its supplier the entire amount of the delayed payment $(1-\alpha)cDT$ at the end of the trade credit M, and then pays down the loan after time N at which the seller starts to receive sales revenue from its customers. For the collateral deposit, the seller deposits the sales revenue into an interest bearing account until the end of the permissible delay M. If $T\ge M$, then the seller pays the supplier all units sold by M, keeps the profit for the use of the other activities, and starts paying for the interest charges on the items sold after M.

3. Mathematical formulation of the model

During the production period $[0,\ t_1]$, the inventory level is affected by production, demand, and deterioration. The evolution of the inventory level can be described by the following differential equation:

$$\frac{dI(t)}{dt}+\theta I(t)=P-D,\ 0\le t\le t_1, \tag{1}$$

with the initial inventory level $I(0)=0$.

Next, during non-production period $[t_1,T]$, the inventory depletes by the combined effect of demand and deterioration. Consequently, the change in the inventory level is described by the following differential equation:

$$\frac{dI(t)}{dt}+\theta I(t)=-D,\ t_1\le t\le T, \tag{2}$$

with the ending inventory level $I(T)=0$.

The solutions of the above differential equations are respectively:

$$I(t)=\frac{P-D}{\theta}(1-e^{-\theta t}),\ 0\le t\le t_1, \tag{3}$$

and

$$I(t)=\frac{D}{\theta}(e^{\theta(T-t)}-1),\ t_1\le t\le T. \tag{4}$$

From the continuity of the inventory level at time t_1 the following relation between t_1 and T prevails:

$$t_1=\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right] \tag{5}$$

For the derivation of the seller's annual total relevant cost, the mathematical expressions of set-up cost, holding cost (excluding interest charges), deterioration cost, and the interest payable and the interest earned are required.

(a) *Annual set−up cost* $= A/T$ (6)

(b) *Annual holding cost excluding interest charges*

$$= \frac{h}{T}\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt\right]$$
$$= \frac{h}{\theta^2 T}\left[P(\theta\ t_1 + e^{-\theta t_1} - 1) + D(e^{\theta(T-t_1)} - \theta T - e^{-\theta t_1})\right]$$
$$= \frac{h}{\theta T}(Pt_1 - DT) \quad (7)$$

which was derived by Mahata (2012).
Notice that the holding cost usually includes capital cost. However in this model we consider trade credit options so we exclude interest charges (note that the per unit per unit time holding cost has been defined excluding interest charges). The capital cost is calculated below separately."

(c) *Annual deterioration cost* $= c(Pt_1 - DT)/T$ (8)

Nevertheless, for the derivation of the seller's annual total relevant cost, the up-stream and the down-stream trade credits should be taken into consideration. From the values of N and M, there are two possible cases: (1) $N < M$ and (2) $N \geq M$. Then, these two cases are examined separately below.

3.1. Annual total relevant cost for the case of $N < M$

Based on values of M, T, and $T+N$ (i.e., the time at which the seller receives the payment from the last customer), three sub-cases can occur: (i) $M \leq T$, (ii) $T \leq M \leq T+N$(i.e., $M-N \leq T \leq M$), and (iii) $T+N \leq M$(i.e., $T \leq M-N$). For these three cases, we derive the annual interest earned and the annual interest payable accordingly.

3.1.1. Sub-case 1.1. $M \leq T$

In this sub-case, the seller accumulates revenue and earns interest: (1) from the portion of instant payment starting time 0 through M, and (2) from the portion of delayed payment starting time N through M. Hence, the interest earned per cycle is I_e times the total area of the triangle OMA and the triangle NMA' as shown in Fig. 1. Therefore, the annual interest earned is given by

$$\frac{pI_eD}{2T}\left[\alpha M^2 + (1-\alpha)(M-N)^2\right]. \quad (9)$$

Notice that Mahata (2012) did not recognize the fact that the seller offers customers a permissible delay of N, and hence receives money from N to T+N, not from 0 to T. Consequently, he miscalculated the annual interest earned as follow:

$$\frac{pI_eD}{2T}\left[M^2 + (1-\alpha)N^2\right]. \quad (10)$$

On the other hand, the seller grants its buyers a permissible delay of N periods, and receives the money from its buyers from time N through T+N. Thus, at time M the seller receives αpDM dollars from instant payment and $(1-\alpha)pD(M-N)$ dollars from delayed payment, and pays its supplier $\alpha cDM + (1-\alpha)cD(M-N)$ dollars. The retailer must finance (1) all items sold after M for the portion of instant payment, and (2) all items sold after $M-N$ for the portion of delayed payment at an interest charged I_c per dollar per year. As a result, the interest payable per cycle is $(c/p)\ I_c$ times the total area of the triangle ABC and the triangle $A'B'C'$ as shown in Fig. 1. Therefore, the annual interest payable is given by

