All-to-all broadcast problem of some classes of graphs under the half duplex all-port model

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ABSTRACT

All-to-all communication occurs in many important applications in parallel processing. In this paper, we study the all-to-all broadcast number (the shortest time needed to complete the all-to-all broadcast) of graphs under the assumption that: each vertex can use all of its links at the same time, and each communication link is half duplex and can carry only one message at a unit of time. We give upper and lower bounds for the all-to-all broadcast number of graphs and give formulas for the all-to-all broadcast number of trees, complete bipartite graphs and double loop networks under this model.

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1. Introduction

Broadcasting refers to the process of message dissemination in a connected network. This problem has been extensively investigated in recent years for many different networks and under a large variety of models. For an account of the history of the area of broadcasting and the intensive research devoted to it, see the surveys [6,13,14] and [15]. For different variations and models of the broadcasting problem, such as the *k-broadcasting* problem (each informed vertex can send the message to at most *k* of its neighbors in a given time unit), the *line broadcasting* problem (an informed vertex can send the message to any other vertex through a path between the two vertices in a given time unit), the *multiple originator broadcasting* problem (broadcast a message from a set of originators), the *multiple message broadcasting* problem (*m* messages, originated by one vertex, are transmitted to all vertices of the network), the *all-to-all personalized communication* problem (each vertex sends a specific message to every other vertex), see [1–5,7–12,16–19] and the references there in.

Broadcasting can be one-to-all or all-to-all. In one-to-all broadcasting, we assume that some messages are sent from one vertex, the *originator*, to all vertices of the graph. And in all-to-all broadcasting, we assume that each vertex in the graph has a message needed to be sent to every other vertices of the graph.

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Different models arise when considering the broadcast problem. The models studied most frequently in the literature are the 1-*port* model and the *all-port* model. In the 1-port model, a vertex can only use one of its edges to transmit or receive messages in a unit of time. And in the all-port model, a vertex can use all its edges to transmit or receive messages in a unit of time.

Here, we study the broadcast problem under the *half duplex* (only one message can travel a link at a time unit) all-port model with the restriction that each edge can transmit at most one message during each time unit (we add this constraint since in real applications, the size of the messages that can be transferred by a link at a time unit may be limited). That is, we assume that at each time unit, vertices exchange their messages under the following rules:

(1) communication links are half duplex,
(2) each edge can transmit at most one message during each time unit,
(3) each vertex can use all of its links at the same time.

We call such model the HA1 model. Note that in this model, a vertex can send and receive messages during the same time unit, and a vertex can receive multiple messages during the same time unit.

In this paper, we study the all-to-all broadcast problem of some classes of graphs which are used frequently in network under the HA1 model. We give upper and lower bounds for all-to-all broadcast number (the shortest time needed to complete all-to-all broadcast) of graphs and give formulas for all-to-all broadcast number of trees, complete bipartite graphs and double loop networks.

2. Preliminary

All graphs we consider in this paper are connected and simple (loopless and without multiple edges). We first fix some notations that will be used later. Given a graph G and a vertex v in $V(G)$, we use $m_0(v) = \{a\}$ to denote that at the beginning, v owns a private message a. And we use $\{a\}^i_{\overrightarrow{uv}}$ to denote that at the ith time unit, u send the message a to vertex v, and call $\{a\}^i_{\overrightarrow{uv}}$ a *call*. A set of calls $B(G)$ is a *scheme* of G if for each $\{a\}^i_{\overrightarrow{uv}} \in B(G)$, $a \in \{b : \{b\}^l_{\overrightarrow{wu}} \in B(G), 1 \le l \le i-1\} \cup m_0(u)$. For a scheme $B(G)$ of G, we use $\Delta_{B(G)}$ to denote the maximum number of time used in $B(G)$ (that is, $\Delta_{B(G)} = \max\{i : \{a\}^i_{\overrightarrow{uv}} \in B(G)\}$), and for all v in $V(G)$ and all i, $1 \le i \le \Delta_{B(G)}$, we use $(m_i(v))_{B(G)}$ (or, simply, $m_i(v)$, if $B(G)$ need not to be specified) to denote the set of messages received by vertex v till time unit i (that is, $(m_i(v))_{B(G)} = \{a : \{a\}^l_{\overrightarrow{uv}} \in B(G), 1 \le l \le i\} \cup m_0(v)$).

