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On a particular Emden–Fowler equation with non-positive energy $u'' - u^3 = 0$ Mathematical model of enterprise competitiveness and performance

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Abstract

In this paper we work with the ordinary equation $u'' - u^3 = 0$ and obtain some interesting phenomena concerning blow-up, blow-up rate and life-span of solutions to those equations under negative energy.

Keywords: Estimate; Life-span; Blow-up; Blow-up rate; Performance

1. Introduction

How to improve the performance and competitiveness of a company is the critical issue of Industrial and Organizational Psychology in Taiwan. We try to design an appropriate mathematical model of the competitiveness and the performance of the 293 benchmark enterprises out of 655 companies ?

Unexpectedly, we discover the correlation of performance and competitiveness is extremely high, some benchmark enterprises present the following phenomena:

Competitiveness (Force, $F(P(n))$) is a cubic function of the performance ($P(n)$); that is, there exist positive constants $P_0 > 0$ and $k > 0$ such that

$$F(P(n)) = k(P(n) - P_0)^3,$$

where n is the number of departments or the decision makers of one of the surveying benchmark enterprise, $P(n) \geq P_0$ and F is proportional to the second derivative of P with respect to n.

Now we consider the special case $F(P(n)) = M\frac{\mathrm{d}^2 P(n)}{\mathrm{d}n^2}$, and let $u(n) := \sqrt{\frac{k}{M}}(P(n) - P_0) \geq 0$, then we obtain a particular Emden–Fowler equation

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$$\begin{cases} u'' - u^3 = 0, \quad n \geq n_0, \\ u(n_0) = u_0 = \sqrt{\dfrac{k}{M}}(P(n_0) - P_0), \quad u'(n_0) = u_1. \end{cases} \tag{1.1}$$

It is clear that the function u^3 is locally Lipschitz; hence by the standard theory, the local existence of classical solutions is applicable to the Eq. (1.1); for the more general case of Lane–Emden type one can read the paper written by Agarward and O'Regean [9].

We shall use our methods used in [1–8] to deal with the estimates for the life-span of the solutions of (1.1), the blow-up rates and blow-up constants.

Notation and Fundamental Lemma. *For a given function u in this work, we use the following abbreviations:*

$$a(n) = u(n)^2, \qquad E(n) = u'(n)^2 - \frac{1}{2}u(n)^4, \qquad J(n) = a(n)^{-\frac{1}{2}}.$$

Definition. A function $g : \mathbb{R} \to \mathbb{R}$ with a blow-up rate q means that g exists only in finite time; i.e., there is a finite number T^* so that $\lim_{t\to T^*} g(t)^{-1} = 0$ and there exists a non-zero $\beta \in \mathbb{R}$ with $\lim_{t\to T^*}(T^* - t)^q g(t) = \beta$, in this case β is called the blow-up constant of g.

After some elementary calculation we obtain the following lemma.

Lemma. *Suppose that u is the solution of (1.1), then $E(n)$ is a constant and we have*

$$6u'(n)^2 = 4E(n_0) + a''(n), \tag{1.2}$$

$$J''(n) = 2E(n_0)J(n)^3 \tag{1.3}$$

and

$$J'(n)^2 = J'(n_0)^2 - E(n_0)J(n_0)^4 + E(n_0)J(n)^4. \tag{1.4}$$

2. Estimates for the life-span, blow-up rate and blow-up constant under $E(n_0) \leq 0$

To estimate the life-span of the solution of the Eq. (1.1), we separate this section into two parts: $E(n_0) < 0$ and $E(n_0) = 0$, $a'(n_0) > 0$. Here the life-span N of u means that u exists and makes sense only in the interval $[n_0, N)$ so that the problem (1.1) possesses the solution $u \in \bar{C}^2(n_0, N)$. Here we have the following result:

Theorem 2.1. *If N is the life-span of the solution u to (1.1) with $E(n_0) < 0$, then N is finite. Further, for $a'(n_0) \geq 0$ we have the estimate*

$$N \leq N_1^* = n_0 + \int_0^{J(n_0)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}}; \tag{2.1}$$

for $a'(n_0) < 0$,

$$N \leq N_2^* = n_0 + \left(\int_0^{\alpha} + \int_{J(n_0)}^{\alpha}\right) \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}}, \tag{2.2}$$

where $\alpha = (\frac{1}{2}\frac{-1}{E(n_0)})^{\frac{1}{4}}$. Furthermore, if $E(n_0) = 0$ and $a'(n_0) > 0$, then

$$N \leq N_3^* := n_0 + \frac{2a(n_0)}{a'(n_0)}. \tag{2.3}$$

Proof. For $E(n_0) < 0$, we know that $a(n_0) > 0$; otherwise we get $a(n_0) = 0$, that is, $u_0 = 0$, then $E(n_0) = u_1^2 \geq 0$, this contradicts $E(n_0) < 0$.
(i) $a'(n_0) \geq 0$. By (1.2) and (1.4) we find that

$$a'(n) \geq a'(n_0) - 4E(n_0)(n - n_0) \quad \forall n \geq n_0, \tag{2.4}$$

$$J'(n) = -\sqrt{\frac{1}{2} + E(n_0)J(n)^4} \leq J'(n_0) \quad \forall n \geq n_0 \tag{2.5}$$

and $J(n) \leq 0$ for large n. Thus there exists a finite number $N_1^* := N_1^*(u_0, u_1) \leq n_0 + \frac{u_0}{u_1}$ such that $J(N_1^*) = 0$ and so $a(n) \to \infty$ as $n \to N_1^*$. This means that $N \leq N_1^*$. By (2.5) and $J(N_1^*) = 0$ we find that

$$\int_{J(n)}^{J(n_0)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}} = n - n_0 \quad \forall n \geq n_0 \tag{2.6}$$

and hence we get the estimate (2.1).