$$\frac{cI_cD}{2T}\left[\alpha(T-M)^2 + (1-\alpha)(T+N-M)^2\right]. \quad (11)$$

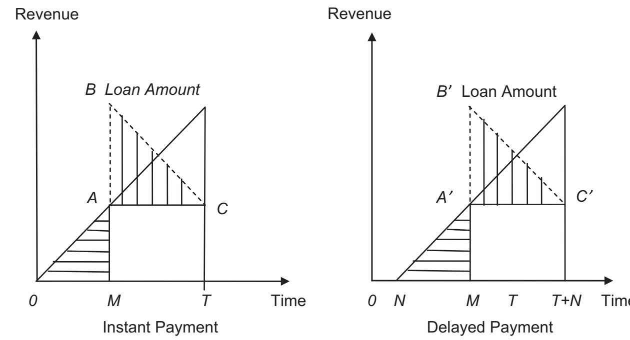

Fig. 1. $M > N$ and $M \leq T$.

Notice that the seller's interest payable should be based on the number of unsold supplier's parts, not the number of finished goods in stock. For example, the seller receives 2000 parts from the supplier at time 0, uses 100 parts to make 100 finished goods per day, sells 50 goods per day, and stores 50 goods per day. On Day 1, the seller has 50 finished goods in stock but 1950 supplier's parts unsold. If there is no trade credit, then the seller's interest charge is based on those 1950 unsold parts, not 50 finished goods in stock. In addition, the seller offers customers a permissible delay of N, and hence receives money from N to T+N, not from 0 to T. Mahata (2012) did not recognize those two facts, and inappropriately calculated the annual interest payable as follow:

$$\frac{cI_c}{\theta^2 T}\{(P-D)[1-\theta M - \exp(-\theta M)] + \theta(Pt_1 - DT)\}, \quad (12)$$

which is significantly different from (11). As a result the seller's annual total relevant cost by using (6)–(9) and (11) is:

$$TRC_1(T) = \frac{A}{T} + \frac{h+\theta\ c}{\theta\ T}(Pt_1 - DT) + \frac{cI_cD}{2T}\left[\alpha(T-M)^2 + (1-\alpha)(T+N-M)^2\right] - \frac{pI_eD}{2T}\left[\alpha M^2 + (1-\alpha)(M-N)^2\right]. \quad (13)$$

3.1.2. Sub-case 1.2. $T \leq M \leq T+N$.

Again, the seller accumulates revenue and earns interest from two accounts: (1) the portion of instant payment starting time 0 through M, and (2) the portion of delayed payment starting time N through M. Hence, the seller's annual interest earned as shown in Fig. 2 as

$$\frac{pI_eD}{2T}\left[\alpha T^2 + 2\alpha T(M-T) + (1-\alpha)(M-N)^2\right]. \quad (14)$$

Likewise, the seller receives all instant payment by time T $(\leq M)$ so that there is no interest payable for the portion of instant payment. However, the seller must finance all items sold during time interval $[M-N,\ T]$. Therefore, the annual interest payable is

$$\frac{cI_cD}{2T}(1-\alpha)(T+N-M)^2. \quad (15)$$

Consequently, the seller's annual total relevant cost by using (6)–(8), (14) and (15) is

$$TRC_2(T) = \frac{A}{T} + \frac{h+\theta\ c}{\theta\ T}(Pt_1 - DT) + \frac{cI_cD}{2T}(1-\alpha)(T+N-M)^2 - \frac{pI_eD}{2T}\left[\alpha T^2 + 2\alpha T(M-T) + (1-\alpha)(M-N)^2\right]. \quad (16)$$

3.1.3. Sub-case 1.3. $T+N \leq M$.

In this sub-case, the seller receives the total revenue before the trade credit period M, and hence there is no interest payable.

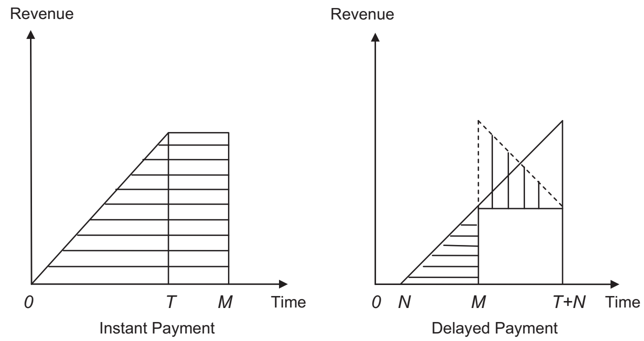

Fig. 2. $M > N$ and $T \le M \le T+N$.

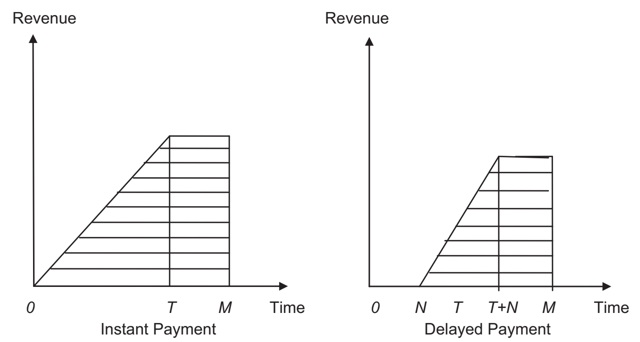

Fig. 3. $M > N$ and $T+N \le M$.