A scheme $B(G)$ of G is a *broadcasting set* of G if $(m_{\Delta_{B(G)}}(v))_{B(G)} = \bigcup_{w \in V(G)} m_0(w)$ for all $v \in V(G)$ (that is, if all vertices receive all the messages in $\bigcup_{w \in V(G)} m_0(w)$ by using the scheme $B(G)$). A broadcasting set of G is a *k-step broadcasting set* of G if $\Delta_{B(G)} = k$. For a graph G, the *all-to-all broadcast number* of G is the number $t(G) = \min\{\Delta_{B(G)} : B(G)$ is a broadcasting set of $G\}$. A broadcasting set $B(G)$ is called an *optimal broadcasting set* if $\Delta_{B(G)} = t(G)$.

From now on, when considering a scheme $B(G)$ of G, we always assume that no vertex receives the same message twice. And, when considering the graphs K_n and C_n, we always assume that $V(K_n) = V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $m_0(v_i) = \{i\}$ for all i, $0 \le i \le n-1$. The following lemma is easy to verify.

Lemma 1. *If H is a spanning subgraph of a graph G, then $t(G) \le t(H)$.*

Lemma 2. *For any graph G with $|V(G)| = n$ and $|E(G)| = m$, $t(G) \ge \left\lceil \frac{n(n-1)}{m} \right\rceil$.*

Proof. The total number of messages need to be transmitted is $n(n-1)$. Since at most m messages can be transmitted at each time unit (see Figs. 1–3), it follows that $\left\lceil \frac{n(n-1)}{m} \right\rceil$ time units are necessary. ■

Lemma 3. *For any graph G with $|V(G)| = n$ and $|E(G)| = m$, if $B(G)$ is a k-step broadcasting set of G such that for some $r > 0$, the number of calls in $B(G)$ at each time unit till time unit r is equal to m, and the total number of calls in $B(G)$ from time unit $r+1$ to time unit k is greater than or equal to $(k-r-1)m + 1$, then $B(G)$ is an optimal broadcasting set of G, and $t(G) = \left\lceil \frac{n(n-1)}{m} \right\rceil$.*

Proof. Note that for any broadcasting set $B(G)$ of G, the number of messages that can be transmitted at each time unit is at most m. If $B(G)$ is a broadcasting set of G such that the number of messages that are transmitted at each time unit till time unit r is equal to m and the total number of messages that are transmitted from time unit $r+1$ to time unit k is greater than or equal to $(k-r-1)m + 1$, then, since the total number of messages need to be transmitted is $n(n-1)$, $n(n-1) \ge rm + (k-r-1)m+1 = (k-1)m+1$. Hence $t(G) \ge \left\lceil \frac{n(n-1)}{m} \right\rceil \ge \left\lceil \frac{(k-1)m+1}{m} \right\rceil = k$ by Lemma 2. Since $B(G)$ is a k-step broadcasting set of G, $B(G)$ is an optimal broadcasting set of G and $t(G) = k = \left\lceil \frac{n(n-1)}{m} \right\rceil$. ■

For a connected graph G, the *distance* between two vertices u and v, written $d_G(u, v)$ (or simply, $d(u, v)$), is the minimum length of all u–v paths in G. For a vertex v in G, the *eccentricity* of v, written $\epsilon_G(v)$ (or simply, $\epsilon(v)$), is $\max_{u \in V(G)} d_G(u, v)$.

