# UNIFORM DISTRIBUTION IN $Z_g$ AND $Z_{g_1} \times \ldots \times Z_{g_t}$

BY

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## 1. INTRODUCTION

Uniform distribution of $g$-adic sequences was studied in Meijer [4], [5], [6] and Shiue [9]. This theory is on one hand a generalization of the theory of uniform distribution of $p$-adic sequences introduced by Cugiani [1]; see Kuipers and Niederreiter [2], chapter 5.2. On the other hand the notion of uniform distribution of a $g$-adic sequence is a generalization of the notion of uniform distribution of a sequence of rational integers introduced by Niven [8]; see Kuipers and Niederreiter [2], chapter 5.1.

In the present paper we explored the fact that a $g$-adic ring $Z_g$, where $g = p_1^{e_1} \ldots p_r^{e_r}$, can be decomposed into a direct product of $p$-adic rings $Z_{p_1} \times \ldots \times Z_{p_r}$. In section 3 we generalize this decomposition in a natural way to a decomposition of $Z_g$ into $Z_{g_1} \times \ldots \times Z_{g_t}$, where $g = g_1 \ldots g_t$ and the $g_1, \ldots, g_t$ are pairwise relatively prime. We prove that a sequence $\{x_n\}$ is uniformly distributed in $Z_g$ if and only if the corresponding sequence in the direct product is uniformly distributed. In section 3 we apply this result to sequences of integers in $Z_g$. Finally in section 4 results are derived on sequences which are uniformly distributed in a direct product of $g$-adic rings $Z_{g_1} \times \ldots \times Z_{g_t}$.

## 2. $g$-ADIC NUMBERS

In this section we give a survey of results on $g$-adic numbers which will be used in the sequel. For a more complete discussion of the theory of $g$-adic numbers we refer to Mahler [3], chapters 1 and 2.

Let $Q$ denote the set of rational numbers.

If $p$ is a prime number and $a \in Q$, then we denote the $p$-adic valuation of $a$ by $|a|_p$. Let $g$ be an integer, $g \geq 2$ and let $g = p_1^{e_1} \ldots p_r^{e_r}$ be its canonical prime factorization. For $a \in Q$ the $g$-adic pseudo-valuation $|a|_g$ is defined by

$$|a|_g = \max \{|a|_{p_1}^{\lambda_1}, \ldots, |a|_{p_r}^{\lambda_r}\}, \tag{1}$$

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where the real numbers $\lambda_1, \ldots, \lambda_r$ are chosen in such a way, that

$$|g|_{p_1}^{\lambda_1} = \ldots = |g|_{p_r}^{\lambda_r} = g^{-1}, \text{ i.e. } \lambda_\varrho = \frac{\log g}{e_\varrho \log p_\varrho} \ (\varrho = 1, \ldots, r).$$

The ring of $g$-adic numbers $Q_g$ is defined as the completion of $Q$ with respect to the $g$-adic pseudo-valuation. The $g$-adic pseudo-valuation of $Q$ has a uniquely determined continuation on $Q_g$. For $a \in Q_g$ we denote this pseudo-valuation by $|a|_g$.

The subset of $Q_g$ consisting of all elements a satisfying $|a|_g \leqq 1$ is called the ring of $g$-adic integers $Z_g$.

Every $a \in Z_g$ has a unique representation

$$a = \sum_{i=0}^{\infty} a_i g^i, \tag{2}$$

where the $a_i$ are taken from the set $\{0, 1, \ldots, g-1\}$; see Mahler [3], chapter 2. Moreover, if $a_0 = \ldots = a_{t-1} = 0$ and $a_t \neq 0$, then $|a|_g = |a_t g^t|_g = |a_t|_g g^{-t}$, where $g^{-1} < |a_t|_g \leqq 1$, since $a_t \in \{0, 1, \ldots, g-1\}$.

The functions $\Psi_k$ mapping $Z_g$ into the set of non-negative integers are defined by: if $a \in Z_g$ has representation (2), then

$$\Psi_k(a) = \sum_{i=0}^{k-1} a_i g^i.$$

We have the following relations for $\Psi_k$;

$$\Psi_k(a+b) \equiv \Psi_k(a) + \Psi_k(b) \pmod{g^k}$$

$$\Psi_k(ab) \equiv \Psi_k(a)\Psi_k(b) \pmod{g^k},$$

for $a, b \in Z_g$.

For $d \in Z_g$ and $k$ a non-negative integer we define the neighbourhood $U_k(d)$ by

$$U_k(d) = \{x | x \in Z_g, \ |x-d|_g \leqq g^{-k}\}.$$

It is easy to show (see [4], p. 14–15) that

$$a \in U_k(d) \text{ if and only if } \Psi_k(a) = \Psi_k(d). \tag{3}$$

This implies $U_k(d) = U_k(\Psi_k(d))$. It is easy to derive that for $h > k$, the neighbourhood $U_k(d)$ can be written as the union of $g^{h-k}$ disjoint neighbourhoods of the form $U_h(c)$ by

$$U_k(d) = \bigcup_{j=0}^{g^{h-k}-1} U_h(\Psi_k(d) + jg^k).$$

As is usual an element $a \in Z_g$ is called a unit of $Z_g$ if $a^{-1}$ exists in $Z_g$. In [5] lemma 4 the following characterization of the units of $Z_g$ is given.