From Fig. 3, we know that the annual interest earned is:

$$\frac{pI_eD}{2T}\left[\alpha T^2 + 2\alpha T(M-T) + (1-\alpha)T^2 + 2(1-\alpha)T(M-T-N)\right] = \frac{pI_eD}{2}[2M-T-2(1-\alpha)N]. \tag{17}$$

Hence, the seller's annual total relevant cost is

$$TRC_3(T) = \frac{A}{T} + \frac{h+\theta c}{\theta T}(Pt_1 - DT) - \frac{pI_eD}{2}[2M-T-2(1-\alpha)N]. \tag{18}$$

Combining (13), (16), and (18), the seller's annual total relevant cost is given as

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } M \le T \\ TRC_2(T), & \text{if } T \le M \le T+N \\ TRC_3(T), & \text{if } T+N \le M \end{cases}$$

Hence, $TRC(T)$ is continuous in T, and has the following properties

$$TRC_1(M) = TRC_2(M), \text{ and } TRC_2(M-N) = TRC_3(M-N). \tag{19}$$

Then we proceed with the case of $N \ge M$.

3.2. Annual total relevant cost for the case of $N \ge M$

Now based on values of M and T, the following two sub-cases can occur: (i) $M \le T$, and (ii) $M \ge T$. Let's discuss them accordingly.

3.2.1. Sub-case 2.1. $M \le T$

From Fig. 4 we know the annual interest earned from the instant payment is

$$\frac{pI_eD}{2T}(\alpha M^2). \tag{20}$$

In this sub-case, for instant payment the seller must finance $\alpha cD(T-M)$ at time M, and pay off the loan at time T. As to delayed payment, the seller must finance $(1-\alpha)cDT$ for delayed payment at time M, and pay off the loan at time $T+N$. Therefore, the annual interest payable is

$$\frac{cI_cD}{2T}\left\{\alpha(T-M)^2 + (1-\alpha)T[T+2(N-M)]\right\}. \tag{21}$$

Consequently, the annual total relevant cost is

$$TRC_4(T) = \frac{A}{T} + \frac{h+\theta c}{\theta T}(Pt_1 - DT) + \frac{cI_cD}{2T}\left\{\alpha(T-M)^2 + (1-\alpha)T[T+2(N-M)]\right\} - \frac{pI_eD}{2T}\alpha M^2. \tag{22}$$

We then discuss the last sub-case in which $N \ge M \ge T$.

3.2.2. Sub-case 2.2. $M \ge T$.

From Fig. 5 we know the annual interest earned from the instant payment is

$$\frac{pI_eD}{2}\alpha[T+2(M-T)]. \tag{23}$$

In this sub-case, there is no interest payable for instant payment. However, the seller must finance $(1-\alpha)cDT$ for delayed payment at time M, and pay off the loan at time $T+N$. Therefore, the annual interest payable is

$$\frac{cI_cD}{2}(1-\alpha)[T+2(N-M)]. \tag{24}$$

Consequently, the seller's annual total relevant cost is

$$TRC_5(T) = \frac{A}{T} + \frac{h+\theta c}{\theta T}(Pt_1 - DT) + \frac{cI_cD}{2}(1-\alpha)[T+2(N-M)] - \frac{pI_eD}{2}\alpha[T+2(M-T)]. \tag{25}$$

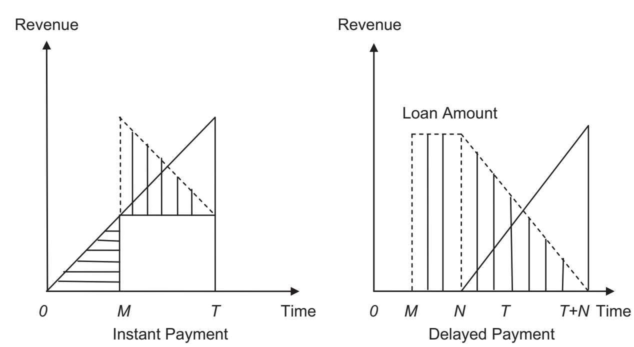

Fig. 4. $M \le N$ and $M \le T$.

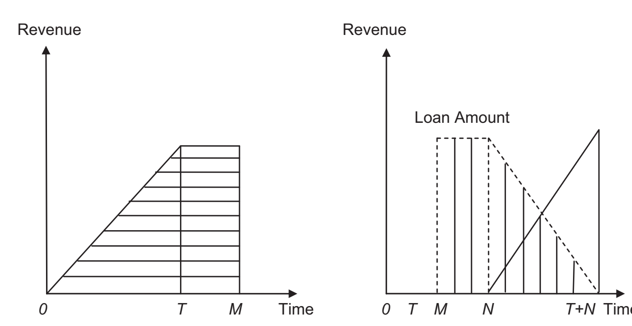

Fig. 5. $M \le N$ and $M \ge T$.