Lemma 4. *Given a graph G with $|V(G)| = n$, if uv is a cut-edge of G and G_1, G_2 are components of $G - uv$ containing u, v, respectively, then*

$$t(G) \ge n + \min\{\epsilon_{G_1}(u), \epsilon_{G_2}(v)\}.$$

Proof. Let $B(G)$ be an optimal broadcasting set of G. Note that since uv is a cut-edge of G, every message in $\cup_{v \in V(G)} m(v)$ must pass through uv at some transmitting step. If in $B(G)$, a is the last message in $\cup_{v \in V(G)} m(v)$ that pass through uv, then there exists k, $k \geq n$, such that $\{a\}_{\overrightarrow{uv}}^k \in B(G)$ or $\{a\}_{\overrightarrow{vu}}^k \in B(G)$. If $\{a\}_{\overrightarrow{uv}}^k \in B(G)$, and x is a vertex in G_2 such that $d_{G_2}(v, x) = \epsilon_{G_2}(v)$, then there exists a vertex y and a number k', such that $\{a\}_{\overrightarrow{yx}}^{k'} \in B(G)$. Clearly, $k' \geq k + d_{G_2}(v, x)$. Thus $t(G) = \Delta_{B(G)} \geq k' \geq k + d_{G_2}(v, x) \geq n + \epsilon_{G_2}(v)$ in this case. By a similar argument, $t(G) \geq n + \epsilon_{G_1}(u)$ if $\{a\}_{\overrightarrow{vu}}^k \in B(G)$. Hence $t(G) \geq n + \min\{\epsilon_{G_1}(u), \epsilon_{G_2}(v)\}$. ∎

Theorem 5. $t(G) \geq 2$ for any graph G with $|V(G)| \geq 2$. And equality holds if and only if $G = K_n, n \geq 2$.

Proof. $t(G) \geq 2$ for any graph G with $|V(G)| \geq 2$ follows from Lemma 2. For the graph K_n, consider the scheme $B(K_n)$ of K_n defined by $B(K_n) = \{\{j\}_{\overrightarrow{v_j v_k}}^1 : 0 \leq j < k \leq n-1\} \cup \{\{k\}_{\overrightarrow{v_k v_j}}^2 : 0 \leq j < k \leq n-1\}$. Since each vertex v_i received all the messages in $\{0, 1, \ldots, i-1\}$ at the first time unit, and received all the messages in $\{i+1, i+2, \ldots, n-1\}$ at the second time unit, $B(K_n)$ is a 2-step broadcasting set of K_n. Thus $t(K_n) \leq 2$. By Lemma 2, we also have $t(K_n) \geq \left\lceil \frac{n(n-1)}{|E(K_n)|} \right\rceil = 2$. Hence $t(K_n) = 2$ for all $n \geq 2$. Note that the equality does not hold if $|E(G)| < \frac{n(n-1)}{2}$. Hence $t(G) = 2$ if and only if $G = K_n, n \geq 2$. ∎

From now on, for convenience, we use the notation $[k]_n$ to denote the number $(k \bmod n)$.

Theorem 6. $t(C_n) = n - 1$ for all $n \geq 3$.

Proof. Consider the scheme $B(C_n)$ of C_n, defined by $B(C_n) = \{\{[j+1-i]_n\}_{\overrightarrow{v_j v_{[j+1]_n}}}^i : 1 \leq i \leq n-1, 0 \leq j \leq n-1\}$. Since each vertex v_j received the message $[j-i]_n$ at the ith time unit, $(m_{n-1}(v_i))_{B(C_n)} = \{0, 1, 2, \ldots, n-1\}$ for each vertex v_i. Hence $B(C_n)$ is an $(n-1)$-step broadcasting set of C_n, and so $t(C_n) \leq n-1$. By Lemma 2, we also have $t(C_n) \geq \left\lceil \frac{n(n-1)}{|E(C_n)|} \right\rceil = n-1$. Hence $t(C_n) = n-1$ for all $n \geq 3$. ∎