LEMMA 1. If $a$ is an element of $Z_g$, then $a$ is a unit of $Z_g$ if and only if $(\Psi_1(a), g) = 1$. Moreover, if $a$ is a unit of $Z_g$, then $|a|_g = |a^{-1}|_g = 1$.

In particular in the $p$-adic case is $a$ a unit of $Z_p$ if and only if $|a|_p = 1$.

Finally we recall the definitions of uniform distribution in $Z_g$ and $Z_{g_1} \times \ldots \times Z_{g_t}$.

If $\{x_n\}$ is a sequence in a space $M$, $V$ is a subset of $M$ and $N$ is a positive integer, we denote the number of points $x_n$ satisfying

$$x_n \in V, \quad 1 \leqq n \leqq N$$

by $A(V, N, \{x_n\})$ or, if there is no risk of confusion, simply by $A(V, N)$. A sequence $\{x_n\}$ in $Z_g$ is called uniformly distributed in $Z_g$, if

$$\lim_{N \to \infty} N^{-1} A(U_k(d), N) = g^{-k}$$

for every neighbourhood $U_k(d)$ in $Z_g$.

COROLLARY 1. Relation (3) implies that $\{x_n\}$ is uniformly distributed in $Z_g$ if and only if for every positive integer $k$ the sequence of rational integers $\{\Psi_k(x_n)\}$ is uniformly distributed mod. $g^k$.

Let $g_1, \ldots, g_t$ denote an ordered $r$-tuple of integers $g_\tau \geqq 2$, $\tau = 1, \ldots, t$. In the direct product

$$Z_{g_1} \times Z_{g_2} \times \ldots \times Z_{g_t}$$

we define the neighbourhoods by

$$U_{k_1, \ldots, k_t}(d_1, \ldots, d_t) = U_{k_1}(d_1) \times \ldots \times U_{k_t}(d_t).$$

Then a sequence $\{x_n\}$ in $Z_{g_1} \times \ldots \times Z_{g_t}$ is called uniformly distributed in the direct product if

$$\lim_{N \to \infty} N^{-1} A(U_{k_1, \ldots, k_t}(d_1, \ldots, d_t), N) = g_1^{-k_1} \ldots g_t^{-k_t}$$

for every neighbourhood in the direct product.

## 3. DECOMPOSITION THEOREM

It is well-known that the $g$-adic ring $Z_g$ can be decomposed into a direct product of $p$-adic rings. We shall state this in lemma 2. In this lemma the following fact is used. If $\{x_n\}$ is a sequence in $Z$, which converges in $Z_g$ to the element $a$ – e.g. the sequence $\{\Psi_n(a)\}$ – then it is a fundamental

sequence with respect to the $g$-adic pseudo-valuation. By (1) it is also a fundamental sequence with respect to each of the $p_\varrho$-adic valuations and therefore converges in each of the $Z_{p_\varrho}$.

LEMMA 2. Let $g = p_1^{e_1} \dots p_r^{e_r}$. There exists a one-to-one mapping $\varphi$ of $Z_g$ onto the direct product $Z_{p_1} \times \dots \times Z_{p_r}$ such that if $\varphi(a) = (a_1, \dots, a_r)$ and $\varphi(b) = (b_1, \dots, b_r)$ then

$$\varphi(a+b) = (a_1+b_1, \dots, a_r+b_r), \tag{4}$$

$$\varphi(a-b) = (a_1-b_1, \dots, a_r-b_r), \tag{5}$$

$$\varphi(ab) = (a_1b_1, \dots, a_rb_r). \tag{6}$$

This mapping is defined by: if $\{Z_n\}$ is a sequence in $Z$ with $a = \lim Z_n$ in the $g$-adic pseudo-valuation, then $a_\varrho = \lim Z_n$ in the $p_\varrho$-adic valuation for $\varrho = 1, \dots, r$.

PROOF. See Mahler [3], p. 23, 24.

LEMMA 3. Let $U_k(d)$ denote a neighbourhood in $Z_g$ and $\varphi(d) = (d_1, \dots, d_r)$, then

$$\varphi U_k(d) = U_{ke_1,\dots,ke_r}(d_1, \dots, d_r).$$

PROOF. Let $a \in Z_g$ and $\varphi(a) = (a_1, \dots, a_r)$. Let $\{Z_n\}$ be a sequence in $Z$ such that

$$a-d = \lim Z_n \text{ in } Z_g, \quad a_\varrho - d_\varrho = (a-d)_\varrho = \lim Z_n \text{ in } Z_{p_\varrho} \ (\varrho = 1, \dots, r).$$

Then the following statements are equivalent.