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Combining (22) and (25), we know that the seller's annual total relevant cost is

$$TRC(T)=\begin{cases} TRC_4(T), & if\ \ M\le T \\ TRC_5(T), & if\ \ M\ge T \end{cases}$$

It is clear that $TRC(T)$ is continuous in T, and has the following properties

$$TRC_4(M)=TRC_5(M). \tag{26}$$

Now, our aim is to determine the optimal replenishment cycle T^* for both cases of $N<M$, and $N\ge M$.

4. Determination of the optimal replenishment cycle

In this section, the necessary and sufficient conditions for the determination of the optimal replenishment cycle are presented for the case of $N<M$ first, and then the case of $N\ge M$. According to Theorem 3.2.10 in Cambini and Martein (2009), the function

$$q(x)=\frac{f(x)}{g(x)} \tag{27}$$

is pseudo-convex, if $f(x)$ is non-negative, differentiable and convex, and $g(x)$ is positive, differentiable and concave. Applying Theorem 3.2.9 in Cambini and Martein (2009), we know that $q(x)$ is strictly pseudo-convex if $f(x)$ is strictly convex. Let's apply the above theoretical results to obtain the optimal solution T^* such that $TRC_i(T^*)$ for $i=1,\ 2,\ldots,\ 5$ is minimized.

4.1. Optimal replenishment cycle time for the case of $N<M$

By applying the above mentioned results, we separately minimize each of $TRC_i(T)$ for $i=1,\ 2,$and 3; and obtain the following theoretical results.

Theorem 1.

(1) $TRC_1(T)$ is a strictly pseudo-convex function in T, and hence exists a unique minimum solution T_1^*.
(2) If $M\le T_1^*$, then $TRC_1(T)$ subject to $M\le T$ is minimized at T_1^*.
(3) If $M\ge T_1^*$, then $TRC_1(T)$ subject to $M\le T$ is minimized at M.

Proof. See Appendix A.

To find T_1^*, taking the first-order derivative of $TRC_1(T)$, setting the result to zero, and re-arranging terms, we get

$$\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\} \\ +\frac{D[\alpha M^2+(1-\alpha)(M-N)^2](pI_e-cI_c)}{2}+\frac{cI_cDT^2}{2}-A=0. \tag{28}$$

From Theorem 1, we know that (28) has a unique solution, T_1^*. If $T_1^*\ge M$, then $TRC_1(T)$ is minimized at T_1^*. Otherwise, $TRC_1(T)$ is minimized at M.

By using the analogous argument, we have the following results.

Theorem 2.

(1) $TRC_2(T)$ is a strictly pseudo-convex function in T, and hence exists a unique minimum solution T_2^*.
(2) If $M-N\le T_2^*\le M$, then $TRC_2(T)$ subject to $T\le M\le T+N$ is minimized at T_2^*.
(3) If $T_2^*\le M-N$, then $TRC_2(T)$ subject to $T\le M\le T+N$ is minimized at $M-N$.
(4) If $T_2^*\ge M$, then $TRC_2(T)$ subject to $T\le M\le T+N$ is minimized at M.

Proof. See Appendix B.

To get T_2^*, taking the first-order derivative of $TRC_2(T)$, setting the result to zero, and re-arranging terms, we get

$$\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\} \\ +\frac{D(1-\alpha)(M-N)^2(pI_e-cI_c)}{2}+\frac{DT^2[\alpha pI_e+(1-\alpha)cI_c]}{2}-A=0. \tag{29}$$

It is clear from Theorem 2 that (29) has a unique solution T_2^*. If $M-N\le T_2^*\le M$, then $TRC_2(T)$ is minimized at T_2^*. If $T_2^*\le M-N$, then $TRC_2(T)$ is minimized at $M-N$. If $M\le T_2^*$, then $TRC_2(T)$ is minimized at M.

Finally, for the case of $N<M$, we have the following similar results for $TRC_3(T)$.

Theorem 3.

(1) $TRC_3(T)$is a strictly pseudo-convex function in T, and hence exists a unique minimum solution T_3^*.
(2) If $T_3^*\le M-N$, then $TRC_3(T)$ subject to $T+N\le M$ is minimized at T_3^*.
(3) If $T_3^*\ge M-N$, then $TRC_3(T)$ subject to $T+N\le M$ is minimized at $M-N$.

Proof. See Appendix C.

Similarly, taking the first-order derivative of $TRC_3(T)$, setting the result to zero, and re-arranging terms, we get

$$\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\}+\frac{DT^2pI_e}{2}-A=0. \tag{30}$$

From Theorem 3, we know that (30) has a unique solution T_3^*. If $T_3^*\le M-N$, then $TRC_3(T)$ is minimized at T_3^*. If $T_3^*\ge M-N$, then $TRC_3(T)$ is minimized at $M-N$.