3. Trees

Given a graph G and a vertex v in $V(G)$, the *(open) neighborhood* of v is $N(v) = \{u \in V(G) : uv \in E(G)\}$, and the *eccentricity* of v, written $\epsilon_G(v)$ (or simply, $\epsilon(v)$), is $\max_{u \in V(G)} d(u, v)$. For a graph G, the *radius* of G, written $\mathrm{rad}(G)$, is $\min_{u \in V(G)} \epsilon(u)$. A vertex v in $V(G)$ is a *central vertex* of G if $\epsilon(v) = \mathrm{rad}(G)$.

A *rooted tree* is a tree with one vertex chosen as the root. We use T_v to denote a tree T rooted at v. The *height* $h(T_v)$ of T_v is defined by $h(T_v) = \max\{d(v, w) : w \in V(T)\}$. For a vertex u in T_v, we use T_u to denote the subtree of T_v rooted at u. For each vertex u in $V(T_v)$, let $P_{T_v}(u)$ be the unique u–v path in T_v. The *parent* of u in T_v, denoted by $u_{p|_{T_v}}$ (or simply, u_p, if T_v is the only rooted tree we considered), is its neighbor on $P_{T_v}(u)$. And the *children* of u in T_v are the neighbors of u different from $u_{p|_{T_v}}$.

Algorithm BT
Input: A tree T with $V(T) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ rooted at v_0.
Output: A broadcasting set $B(T_{v_0})$ of T_{v_0} and the time units t needed to complete the broadcast.
**Method:**
1. Set $t = 0, B(T_{v_0}) = \emptyset$, and for each vertex v_i, set $m_0(v_i) = \{i\}$.
2. While $\cap_{v \in V(T)} m_t(v) \neq \{0, \ldots, n-1\}$, //at least one vertex did not receive all the messages in $\{0, \ldots, n-1\}$//
    do {
    For each $v_i v_j \in E(T)$, and $d(v_i, v_0) > d(v_j, v_0)$, do {
      If $m_t(v_i) \setminus m_t(v_j) \neq \emptyset$, choose any $x \in m_t(v_i) \setminus m_t(v_j)$,
        set $B(T_{v_0}) = B(T_{v_0}) \cup \{\{x\}_{\overrightarrow{v_i v_j}}^t\}$, and set $m_{t+1}(v_j) = m_t(v_j) \cup \{x\}$;
      Else if $m_t(v_j) \setminus m_t(v_i) \neq \emptyset$, choose any $x \in m_t(v_j) \setminus m_t(v_i)$,
        set $B(T_{v_0}) = B(T_{v_0}) \cup \{\{x\}_{\overrightarrow{v_j v_i}}^t\}$, and set $m_{t+1}(v_i) = m_t(v_i) \cup \{x\};\}$
    $t = t + 1;\}$.
3. Return $B(T_{v_0})$ and t.

In Algorithm BT, we first choose a vertex v_0 as a root, and consider the rooted tree T_{v_0}. At each time unit t, $t \geq 1$, for a vertex u in T_{v_0}, if $m_{t-1}(u) \nsubseteq m_{t-1}(u_p)$, then we choose a message in $m_{t-1}(u) \setminus m_{t-1}(u_p)$ and send it to u_p at time unit t. And for a child w of u, if $m_{t-1}(w) \subseteq m_{t-1}(u)$ and $m_{t-1}(w) \neq m_{t-1}(u)$, then we choose a message in $m_{t-1}(u) \setminus m_{t-1}(w)$ and send it to w at time unit t. Clearly, when the algorithm terminate, the set $B(T_{v_0})$, produced by this algorithm, is a broadcasting set of T_{v_0}.

Lemma 7. For a rooted tree T_{v_0} with $V(T_{v_0}) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, the set $B(T_{v_0})$, produced by Algorithm BT, is an $(n + h(T_{v_0}) - 1)$-step broadcasting set of T_{v_0}.