1) $a \in U_k(d)$.
2) $|a-d|_g \leqq g^{-k}$.
3) $|Z_n|_g \leqq g^{-k}$ if $n$ is sufficiently large.
4) $|Z_n|_{p_\varrho}^{\lambda_\varrho} \leqq g^{-k}$ $(\varrho = 1, \dots, r)$ if $n$ is sufficiently large.
5) $|Z_n|_{p_\varrho} \leqq p_\varrho^{-ke_\varrho}$ $\varrho = 1, \dots, r$, $n$ sufficiently large.
6) $|a_\varrho - d_\varrho| \leqq p_\varrho^{-ke_\varrho}$ $(\varrho = 1, \dots, r)$.
7) $a_\varrho \in U_{ke_\varrho}(d_\varrho)$ $(\varrho = 1, \dots, r)$.
8) $\varphi(a) \in U_{ke_1,\dots,ke_r}$ $(d_1, \dots, d_r)$.

This proves the lemma.

THEOREM 1. A sequence $\{x_n\}$ in $Z_g$ is uniformly distributed in $Z_g$ if and only if $\{\varphi(x_n)\}$ is uniformly distributed in $Z_{p_1} \times \dots \times Z_{p_r}$.

PROOF. a) Suppose $\{\varphi(x_n)\}$ is uniformly distributed in the direct product. Let $U_k(d)$ be a neighbourhood in $Z_g$.

Then, using lemma 3,

$$x_n \in U_k(d) \text{ if and only if } \varphi(x_n) \in U_{ke_1,\ldots,ke_r}(d_1, \ldots, d_r).$$

Hence

$$A(U_k(d), N, \{x_n\}) = A(U_{ke_1,\ldots,ke_r}(d_1, \ldots, d_r), N, \{\varphi(x_n)\}).$$

By hypothesis

$$\lim_{N\to\infty} N^{-1}A(U_{ke_1,\ldots,ke_r}(d_1, \ldots, d_r), N, \{\varphi(x_n)\}) = p_1^{-ke_1} \ldots p_r^{-ke_r} = g^{-k}.$$

Therefore

$$\lim_{N\to\infty} N^{-1}A(U_k(d), N, \{x_n\}) = g^{-k}$$

and the sequence $\{x_n\}$ is uniformly distributed in $Z_g$.

b) Suppose $\{x_n\}$ is uniformly distributed in $Z_g$.

Let $U_{k_1,\ldots,k_r}(d_1, \ldots, d_r)$ denote a neighbourhood in the direct product. Choose an integer $k$ such that $ke_\varrho \geqq k_\varrho$ for $\varrho = 1, \ldots, r$. Then the neighbourhood $U_{k_\varrho}(d_\varrho)$ in $Z_{p_\varrho}$ is the union of $p_\varrho^{ke_\varrho - k_\varrho}$ disjoint neighbourhoods in $Z_{p_\varrho}$ of the form $U_{ke_\varrho}(c_\varrho)$. Therefore the neighbourhood $U_{k_1,\ldots,k_r}(d_1, \ldots, d_r)$ in the direct product is the union of $s = p_1^{ke_1 - k_1} \ldots p_r^{ke_r - k_r}$ disjoint neighbourhoods of the form $U_{ke_1,\ldots,ke_r}(c_1, \ldots, c_r)$. Every neighbourhood of the last form is the image under $\varphi$ of a neighbourhood $U_k(c)$ in $Z_g$. Hence, $U_{k_1,\ldots,k_r}(d_1, \ldots, d_r)$ is the image of $s$ neighbourhoods of the form $U_k(c)$ in $Z_g$. Since $\{x_n\}$ is uniformly distributed in $Z_g$ we have for every neighbourhood $U_k(c)$ in $Z_g$

$$\lim_{N\to\infty} N^{-1}A(U_k(c), N, \{x_n\}) = g^{-k}.$$

Hence

$$\lim_{N\to\infty} N^{-1}A(U_{k_1,\ldots,k_r}(d_1, \ldots, d_r), N, \{\varphi(x_n)\}) = sg^{-k} = p_1^{-k_1} \ldots p_r^{-k_r}.$$

Therefore $\{\varphi(x_n)\}$ is uniformly distributed in the direct product.

It is possible to generalize theorem 1. Let $g_1, \ldots, g_t$ be integers, $g_\tau \geqq 2$, $\tau = 1, \ldots, t$, which are pairwise relatively prime and put $g = g_1 \ldots g_t = p_1^{e_1} \ldots p_r^{e_r}$. By lemma 2 there is a one-to-one correspondance between $Z_g$ and $Z_{p_1} \times \ldots \times Z_{p_r}$. On the other hand each of the $Z_{g_\varrho}$ can be decomposed into a direct product of $p$-adic rings $Z_p$. Since $g_1, \ldots, g_t$ are pairwise relatively prime this induces a one-to-one correspondance between $Z_{g_1} \times \ldots \times Z_{g_t}$ and $Z_{p_1} \times \ldots \times Z_{p_r}$. Hence there exists a one-to-one mapping of $Z_g$ onto $Z_{g_1} \times \ldots \times Z_{g_t}$.