For simplicity, let's define two discrimination terms.

$$\Delta_1=\frac{P(h+\theta\ c)}{\theta}\left\{\frac{D(M-N)e^{\theta(M-N)}}{P+D[e^{\theta(M-N)}-1]}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta(M-N)}-1)\right]\right\} \\ +\frac{D(M-N)^2pI_e}{2}-A, \tag{31}$$

and

$$\Delta_2=\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DMe^{\theta\ M}}{P+D[e^{\theta\ M}-1]}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta\ M}-1)\right]\right\} \\ +\frac{D[\alpha M^2+(1-\alpha)(M-N)^2](pI_e-cI_c)}{2}+\frac{DM^2cI_c}{2}-A. \tag{32}$$

Then we have the following results.

Lemma 1. $\Delta_1<\Delta_2$.

Proof. See Appendix D.

Theorem 4.

(1) If $\Delta_2<0$, then $TRC(T)$ is minimized at T_1^*.
(2) If $\Delta_2=0$, then $TRC(T)$ is minimized at M.
(3) If $\Delta_1<0$ and $\Delta_2>0$, $TRC(T)$ is minimized at T_2^*.
(4) If $\Delta_1=0$, then $TRC(T)$ is minimized at $M-N$.
(5) If $\Delta_1>0$, then $TRC(T)$ is minimized at T_3^*.

Proof. See Appendix E.

4.2. Optimal replenishment cycle time for the case of $N \geq M$

Similar to the case of $N<M$, we have the following results.

Theorem 5.

(1) $TRC_4(T)$ is a strictly pseudo-convex function in T, and hence exists a unique minimum solution T_4^*.
(2) If $M \leq T_4^*$, then $TRC_4(T)$ subject to $M \leq T$ is minimized at T_4^*.
(3) If $M \geq T_4^*$, then $TRC_4(T)$ subject to $M \leq T$ is minimized at M.

Proof. The proof is similar to that in Theorem 1, and hence omitted.

To find T_4^*, taking the first derivative of $TRC_4(T)$, setting the result to zero, and re-arranging terms, we derive

$$\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\}+\frac{\alpha DM^2(pI_e-cI_c)}{2}+\frac{cI_cDT^2}{2}-A=0. \quad (33)$$

It is obvious from Theorem 5 that (33) has a unique solution, T_4^*. If $T_4^* \geq M$, then $TRC_4(T)$ is minimized at T_4^*. Otherwise, $TRC_4(T)$ is minimized at M.

By using the analogous argument, we have the following results.

Theorem 6.

(1) $TRC_5(T)$ is a strictly pseudo-convex function in T, and hence exists a unique minimum solution T_5^*.
(2) If $M \geq T_5^*$, then $TRC_5(T)$ subject to $M \geq T$ is minimized at T_5^*.
(3) If $M \leq T_5^*$, then $TRC_5(T)$ subject to $M \geq T$ is minimized at M.

Proof. The proof is omitted.

To get T_5^*, taking the first derivative of $TRC_5(T)$, setting the result to zero, and re-arranging terms, we derive

$$\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\}+\frac{\alpha pI_eDT^2}{2}+\frac{(1-\alpha)cI_cDT^2}{2}-A=0. \quad (34)$$

It is clear from Theorem 6 that (34) has a unique solution, T_5^*. If $T_5^* \leq M$, then $TRC_5(T)$ is minimized at T_5^*. Otherwise, $TRC_5(T)$ is minimized at M.

Again, let's define the third discrimination term.

$$\Delta_3=\frac{P(h+\theta\ c)}{\theta}\left\{\frac{DMe^{\theta\ M}}{P+D[e^{\theta\ M}-1]}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta M}-1)\right]\right\}+\frac{\alpha pI_eDM^2}{2}+\frac{(1-\alpha)cI_cDM^2}{2}-A. \quad (35)$$

We then have the following results.

Theorem 7.

(1) If $\Delta_3<0$, then $TRC(T)$ is minimized at T_4^*.
(2) If $\Delta_3=0$, then $TRC(T)$ is minimized at M.
(3) If $\Delta_3>0$, $TRC(T)$ is minimized at T_5^*.

Proof. The proof is omitted.

5. Numerical examples

In this section, we provide some numerical examples to illustrate several distinct theoretical results as well as to gain some managerial insights.

Table 1
Numerical example for $N<M$.

Case i	P	M	A	p	Δ_1	Δ_2	T_i^*	TRC_i^*
1	3000	0.10	150	75.00	−117.46	−19.87	0.107(T_1^*)	1810.24
2	4000	0.10	100	75.00	−56.06	75.71	0.075(T_2^*)	1667.08
3	3500	0.15	150	100.00	37.41	234.47	0.089(T_3^*)	791.25

Table 2
Numerical example for $N \geq M$.

Case i	P	M	A	N	Δ_3	T_i^*	TRC_i^*
4	3000	0.06	150	1.00	−103.14	0.107(T_4^*)	19,482.10
5	3500	0.09	100	1.10	26.50	0.080(T_5^*)	20,406.00

Example 1. For the case of $N<M$, let's assume that $D=2500$ units per year, $N=0.05$ years, $c=\$50$ per unit, $h=\$15$ per unit per year, $\theta=0.05$, $\alpha=0.05$, $I_c=0.15$ per dollar per year, and $I_e=0.10$ per dollar per year. Since we have three possible alternatives, we provide three cases for each alternative. By using Theorem 4, we easily obtain the optimal solutions on three different sets of parameters as shown in Table 1.