Proof. Let $B(T_{v_0})$ and $m_t(v_i), 0 \leq i \leq n-1, 0 \leq t \leq \Delta_{B(T_{v_0})}$, be the sets produced by Algorithm BT. By step 2 of the algorithm, each vertex v_i satisfies the condition that $m_{t_i}(v_i) = \cup_{v_k \in V(T_{v_i})} m_0(v_k)$, where $t_i = |V(T_{v_i})|$. Moreover, if v_j is the parent of v_i,
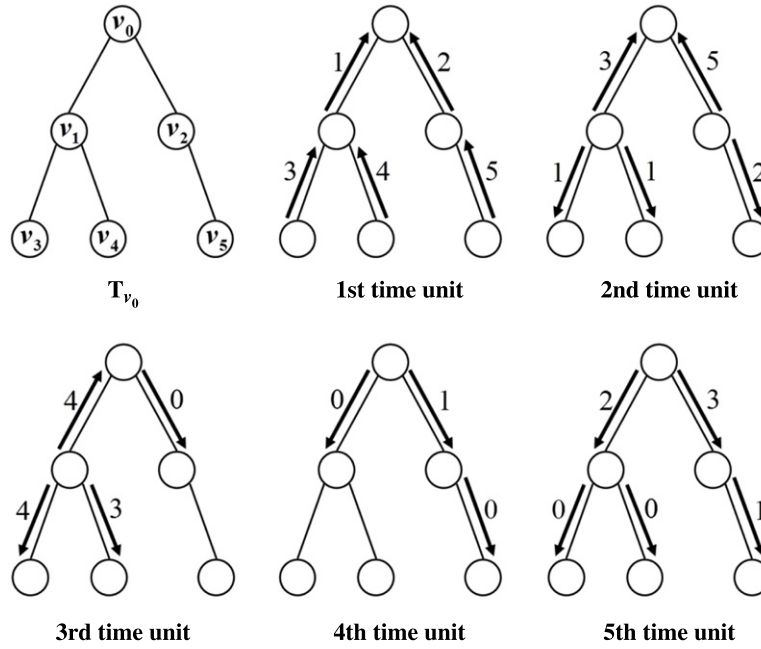
Fig. 1. A tree T and the first few transmission steps of T.
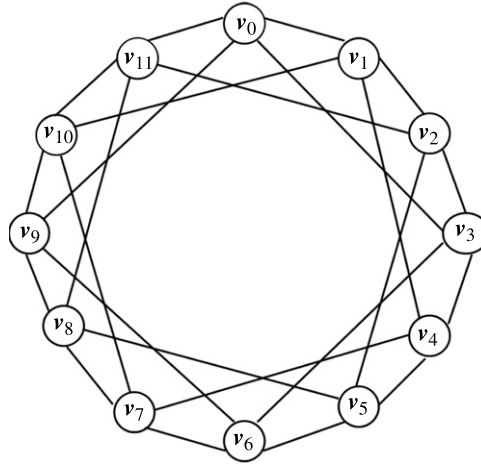
Fig. 2. $D_{12}(1, 3)$.

then $|m_{t_j}(v_k)| \geq |m_{t_j}(v_j)| - d_{T_{v_j}}(v_k, v_j) + 1$ for all vertices v_k in T_{v_i}. Hence $m_{t^*}(v_i) = \{0, 1, 2, \ldots, n-1\}$ for all vertices v_i in T_{v_0}, where $t^* = |m_{t_0}(v_0)| + \max\{d_{T_{v_0}}(v_k, v_0) : v_k \in V(T_{v_0})\} - 1 = n + h(T_{v_0}) - 1$. Therefore, $B(T_{v_0})$ is an $(n+h(T_{v_0})-1)$-step broadcasting set of T_{v_0}. ∎