In the sequel we shall denote this mapping by $\gamma$. Obviously the mapping $\varphi$ may be regarded as a special case of $\gamma$ which can be obtained by taking $g_1 = p_1^{e_1}, \ldots, g_r = p_r^{e_r}$ and observing that $Z_{p_1}^{e_1} \times \ldots \times Z_{p_r}^{e_r} = Z_{p_1} \times \ldots \times Z_{p_r}$. Note

that for $n \in Z$, $\gamma(n) = (n, \ldots, n)$. It is easy to check that $\gamma$ satisfies relations similar to (4), (5) and (6).

Moreover if $U_k(d)$ is a neighbourhood in $Z_g$ and $\gamma(d) = (d_1, \ldots, d_t)$, then it is easy to prove that $\gamma U_k(d) = U_{k,\ldots,k}(d_1, \ldots, d_t)$.

Repeating the proof of theorem 1 with $p_\varrho$ replaced by $g_\tau$ and $e_1 = \ldots = e_t = 1$ we obtain the following result.

THEOREM 1a. Let $g_1, \ldots, g_t$ denote integers $\geqq 2$ which are pairwise relatively prime, put $g = g_1 \ldots g_t$ and let $\gamma$ be the one-to-one mapping of $Z_g$ onto the direct product $Z_{g_1} \times \ldots \times Z_{g_t}$ induces by the mapping $\varphi$ of lemma 2. A sequence $\{x_n\}$ in $Z_g$ is uniformly distributed in $Z_g$ if and only if $\{\gamma(x_n)\}$ is uniformly distributed in the direct product.

EXAMPLE 1. It is easy to prove that the sequence of positive integers $\{x_n = n\}$ is uniformly distributed in $Z_g$ (see [5], theorem 2). It follows from theorem 1a that the sequence $\{(n, \ldots, n)\}$ is uniformly distributed in $Z_{g_1} \times \ldots \times Z_{g_t}$ if $g_1, \ldots, g_t$ are pairwise relatively prime. (See [4], theorem 5.3).

REMARK 1. If $g_1, \ldots, g_t$ are *not* pairwise relatively prime, then the sequence $\{x_n\} = \{(n, \ldots, n)\}$ is not uniformly distributed in $Z_{g_1} \times \ldots \times Z_{g_t}$. Suppose e.g. $(g_1, g_2) > 1$. Consider the neighbourhood

$$U = U_{1,1,0,\ldots,0}(1, 0, \ldots, 0).$$

Then $x_n \in U$ is equivalent to $n \equiv 1 \pmod{g_1}$ and $n \equiv 0 \pmod{g_2}$. It is easy to see that this system of congruences has no solution since $(g_1, g_2) > 1$. Hence $A(U, N) = 0$ and the sequence cannot be uniformly distributed in $Z_{g_1} \times \ldots \times Z_{g_t}$. (compare [4], theorem 5.3).

LEMMA 4. Let $a \in Z_g$ and $\varphi(a) = (a_1, \ldots, a_r) \in Z_{p_1} \times \ldots \times Z_{p_r}$. Then $a$ is a unit of $Z_g$ if and only if all $a_\varrho$ are units of $Z_{p_\varrho}$ $(\varrho = 1, \ldots, r)$.

PROOF. By lemma 1 the element $a$ is a unit of $Z_g$ if and only if $(\Psi_1(a), g) = 1$ which is equivalent to $(\Psi_k(a), g) = 1$ for every $k$.

The sequence $\{\Psi_k(a)\}$ is a sequence in $Z$ which converges to $a$ in $Z_g$. Then, by lemma 2, $a_\varrho = \lim_{k \to \infty} \Psi_k(a)$ in $Z_{p_\varrho}$ $(\varrho = 1, \ldots, r)$.

Now $(\Psi_k(a), g) = 1$ is equivalent to $(\Psi_k(a), p_\varrho) = 1$ for $\varrho = 1, \ldots, r$ or $|\Psi_k(a)|_{p_\varrho} = 1$ and $|a_\varrho|_{p_\varrho} = \lim |\Psi_k(a)|_{p_\varrho} = 1$ for $\varrho = 1, \ldots, r$. The last relation is equivalent to: $a_\varrho$ is a unit of $Z_{p_\varrho}$ for $\varrho = 1, \ldots, r$.

From lemma 4 and the construction of $\gamma$ the following result follows.