(ii) $a'(n_0) < 0$. By (2.4), $a'(n_0) < 0$ and the convexity of a we can find a unique finite number $n_1 = n_1(u_0, u_1)$ such that

$$a'(n) > 0 \quad \text{for } n > n_1, \tag{2.7}$$

and $a(n_1) > 0$; we conclude that $a(n) > 0\ \forall n \geq n_0$, $u'(n_1) = 0$, $E(n_0) = -\frac{1}{2}u(n_1)^4$ and $J(n_1)^4 = \frac{-1}{2E(n_1)}$.

After arguments similar to step (i), there exists a finite number $N_2^* := N_2^*(u_0, u_1)$ such that the life-span N of u is bounded by N_2^*, that is, $N \leq N_2^*$. By an analogous argument, using (2.7) and (1.4) and the fact that $J(n_1)^4 = \frac{-1}{2E(n_0)}$ and $J(N_2^*) = 0$, we conclude that

$$J'(n) = -\sqrt{\frac{1}{2} + E(n_0)J(n)^4} \quad \forall n \geq n_1, \tag{2.8a}$$

$$J'(n) = \sqrt{\frac{1}{2} + E(n_0)J(n)^4} \quad \forall n \in [n_0, n_1], \tag{2.8b}$$

$$\int_{J(n)}^{J(n_1)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}} = n - n_1 \quad \forall n \geq n_1, \tag{2.9a}$$

$$\int_{J(n_0)}^{J(n_1)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}} = n_1 - n_0 \tag{2.9b}$$

and

$$N_2^* = n_1 + \int_0^{\alpha} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}}. \tag{2.10}$$

This estimate (2.10) is equivalent to (2.2).

(iii) For $E(n_0) = 0$, by (1.3) and $a'(n_0) > 0$ we get that $J'(n_0) < 0$, $J''(n) = 0$ and $J(n) = a(n_0)^{-\frac{3}{2}}(a(n_0) - \frac{1}{2}a'(n_0)(n - n_0))\ \forall n \geq n_0$. Thus we conclude that

$$a(n) = a(n_0)^3 \left(a(n_0) - \frac{1}{2}a'(n_0)(n - n_0)\right)^{-2} \quad \forall n \geq n_0,$$

and (2.3) is proved. □

We also study the blow-up rate and blow-up constant for a, a' and a'' under the conditions in Theorem 2.1. We have got the following results.

Theorem 2.2. *If u is the solution of the problem (1.1) with one of the following properties that*

(i) $E(n_0) < 0$ *or*
(ii) $E(n_0) = 0,\ a'(n_0) > 0.$

Then the blow-up rate of a is 2, and the blow-up constant K_1 of a is 2, that is, for $m = 1$–3,

$$\lim_{n\to N_m^*} (N_m^* - n)^2 a(n) = 2. \tag{2.11}$$

The blow-up rate of a' is 3, and the blow-up constant K_2 of a' is 4, that is, for $m = 1$–3,

$$\lim_{n\to N_m^*} (N_m^* - n)^3 a'(n) = 4. \tag{2.12}$$

The blow-up rate of a'' is 4, and the blow-up constant K_3 of a'' is 12, that is, $m = 1$–3,

$$\lim_{n\to N_m^*} a''(n)(N_m^* - n)^4 = 12. \tag{2.13}$$

Proof. (i) Under this condition, $E(n_0) < 0,\ a'(n_0) \geq 0$ by (2.1) and (2.6) we get

$$\int_0^{J(n)} \frac{1}{N_1^* - n} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}} = 1 \quad \forall n \geq n_0, \tag{2.14}$$

$$\lim_{n\to N_1^*} \sqrt{2}\frac{J(n)}{N_1^* - n} = 1. \tag{2.15}$$

This identity (2.15) is equivalent to (2.11) for $m = 1$.

For $E(n_0) < 0,\ a'(n_0) < 0$ by (2.9) we also have

$$\int_0^{J(n)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}} = N_2^* - n \quad \forall n \geq n_0; \tag{2.16}$$

therefore we get (2.11) for $m = 2$.

Seeing (2.5) and (2.8), we find

$$\lim_{n\to N_m^*} J'(n) = -\frac{1}{2}\sqrt{2}, \tag{2.17}$$

$$\lim_{n\to N_m^*} a'(n)(N_m^* - n)^3 = 4, \tag{2.18}$$

$$\lim_{n\to N_m^*} u'(n)^2(N_m^* - n)^4 = 2 \tag{2.19}$$

for $m = 1, 2$. Using (1.2) and (2.19) we obtain for $m = 1, 2$,

$$\lim_{n\to N_m^*} a''(n)(N_m^* - n)^4 = 6 \lim_{n\to N_m^*} u'(n)^2(N_m^* - n)^4. \tag{2.20}$$

Thus, (2.20) and (2.13) are equivalent.

(ii) For $E(n_0) = 0,\ a'(n_0) > 0$, by (2.11) we get for $m = 1, 2$,

$$a(n) = a(n_0)^3 \left(\frac{1}{2}a'(n_0)\right)^{-2} \cdot (N_3^* - n) \quad \forall n \geq n_0. \tag{2.21}$$

Therefore, the estimates (2.11)–(2.13) for $m = 3$ follow from (2.21). □

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