Example 2. For the case of $N \geq M$, we assume that $D=2500$ units per year, $p=\$75$, $c=\$50$ per unit, $h=\$15$ per unit per year, $\theta=0.05$, $\alpha=0.05$, $I_c=0.15$ per dollar per year, and $I_e=0.10$ per dollar per year. Since we have only two possible alternatives, we provide two cases for each alternative. Applying Theorem 7, we can easily get the optimal solutions on two different sets of parameters as shown in Table 2.

6. Conclusions

The use of a down-stream partial trade credit to reduce default risks with credit-risk customers has received a very little attention by the researchers. In this paper, we have built an appropriate EPQ model for deteriorating items in which the seller (e.g., a manufacturer or a retailer) receives an up-stream full trade credit from its supplier while offers a down-stream partial trade credit to its credit-risk customers. In fact, our proposed inventory model forms a general framework that includes many previous models as special cases such as Goyal (1985), Teng (2002), Teng (2009), Teng and Goyal (2009), and others. In addition, we have pointed some inappropriate manipulations on interest earned and payable by Mahata (2012). By applying the theoretical results in convex fractional programs, we have obtained the necessary and sufficient conditions for finding a unique optimal solution. Furthermore, we have proposed three discrimination terms to identify the optimal solution among alternatives. Finally, we have used several numerical examples to show all possible alternatives.

For further research, this paper can be extended in several ways. For instance, we may consider an integrated solution for both the seller and the buyer, or a non-cooperative Nash solution. Also, we could generalize the model to allow for shortages, quantity discount, backlogging, etc. Finally, we could consider the effect of inflation rates on the optimal credit period and cycle time simultaneously.

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Appendix A. Proof of Theorem 1

From (13), let

$$f_1(T)=A+\frac{h+\theta c}{\theta}(Pt_1-DT)+\frac{cI_cD}{2}\left[\alpha(T-M)^2+(1-\alpha)(T+N-M)^2\right]$$
$$-\frac{pI_eD}{2}\left[\alpha M^2+(1-\alpha)(M-N)^2\right]\geq 0,\ \text{if } T>0; \quad \text{(A1)}$$

and

$$g_1(T)=T>0. \quad \text{(A2)}$$

Taking the first- and second-order derivatives of $f_1(T)$, we have:

$$f_1'(T)=\frac{h+\theta c}{\theta}\left[\frac{PDe^{\theta T}}{P+D(e^{\theta T}-1)}-D\right]+cI_cD[T-M+(1-\alpha)N], \quad \text{(A3)}$$

and

$$f_1''(T)=\frac{h+\theta c}{\theta}\left\{\frac{P^2D\theta\, e^{\theta T}(1-D/P)}{[P+D(e^{\theta T}-1)]^2}\right\}+cI_cD>0. \quad \text{(A4)}$$

Therefore, $q_1(T)=(f_1(T)/g_1(T))=TRC_1(T)$ is a strictly pseudo-convex function in T, which completes the proof of Part (1) of Theorem 1. The proof of Parts (2) and (3) immediately follows from Part (1) of Theorem 1. This completes the proof of Theorem 1.

Appendix B. Proof of Theorem 2

From (16), let

$$f_2(T)=A+\frac{h+\theta c}{\theta}(Pt_1-DT)+\frac{cI_cD}{2}(1-\alpha)(T+N-M)^2$$
$$-\frac{pI_eD}{2}\left[\alpha M^2+2\alpha T(M-T)+(1-\alpha)(M-N)^2\right]\geq 0 \text{ if } T>0; \quad \text{(B1)}$$

and

$$g_2(T)=T>0. \quad \text{(B2)}$$

Taking the first- and second-order derivatives of $f_2(T)$, we have:

$$f_2'(T)=\frac{h+\theta c}{\theta}\left[\frac{PDe^{\theta T}}{P+D(e^{\theta T}-1)}-D\right]+cI_cD(1-\alpha)(T+N-M)-pI_eD\alpha(M-T), \quad \text{(B3)}$$

and

$$f_1''(T)=\frac{h+\theta c}{\theta}\left\{\frac{P^2D\theta\, e^{\theta T}(1-D/P)}{[P+D(e^{\theta T}-1)]^2}\right\}+D[\alpha pI_e+(1-\alpha)cI_c]>0. \quad \text{(B4)}$$

Therefore, $q_2(T)=(f_2(T)/g_2(T))=TRC_2(T)$ is a strictly pseudo-convex function in T, which completes the proof of Part (1) of Theorem 2. The proof of Parts (2)–(4) immediately follows from Part (1) of Theorem 2. This completes the proof of Theorem 2.