LEMMA 4a. Let $a \in Z_g$ and $\gamma(a) = (a_1, \ldots, a_t) \in Z_{g_1} \times \ldots \times Z_{g_t}$, where $g_1, \ldots, g_t$ are pairwise relatively prime. Then $a$ is a unit of $Z_g$ if and only if all $a_\tau$ are units of $Z_{g_\tau}$ $(\tau = 1, \ldots, t)$.

## 4. SEQUENCES OF INTEGERS

Let $\{x_n\}$ denote a sequence of integers then we can make the following observations.

1. Since $\gamma(x_n)=(x_n, \ldots, x_n)$ we obtain from theorem 1a: the sequence $\{x_n\}$ is uniformly distributed in $Z_g$ if and only if the sequence $\{(x_n, \ldots, x_n)\}$ is uniformly distributed in $Z_{g_1} \times \ldots \times Z_{g_t}$, where $g_1, \ldots, g_t$ are pairwise relatively prime and $g=g_1 \ldots g_t$.
2. It follows from corollary 1 that the sequence $\{x_n\}$ is uniformly distributed in $Z_g$ if and only if for every positive integer $k$ the sequence $\{x_n\}$ is uniformly distributed mod. $g^k$.
3. As a consequence of the last fact we obtain (compare [5], theorem 7) that the sequence $\{x_n\}$ is a uniformly distributed sequence of integers in the sense of Niven [8] if and only if for every integer $g \geqq 2$ the sequence is uniformly distributed in $Z_g$.

We will use the following notion of Niederreiter [7] of uniform distribution mod. $(m_1, \ldots, m_t)$.

DEFINITION 1. Let $\underline{m}=(m_1, \ldots, m_t)$ be a $t$-tuple of natural numbers, and let $\sigma=\{(a_{n1}, \ldots, a_{nt})\}$, $n=1, 2, \ldots$, be a given sequence of $t$-dimensional lattice points. For a natural number $N$ and a $t$-tuple $\underline{d}=(d_1, \ldots, d_t)$ of integers, the counting function $A(\underline{d}, \underline{m}, N, \sigma)$ is defined to be the number of indices $n$, $1 \leqq n \leqq N$, such that simultaneously

$$a_{n1} \equiv d_1 (\text{mod. } m_1), \ldots, a_{nt} \equiv d_t (\text{mod. } m_t).$$

Then $\sigma$ is uniformly distributed mod. $(m_1, \ldots, m_t)$ or mod. $\underline{m}$ if

$$\lim_{N \to \infty} N^{-1} A(\underline{d}, \underline{m}, N, \sigma) = m_1^{-1} m_2^{-1} \ldots m_t^{-1} \text{ for all } \underline{d}.$$

Furthermore, $\sigma$ is uniformly distributed in $Z^t$ if $\sigma$ is uniformly distributed mod. $\underline{m}$ for all possible $\underline{m}$.

Now we can state the following result as a consequence of the observations 1, 2 and 3.

THEOREM 2. The following statements are equivalent for a sequence of integers $\{x_n\}$.

1. $\{x_n\}$ is a uniformly distributed sequence of integers.
2. $\{x_n\}$ is uniformly distributed in $Z_g$ for every $g \geqq 2$.
3. $\{(x_n, \ldots, x_n)\}$ is uniformly distributed in $Z_{g_1} \times \ldots \times Z_{g_t}$ for every set of pairwise relatively prime integers $g_1, \ldots, g_t$ with $g_\tau \geqq 2$, $\tau=1, \ldots, t$.
4. For every set of pairwise relatively prime integers $g_1, \ldots, g_t$ with $g_\tau \geqq 2$, $\tau=1, \ldots, t$ is $\{(x_n, \ldots, x_n)\}$ uniformly distributed mod. $(g_1, \ldots, g_t)$.

REMARK 2. The equivalence of 1. and 4. can also be derived as a direct consequence of the Chinese remainder theorem: if $g_1, \ldots, g_t$ are

pairwise relatively prime and $g=g_1 \dots g_t$, there is a one-to-one correspondance between the residue classes

$$n \equiv a \pmod{g} \text{ and } n \equiv a_1 \pmod{g_1}, \dots, n \equiv a_t \pmod{g_t}.$$

Moreover remark 1 shows that the condition that $g_1, \dots, g_t$ are pairwise relatively prime cannot be omitted in theorem 2.

EXAMPLES 2. Theorem 2 can be applied to the following well-known sequences of integers, which are uniformly distributed (compare Kuipers-Niederreiter [2], p. 307–308).

1. $([\alpha n])$, where $\alpha$ is irrational or $\alpha = 1/d$ for some nonzero integer $d$.
2. $([f(n)])$, where $f$ denotes a polynomial with real coefficients of which at least one, different from $f(0)$, is irrational.
3. $([\alpha n^\sigma])$, where $\alpha$, $\sigma$ real, $\alpha \neq 0$, $\sigma > 0$, $\sigma$ non integral.
4. $([\alpha (\log n)^\tau])$, where $\alpha$, $\tau$ real, $\alpha \neq 0$, $\tau > 1$.