Theorem 8. *For any tree T with $|V(T)| = n \geq 2$, $t(T) = n + \mathrm{rad}(T) - 1$.*

Proof. Choose a central vertex v in $V(T)$, and consider the rooted tree T_v. By Lemma 7, there exists a $(n + h(T_v) - 1)$-step broadcasting set $B(T_v)$ of T_v. Hence $t(T) \leq \Delta_{B(T_v)} = n + h(T_v) - 1$. Since v is a central vertex, $h(T_v) = \mathrm{rad}(T)$, thus $t(T) \leq n + \mathrm{rad}(T) - 1$. To prove the lower bound, consider a vertex w in $V(T_v)$ with $d(v, w) = \mathrm{rad}(T)$, and let u be a vertex in $N(v)$ so that the rooted subtree T_u of T_v contains the vertex w. Then, by Lemma 4, $t(G) \geq n + d(u, w) = n + \mathrm{rad}(T) - 1$. Hence $t(T) = n + \mathrm{rad}(T) - 1$ for any tree T with $|V(T)| = n \geq 2$. ∎

Corollary 9. $t(P_n) = n + \lfloor \frac{n}{2} \rfloor - 1$ *for all $n \geq 2$.*

Combining Lemma 1 and Theorem 8, we have

Corollary 10. *For any graph G with $|V(G)| \geq 2$, $t(G) \leq |V(G)| + \mathrm{rad}(G) - 1$.*
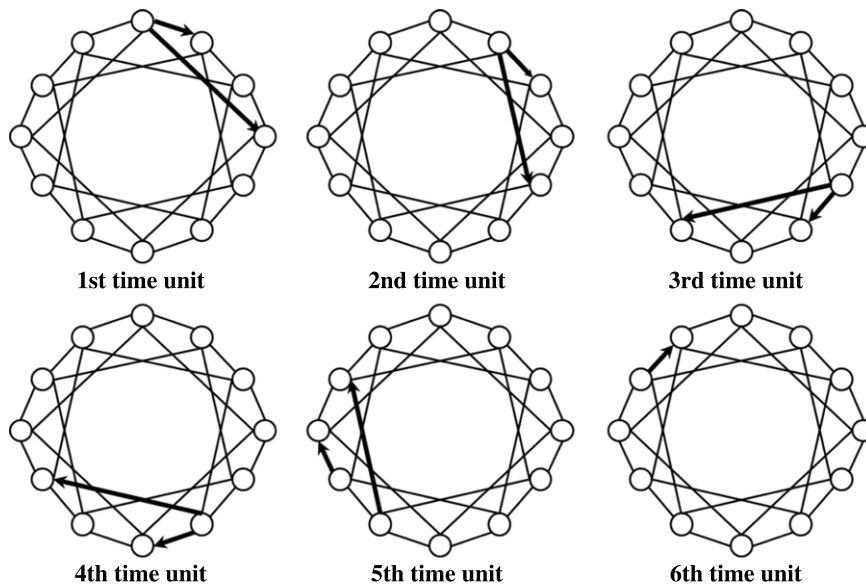
Fig. 3. Transmission of the message 0 in $D_{12}(1, 3)$.

For a cut-edge uv of G, we use $\alpha(u, v)$ to denote the number $\min\{\epsilon_{G_1}(u), \epsilon_{G_2}(v)\}$, where G_1, G_2 are components of $G - uv$ containing u, v, respectively. By Lemma 4 and Corollary 10, we have

Corollary 11. *If G is a graph with cut-edges, then $|V(G)| + \max\{\alpha(u, v) : uv \text{ is a cut-edge of } G\} - 1 \leq t(G) \leq |V(G)| + \mathrm{rad}(G) - 1$.*

4. Complete bipartite graphs

In this section, we study the all-to-all broadcast number of complete bipartite graphs. For convenience, when considering the complete bipartite graph $K_{m,n}$, $m \geq n$, we always assume that $V(K_{m,n}) = \{v_i : 0 \leq i \leq m + n - 1\}$, $E(K_{m,n}) = \{v_iv_j : 0 \leq i \leq m - 1, m \leq j \leq m + n - 1\}$, and $m_0(v_i) = \{i\}$ for all i, $0 \leq i \leq m + n - 1$.