Appendix C. Proof of Theorem 3

Again, by using (18), we let

$$f_3(T)=A+\frac{h+\theta c}{\theta}(Pt_1-DT)-\frac{pI_eD}{2}\left[2TM-T^2-2(1-\alpha)TN\right]\geq 0,\ \text{if } T>0; \quad \text{(C1)}$$

and

$$g_3(T)=T>0. \quad \text{(C2)}$$

Taking the first- and second-order derivatives of $f_3(T)$, we have:

$$f_3'(T)=\frac{h+\theta c}{\theta}\left[\frac{PDe^{\theta T}}{P+D(e^{\theta T}-1)}-D\right]-pI_eD[M-T-(1-\alpha)N], \quad \text{(C3)}$$

and

$$f_3''(T)=\frac{h+\theta c}{\theta}\left\{\frac{P^2D\theta\, e^{\theta T}(1-D/P)}{[P+D(e^{\theta T}-1)]^2}\right\}+pI_eD>0. \quad \text{(C4)}$$

Therefore, $q_3(T)=(f_3(T)/g_3(T))=TRC_3(T)$ is a strictly pseudo-convex function in T, which completes the proof of Part (1) of Theorem 3. The proof of Parts (2) and (3) immediately follows from Part (1) of Theorem 3. This completes the proof of Theorem 3.

Appendix D. Proof of $\Delta_1<\Delta_2$

Since $TRC_2(T)$ is a strictly pseudo-convex function in T, we know from (29) that

$$\frac{d}{dT}TRC_2(T)=\frac{1}{T^2}\left(\frac{P(h+\theta c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\}\right.$$
$$\left.+\frac{D(1-\alpha)(M-N)^2(pI_e-cI_c)}{2}+\frac{DT^2[\alpha pI_e+(1-\alpha)cI_c]}{2}-A\right) \quad \text{(D1)}$$

is an increasing function in T, and hence we get

$$\frac{d}{dT}TRC_2(M-N)=\frac{\Delta_1}{(M-N)^2}<\frac{d}{dT}TRC_2(M)=\frac{\Delta_2}{M^2}. \quad \text{(D2)}$$

Thus, $\Delta_1<\Delta_2$. This completes the proof.

Appendix E. Proof of Theorem 4

From (28), we have

$$\frac{d}{dT}TRC_1(T)=\frac{1}{T^2}\left(\frac{P(h+\theta c)}{\theta}\left\{\frac{DTe^{\theta T}}{P+D(e^{\theta T}-1)}-\frac{1}{\theta}\ln\left[1+\frac{D}{P}(e^{\theta T}-1)\right]\right\}\right.$$
$$\left.+\frac{D[\alpha M^2+(1-\alpha)(M-N)^2](pI_e-cI_c)}{2}+\frac{cI_cDT^2}{2}-A\right). \quad \text{(E1)}$$

If $\Delta_2<0$, then it is clear from (E1) that

$$\lim_{T\to\infty}\frac{d}{dT}TRC_1(T)=\frac{cI_cD}{2}>0,\ \text{and}\ \frac{d}{dT}TRC_1(T)\Big|_{T=M}=\frac{\Delta_2}{M^2}<0 \quad \text{(E2)}$$

By applying the Mean Value Theorem and Theorem 1, we know that there exists a unique $T_1^*\in(M,\infty)$ such that $(d/dT)TRC_1(T_1^*)=0$. Hence, $TRC_1(T)$ is minimizing at the unique point T_1^*, which satisfies (28). By using the analogous argument, we have

$$\frac{d}{dT}TRC_2(M-N)=\frac{\Delta_1}{(M-N)^2}<\frac{d}{dT}TRC_2(M)=\frac{\Delta_2}{M^2}<0, \quad \text{(E3)}$$

which implies $TRC_2(T)$ is minimizing at M. Likewise, we get

$$\lim_{\xi\to 0}\frac{d}{dT}TRC_3(\xi)<\frac{d}{dT}TRC_3(M-N)=\frac{\Delta_1}{(M-N)^2}<0, \quad \text{(E4)}$$

which implies $TRC_3(T)$ is minimizing at $M-N$. Consequently, by using (19), (E3), and (E4), we obtain

$$TRC_1(T_1^*)<TRC_1(M)=TRC_2(M)<TRC_2(M-N)=TRC_3(M-N). \quad \text{(E5)}$$

As a result, we complete the proof that if $\Delta_2<0$, then $TRC(T)$ is minimized at T_1^*. By using the analogous argument, one can easily prove the rest of Theorem 4.

References

Aggarwal, S.P., Jaggi, C.K., 1995. Ordering policies of deteriorating items under permissible delay in payments. Journal of the Operational Research Society 46, 658–662.

Cambini, A., Martein, L., 2009. Generalized Convexity and Optimization: Theory and Applications. Springer, Berlin.

Chang, C.-T., Ouyang, L.-Y., Teng, J.-T., 2003. An EOQ model for deteriorating items under supplier credits linked to ordering quantity. Applied Mathematical Modelling 27, 983–996.