## 5. SPECIAL SEQUENCES

In this section we derive some results on uniformly distributed sequences in $Z_{g_1} \times \dots \times Z_{g_t}$ where $g_1, \dots, g_t$ are integers $\geqq 2$. We point out that in most results of this section the integers $g_1, \dots, g_t$ had not to be relatively prime.

LEMMA 5. Let $\{x_n\}$ be a sequence, uniformly distributed in $Z_g$ and let $a, b \in Z_g$. Then the sequence $\{ax_n + b\}$ is uniformly distributed in $Z_g$ if and only if $a$ is a unit of $Z_g$.

PROOF. See [5], theorem 3.

THEOREM 3. Suppose that $\{X_n\} = \{(x_n, \dots, x_{nt})\}$ is a sequence uniformly distributed in $Z_{g_1} \times \dots \times Z_{g_t}$ and let $(a_1, \dots, a_t), (b_1, \dots, b_t) \in Z_{g_1} \times \dots \times Z_{g_t}$. Then the sequence

$$\{Y_n\} = \{(a_1 x_{n1} + b_1, \dots, a_t x_{nt} + b_t)\}$$

is uniformly distributed in the direct product if and only if $a_1, \dots, a_t$ are units of respectively $Z_{g_1}, \dots, Z_{g_t}$.

PROOF. If $g_1, \dots, g_t$ are relatively prime the theorem is a direct consequence of theorem 1a, lemma 5 and lemma 4a. In the general case the proof proceed as follows.

Suppose $a$ is a unit of some $g$-adic ring $Z_g$. Then by lemma 1

$$|a|_g = |a^{-1}|_g = 1.$$

If $x$, $b$ and $d$ are arbitrary elements of $Z_g$, then

$$|ax + b - d|_g \leqq |a|_g |x + a^{-1}b - a^{-1}d|_g \leqq |a|_g |a^{-1}|_g |ax + b - d|_g.$$

Hence

$$|ax+b-d|_g=|x+a^{-1}b-a^{-1}d|_g.$$

i.e.

$$ax+b\in U_k(d) \text{ if and only if } x\in U_k(a^{-1}b-a^{-1}d).$$

Observe that $a^{-1}d-a^{-1}b\in Z_g$.

Let $a_1, \ldots, a_t$ be units of respectively $Z_{g_1}, \ldots, Z_{g_t}$. If $(d_1, \ldots, d_t)$ is an element in the direct product $Z_{g_1}\times\ldots\times Z_{g_t}$, then

$$Y_n\in U_{k_1,\ldots,k_t}(d_1, \ldots, d_t) \text{ if and only if}$$

$$X_n\in U_{k_1,\ldots,k_t}(a_1^{-1}b_1-a_1^{-1}d_1, \ldots, a_t^{-1}b_t-a_t^{-1}d_t).$$

Since $\{X_n\}$ is uniformly distributed in the direct product it follows that $\{Y_n\}$ is uniformly distributed in the direct product.

Suppose conversely that the sequence $\{Y_n\}$ is uniformly distributed in the direct product.

Considering the neighbourhoods $U_{k_1,\ldots,k_t}(d_1, \ldots, d_t)$ of the form $k_i=k$ for $i=\tau$, $k_i=0$ for $i\neq\tau$, we obtain that the sequences $\{a_\tau x_{n\tau}+b_\tau\}$ are uniformly distributed in $Z_{g_\tau}$. Then, by lemma 5, the elements $a_\tau$ are units of $Z_{g_\tau}$ $(\tau=1, \ldots, t)$.

THEOREM 4. Let $\{x_n\}$ be a sequence uniformly distributed in $Z_g$ and let $a, b, c\in Z_g$. The sequence $\{ax_n^2+bx_n+c\}$ is uniformly distributed in $Z_g$ if and only if $|a|_g<1$ and $b$ is a unit of $Z_g$.

PROOF. Suppose $|a|_g<1$ and $b$ is a unit of $Z_g$.

If $k$ is a positive integer and $d$ is an arbitrary element of $Z_g$, then by (3), the relation

$$ax_n^2+bx_n+c\in U_k(d)$$

is equivalent to

$$\Psi_k(a)\Psi_k(x_n)^2+\Psi_k(b)\Psi_k(x_n)+\Psi_k(c)\equiv\Psi_k(d) \pmod{g^k}.$$

Let $f_k$ be the mapping of the set of integers $\{0, 1, \ldots, g^k-1\}$ into itself defined by

$$f_k(u)\equiv\Psi_k(a)u^2+\Psi_k(b)u+\Psi_k(c) \pmod{g^k}.$$

We prove that $f_k$ is an injection.