Theorem 12. $t(K_{m,n}) = \left\lceil \frac{(m+n)(m+n-1)}{mn} \right\rceil$ *for all $m \geq n \geq 1$.*

Proof. Let $m - 1 = nq + r$, where $0 \leq r \leq n - 1$, and let $r' = [mr]_n$. Let

$$B^1(K_{m,n}) = \left\{ \{j\}\underset{v_jv_{m+i}}{\xrightarrow{1}} : 0 \leq j \leq m - 1, 0 \leq i \leq n - 1 \right\},$$

$$B^2(K_{m,n}) = \left\{ \{m + i\}\underset{v_{m+i}v_j}{\xrightarrow{2}} : 0 \leq j \leq m - 1, 0 \leq i \leq n - 1 \right\},$$

and for all l, $3 \leq l \leq q + 2$, let

$$B^l(K_{m,n}) = \left\{ \{[(l - 3)n + j + i + 1]_m\}\underset{v_{m+i}v_j}{\xrightarrow{l}} : 0 \leq j \leq m - 1, 0 \leq i \leq n - 1 \right\}.$$

Consider the following cases:

Case 1. $mr + n(n - 1) \leq mn$.

In this case, let

$$B^{q+3}(K_{m,n}) = \left\{ \left\{ \left[nq + i + \left\lfloor \frac{i}{m} \right\rfloor + 1 \right]_m \right\}\underset{v_{m+\left[i+\left\lfloor \frac{id}{mn} \right\rfloor\right]_n}v_{[i]_m}}{\xrightarrow{q+3}} : 0 \leq i \leq mr - 1 \right\}$$

$$\bigcup \left\{ \left\{ m + \left[r' + i + \left\lfloor \frac{rd}{n} \right\rfloor + \left\lfloor \frac{i}{n} \right\rfloor + 1 \right]_n \right\}\underset{v_{[i]_m}v_{m+\left[r'+i+\left\lfloor \frac{rd}{n} \right\rfloor\right]_n}}{\xrightarrow{q+3}} : 0 \leq i \leq n^2 - n - 1 \right\},$$

and let $B(K_{m,n}) = \bigcup_{i=1}^{q+3} B^i(K_{m,n})$.

Case 2. $mr + n(n - 1) > mn$.

In this case, let

$$B^{q+3}(K_{m,n}) = \left\{ \{[qn + j + i + 1]_m\}_{\overrightarrow{v_{m+i}v_j}}^{q+3} : 0 \leq j \leq m - 1, 0 \leq i \leq r - 1 \right\},$$

$$B^{q+4}(K_{m,n}) = \left\{ \{m + [j + i + 1]_n\}_{\overrightarrow{v_j v_{m+i}}}^{q+4} : 0 \leq j \leq n - 2, 0 \leq i \leq n - 1 \right\},$$

and let $B(K_{m,n}) = \cup_{i=1}^{q+4} B^i(K_{m,n})$.

It is easy to verify that for each of the two cases above, $B(K_{m,n})$ is a broadcasting set of $K_{m,n}$. Since $|\{\{a\}_{\overrightarrow{uv}}^i : \{a\}_{\overrightarrow{uv}}^i \in B(K_{m,n})\}| = mn$ for all i, $1 \leq i \leq q+2$, and $|\{\{a\}_{\overrightarrow{uv}}^{q+3} : \{a\}_{\overrightarrow{uv}}^{q+3} \in B(K_{m,n})\}| + |\{\{a\}_{\overrightarrow{uv}}^{q+4} : \{a\}_{\overrightarrow{uv}}^{q+4} \in B(K_{m,n})\}| \geq mn+1$ when $mr + n(n-1) > mn$, by Lemma 3, $B(K_{m,n})$ is an optimal broadcasting set of $K_{m,n}$, and $t(K_{m,n}) = \left\lceil \frac{(m+n)(m+n-1)}{mn} \right\rceil$ in either cases. ∎