Chen, S.-C., Cárdenas-Barrón, L.E., Teng, J.-T., 2013a. Retailer′s economic order quantity when the supplier offers conditionally permissible delay in payments link to order quantity. International Journal of Production Economics, In press, http://dx.doi.org/10.1016/j.ijpe.2013.05.032.

Chen, S.-C., Chang, C.-T., Teng, J.-T., 2013b. A comprehensive note on "Lot-sizing decisions for deteriorating items with two warehouses under an order-size-dependent trade credit". International Transactions in Operational Research. (In press).

Chern, M.-S., Pan, Q., Teng, J.-T., Chan, Y.-L., Chen, S.-C., 2013. Stackelberg solution in a vendor-buyer supply chain model with permissible delay in payments. International Journal of Production Economics 144 (1), 397–404.

Chung, K.-J., Cárdenas-Barrón, L.E., 2013. The simplified solution procedure for deteriorating items under stock-dependent demand and two-level trade credit in the supply chain management. Applied Mathematical Modelling 37 (7), 4653–4660.

Goyal, S.K., 1985. Economic order quantity under conditions of permissible delay in payments. Journal of the Operational Research Society 36, 335–338.

Grubbstrom, R.W., 1980. A principle for determining the correct capital costs of work-in-progress and inventory. International Journal of Production Research 18 (2), 259–271.

Harris, F.W., 1913. How many parts to make at once. Factory, The Magazine of Management 10, 135–136. (152).

Hu, F., Liu, D., 2010. Optimal replenishment policy for the EPQ model with permissible delay in payments and allowable shortages. Applied Mathematical Modelling 34 (10), 3108–3117.

Huang, Y.-F., 2003. Optimal retailer′s replenishment policy for the EPQ model under the supplier′s trade credit policy. Production Planning and Control 15 (1), 27–33.

Jamal, A.M.M., Sarker, B.R., Wang, S., 1997. An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. Journal of the Operational Research Society 48, 826–833.

Liao, J.-J., 2007. On an EPQ model for deteriorating items under permissible delay in payments. Applied Mathematical Modelling 31 (3), 393–403.

Liao, J.-J., Huang, K.-N., Chung, K.-J., 2012. Lot-sizing decisions for deteriorating items with two warehouses under an order-size-dependent trade credit. International Journal of Production Economics 137, 102–115.

Mahata, G.C., 2012. An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain. Expert Systems with Applications 39 (3), 3537–3550.

Min, J., Zhou, Y.W., Zhao, J., 2010. An inventory model for deteriorating items under stock-dependent demand and two-level trade credit. Applied Mathematical Modelling 34, 3273–3285.

Min, J., Zhou, Y.W., Liu, G.-Q., Wang, S.-D., 2012. An EPQ model for deteriorating items with inventory-level-dependent demand and permissible delay in payments. International Journal of Systems Science 43 (6), 1039–1053.

Ouyang, L.-Y., Chang, C.-T., 2013. Optimal production lot with imperfect production process under permissible delay in payments and complete backlogging. International Journal of Production Economics 144 (2), 610–617.

Ouyang, L.-Y., Teng, J.-T., Chen, L.-H., 2006. Optimal ordering policy for deteriorating items with partial backlogging under permissible delay in payments. Journal of Global Optimization 34, 245. (–71).

Skouri, K., Konstantaras, I., Papachristos, S., Teng, J.-T., 2011. Supply chain models for deteriorating products with ramp type demand rate under permissible delay in payments. Expert Systems with Applications 38, 14861–14869.

Teng, J.-T., 2002. On the economic order quantity under conditions of permissible delay in payments. Journal of the Operational Research Society 53, 915–918.

Teng, J.-T., 2009. Optimal ordering policies for a retailer who offers distinct trade credits to its good and bad credit customers. International Journal of Production Economics 119 (2), 415–423.

Teng, J.-T., Chang, C.-T., Chern, M.-S., 2012a. Supple chain vendor-buyer inventory models with trade credit financing under a non-cooperative and an integrated environments. International Journal of Systems Science 43 (11), 2050–2061.

Teng, J.-T., Chen, J., Goyal, S.K., 2009. A comprehensive note on: an inventory model under two levels of trade credit and limited storage space derived without derivatives. Applied Mathematical Modelling 33, 4388–4396.

Teng, J.-T., Goyal, S.K., 2009. A note on: optimal inventory replenishment policy for the EPQ model under trade credit derived without derivatives. International Journal of Systems Science 40, 1095–1098.

Teng, J.-T., Krommyda, I.P., Skouri, K., Lou, K.-R., 2011. A comprehensive extension of optimal ordering policy for stock-dependent demand under progressive payment scheme. European Journal of Operational Research 215, 97–104.

Teng, J.-T., Min, J., Pan, Q., 2012b. Economic order quantity model with trade credit financing and non-decreasing demand. Omega 40, 328–335.

Tsao, Y.-C., 2012. Determination of production run time and warranty length under system maintenance and trade credits. International Journal of Systems Science 43 (12), 2351–2360.