Suppose $f_k(u)=f_k(v)$, then it follows

$$(u-v)(\Psi_k(a)(u+v)+\Psi_k(b))\equiv 0 \pmod{g^k}. \tag{7}$$

Let $p_1, \ldots, p_r$ be the different primes in the prime factorization of $g$. Since $|a|_g<1$ we have for the rational integer $\Psi_k(a)$ that $|\Psi_k(a)|_g<1$,

and then

$$\Psi_k(a)=0 \text{ or } p_\varrho|\Psi_k(a) \text{ for } \varrho=1, \ldots, r.$$

Further $b$ is a unit of $Z_g$ and then by lemma 1: $(\Psi_k(b), g)=1$. Hence

$$(\Psi_k(a)(u+v)+\Psi_k(b), g)=1 \text{ for all integers } u, v.$$

Then it follows from (7) that $u \equiv v \pmod{g^k}$, i.e. $f$ is an injection indeed. This implies that there exists exactly one integer $u_0 \in \{0, 1, \ldots, g^k-1\}$ such that

$$\Psi_k(a)u_0^2+\Psi_k(b)u_0+\Psi_k(c) \equiv \Psi_k(d) \pmod{g^k}.$$

Then the relation

$$ax_n^2+bx_n+c \in U_k(d)$$

is equivalent to $\Psi_k(x_n)=u_0$.

Since the sequence $\{x_n\}$ is uniformly distributed in $Z_g$, the sequence $\{\Psi_k(x_n)\}$ is, by corollary 1, uniformly distributed in $\{0, 1, \ldots, g^k-1\}$. Then we obtain

$$\lim_{N\to\infty} N^{-1}A(u_0, N, \{\Psi_k(x_n)\})=g^{-k}.$$

Hence

$$\lim_{N\to\infty} N^{-1}A(U_k(d), N, \{ax_n^2+bx_n+c\}) \text{ is uniformly distributed in } Z_g.$$

and the sequence $\{ax_n^2+bx_n+c\}$ is uniformly distributed in $Z_g$.

Suppose conversely that $\{ax_n^2+bx_n+c\}$ is uniformly distributed in $Z_g$. Then, by lemma 5, $\{ax_n^2+bx_n\}$ is uniformly distributed in $Z_g$. This implies, by corollary 1, that for every positive integer $k$ $\{\Psi_k(a)\Psi_k(x_n)^2+\Psi_k(b)\Psi_k(x_n)\}$ is uniformly distributed mod. $g^k$.

Therefore $\{\Psi_k(a)\Psi_k(x_n)^2+\Psi_k(b)\Psi_k(x_n)\}$ is uniformly distributed mod. $p_\varrho^k$ for $\varrho=1, \ldots, r$ if $p_1, \ldots, p_r$ are the different primes in the prime factorization of $g$. We remark that, by corollary 1, also $\{\Psi_k(x_n)\}$ is uniformly distributed mod. $g^k$ and therefore mod. $p_\varrho^k$ for $\varrho=1, \ldots, r$.

Suppose now $b$ is not a unit of $Z_g$. Then, by lemma 1, $(\Psi_1(b), g)\neq 1$ and there exists a $p_\varrho$ such that $p_\varrho|\Psi_1(b)$. Then also $p_\varrho|\Psi_2(b)$. Now $\Psi_1(x_n) \equiv 0 \pmod{p_\varrho}$ implies $\Psi_2(x_n) \equiv 0 \pmod{p_\varrho}$ and

$$\Psi_2(a)\Psi_2(x_n)^2+\Psi_2(b)\Psi_2(x_n) \equiv 0 \pmod{p_\varrho^2}.$$

Since $\{\Psi_1(x_n)\}$ is uniformly distributed mod. $p_\varrho$. we have

$$\lim_{N\to\infty} N^{-1}A(0, p_\varrho, N, \{\Psi_1(x_n)\})=p_\varrho^{-1}.$$

(We use here the notation of definition 1 with $t=1$). This implies

$$\lim_{N\to\infty} N^{-1}A(0, p_\varrho^2, N, \{\Psi_2(a)\Psi_2(x_n)^2+\Psi_2(b)\Psi_2(x_n)\}) \geqq p_\varrho^{-1}$$

and the sequence $\{\Psi_2(a)\Psi_2(x_n)^2+\Psi_2(b)\Psi_2(x_n)\}$ cannot be uniformly distributed mod. $p_\varrho^2$, which is a contradiction. Therefore $b$ had to be a unit of $Z_g$. Suppose now there is a $p_\varrho$ such that $(\Psi_1(a), p_\varrho)=1$.

Then there is a $u_0$ in the set $\{1, 2, \ldots, p_\varrho-1\}$ such that $\Psi_1(a)u_0+$ $+\Psi_1(b)\equiv 0 \pmod{p_\varrho}$. Then the equation $\Psi_1(a)u^2+\Psi_1(b)u\equiv 0 \pmod{p_\varrho}$ has the two different solutions $u=0$ and $u=u_0$ in $\{0, 1, \ldots, p_\varrho-1\}$.