5. Double loop network

A *double-loop network* $\overrightarrow{D_n}(a, b)$ with n being positive integer, $0 < a < n$, $0 < b < n$, and $a \neq b$ can be viewed as a directed graph with n vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and $2n$ directed edges of the form $\overrightarrow{v_i v_{[i+a]_n}}$ and $\overrightarrow{v_i v_{[i+b]_n}}$, referred to as a-links and b-links. The underlying graph of the directed graph $\overrightarrow{D_n}(a; b)$ is denoted by $D_n(a, b)$. In this section, we study the all-to-all broadcast number of $D_n(1, b)$. Note that $D_n(1, b)$ is isomorphic to $D_n(1, n - b)$ for all b, $2 \leq b \leq n - 2$. Hence, when consider the graph $D_n(1, b)$, we always assume that $2 \leq b \leq \left\lfloor \frac{n}{2} \right\rfloor$.

For convenience, when considering the graph $D_n(1, b)$, we always assume that $m_0(v_i) = \{i\}$ for all i, $0 \leq i \leq n - 1$.

Theorem 13. $t(D_n(1, b)) = \left\lceil \frac{n-1}{2} \right\rceil$ for all $n \geq 5$, $2 \leq b \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. Let $n - 1 = (2b - 2)q + r$, where $0 \leq r \leq 2b - 3$, and let $j_b = [j - 1]_{b-1} + \left\lfloor \frac{j-1}{b-1} \right\rfloor (2b - 2)$, $j_b^* = 2j - 2$ for all positive integer j. For each i, $0 \leq i \leq n - 1$, let

$$B^i(D_n(1, b)) = \left\{ \{i\}_{\overrightarrow{v_{[i+j_b]_n} v_{[i+j_b+1]_n}}}^j : 1 \leq j \leq (b-1)q \right\}$$

$$\bigcup \left\{ \{i\}_{\overrightarrow{v_{[i+j_b]_n} v_{[i+j_b+b]_n}}}^j : 1 \leq j \leq (b-1)q \right\}$$

$$\bigcup \left\{ \{i\}_{\overrightarrow{v_{[i+j_b^*]_n} v_{[i+j_b^*+1]_n}}}^j : (b-1)q + 1 \leq j \leq (b-1)q + \left\lceil \frac{r}{2} \right\rceil \right\}$$

$$\bigcup \left\{ \{i\}_{\overrightarrow{v_{[i+j_b^*+2-b]_n} v_{[i+j_b^*+2]_n}}}^j : (b-1)q + 1 \leq j \leq (b-1)q + \left\lfloor \frac{r}{2} \right\rfloor \right\},$$

and let $B(D_n(1, b)) = \cup_{i=0}^{n-1} B^i(D_n(1, b))$. By the definition of $B^i(D_n(1, b))$, all vertices v_j own the message i after the $[(b-1)q + \left\lceil \frac{r}{2} \right\rceil]^{\text{th}}$ step. Hence $B(D_n(1, b))$ is a broadcasting set of $D_n(1, b)$. Therefore, $t(D_n(1, b)) \leq \Delta_{B(D_n(1,b))} = (b-1)q + \left\lceil \frac{r}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$. Since $|E(D_n(1, b))| = 2n$, by Lemma 2, we also have $t(D_n(1, b)) \geq \left\lceil \frac{n-1}{2} \right\rceil$. Thus $t(D_n(1, b)) = \left\lceil \frac{n-1}{2} \right\rceil$ for all $n \geq 5$, $2 \leq b \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. ∎

Acknowledgments

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