Now $\Psi_1(x_n)=0$ and $\Psi_1(x_n)=u_0$ both imply

$$\Psi_1(a)\Psi_1(x_n)^2+\Psi_1(b)\Psi_1(x_n)\equiv 0 \pmod{p_\varrho}.$$

Hence

$$\lim_{N\to\infty} N^{-1}A(0, p_\varrho, N, \{\Psi_1(a)\Psi_1(x_n)^2+\Psi_1(b)\Psi_1(x_n)\}) =$$

$$\lim_{N\to\infty} N^{-1}A(0, p_\varrho, N, \{\Psi_1(x_n)\}) + \lim_{N\to\infty} N^{-1}A(u_0, p_\varrho, N, \{\Psi_1(x_n)\}).$$

Since $\{\Psi_1(x_n)\}$ is uniformly distributed mod. $p_\varrho$, the sequence

$$\{\Psi_1(a)\Psi_1(x_n)^2+\Psi_1(b)\Psi_1(x_n)\}$$

cannot be uniformly distributed mod. $p_\varrho$. This is a contradiction. Therefore $(\Psi_1(a), p_\varrho)>1$ for $\varrho=1, \ldots, r$ and then $|a|_g<1$.

COROLLARY 2. Since the sequence $\{n\}$ is uniformly distributed in $Z_g$ (see [5], theorem 2), the sequence $\{an^2+bn+c\}$ is uniformly distributed in $Z_g$ if and only if $|a|_g<1$ and $b$ is a unit of $Z_g$. This result improves [5], theorem 5.

THEOREM 5. Suppose that $\{X_n\}=\{(x_n, \ldots, x_{nt})\}$ is a sequence uniformly distributed in $Z_{g_1}\times\ldots\times Z_{g_t}$ and let

$$(a_1, \ldots, a_t),\ (b_1, \ldots, b_t),\ (c_1, \ldots, c_t)\in Z_{g_1}\times\ldots\times Z_{g_t}.$$

Then the sequence

$$\{Y_n\}=\{(a_1x_{n1}^2+b_1x_{n1}+c_1, \ldots, a_tx_{nt}^2+b_tx_{nt}+c_t)\}$$

is uniformly distributed in the direct product if and only if $|a_\tau|_{g_\tau}<1$ for $\tau=1, \ldots, t$ and $b_\tau$ is a unit of $Z_{g_\tau}$ for $\tau=1, \ldots, t$.

PROOF. Suppose $|a_\tau|_{g_\tau}<1$ for $\tau=1, \ldots, t$ and $b_\tau$ is a unit of $Z_{g_\tau}$ for $\tau=1, \ldots, t$. Consider a neighbourhood $U_{k_1,\ldots,k_t}(d_1, \ldots, d_t)$ in the direct product.

In the proof of theorem 4 we have shown that there exists exactly one integer $u_\tau \in \{0, 1, \ldots, g_\tau^{k_\tau}-1\}$ such that

$$\Psi_k(a_\tau)u_\tau^2+\Psi_k(b_\tau)u_\tau+\Psi_k(c_\tau)=\Psi_{k_\tau}(d_\tau) \pmod{g_\tau^{k_\tau}}$$

in $Z_{g_\tau}$ for $\tau=1, \ldots, t$. Moreover the relation $Y_n \in U_{k_1,\ldots,k_t}(d_1, \ldots, d_t)$ is equivalent to

$$\Psi_{k_1}(x_{n1})=u_1, \ldots, \Psi_{k_t}(x_{nt})=u_t.$$

Since the sequence $\{X_n\}$ is uniformly distributed in the direct product, we obtain that the sequence $\{Y_n\}$ is also uniformly distributed in the direct product.

If, conversely, the sequence $\{Y_n\}$ is uniformly distributed in the direct product for each $\tau \in \{1, \ldots, t\}$ the sequence $\{a_\tau x_{n\tau}^2+b_\tau x_{n\tau}+c_\tau\}$ is uniformly distributed in $Z_{g_\tau}$ and then, using theorem 4, we have $|a_\tau|_{g_\tau}<1$ and $b_\tau$ is a unit of $Z_{g_\tau}$ for $\tau=1, \ldots, t$.

EXAMPLES 3.

1. If the $g_1, \ldots, g_t$ are pairwise relatively prime we can apply theorem 3 and theorem 5 to the sequences $\{X_n\}$ mentioned in examples 2.
2. Niederreiter [7] has proved the following result: if the real numbers $1, \alpha_1, \ldots, \alpha_t$ are linearly independent over the rationals, then the sequence $\{X_n\}=\{([n\alpha_1], \ldots, [n\alpha_t])\}$ is uniformly distributed in $Z^t$. This implies that $\{X_n\}$ is uniformly distributed in each direct product $Z_{g_1} \times \ldots \times Z_{g_t}$ and we can apply theorem 3 and theorem 5 to this sequence.

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