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Evaluating forecasting performance for interval data

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ABSTRACT

From the overlapping parts and the non-overlapping parts of the actual intervals and the forecast intervals, it should be defined a criterion which is more efficient to evaluate forecasting performance for interval data. In this paper, we present evaluation techniques for interval time series forecasting. The forecast results are compared by the mean squared error of the interval, mean relative interval error and mean ratio of exclusive-or. Simulation and empirical studies show that our proposed evaluation techniques for interval forecasting can provide a more objective decision space in interval forecasting to policymakers.

1. Introduction

In time series analysis research, the most difficult work may be how to choose an appropriate model from the model base (model family), which can honestly explain the trend of an underlying time series, such as exchange rate or index of stock volume. Two fundamental questions that often arise are: (1) Does there exist an appropriate statistical model that can account for this underlying process? (2) Does the dynamic model follow a linear or non-linear equation? (Need we use more than one equation, e.g. threshold model, to fit the time series?).

On the other hand, we often confront the uncertainty data problems. For instance, should we count the number of yearly-enrolled students at the beginning of the year? at midyear? or at the end of the year? The obtained number is often different at different time. For another example, what is the exchange rate of U.S. dollar to Japanese Yen last week? is with the opening quotation? the closing price? or the average of the highest price and the bottom price? The results are also quite different. Wu and Chen [1] have given an extensive review of literature on this topic.

Recently, the interval data analysis is paid more and more attention, such as daily temperature changes, the fluctuation of the exchange rate, the level price of petroleum etc. Due to the uncertainty of the predicted points, intervals are used as the estimated prediction values. Taking stock market as an example, if it is desired to make a prediction analysis to a certain stock, the daily highest and lowest prices of the stock are regarded as the boundary values of intervals. Then the future price intervals of the stock can be predicted by means of interval time series forecasting. Consequently, we can make comparatively objective decision by the predicted price interval rather than by the closing value or mean value.

Abraham and Ledolter [2], Chatfield [3] proposed prediction interval by time series of points to carry out the prediction. Granger, White and Kamstra [4] elaborated the architecture of interval forecasting. Chatfield [5] made a comparison of several different methods. And Diebold and Mariano [6] proposed the discussion in the respect that different prediction methods have their own pros and cons depending on the time series having a steady tendency or a severe fluctuation. Christoffersen [7] provided the calculation method of interval forecasting for the risk measurement. Nguyen and Wu [8] introduced (fuzzy) interval time series to forecast intervals. Despite there are various methods of interval forecasting due to the variety of the backgrounds and the purposes of researches, the data collections are mostly in the basic form of single real numerical variable.

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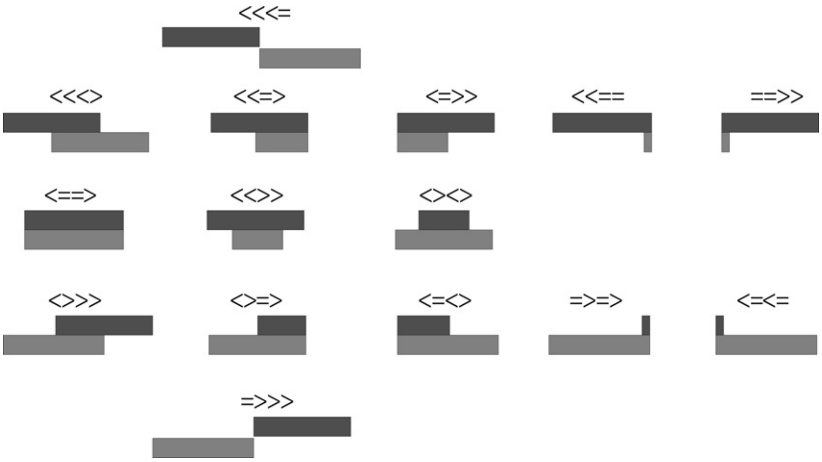

Fig. 1. One encoding lists the relations of the four pairs of end points in a fixed sequence.

When the interval calculation technology is applied to explore the model construction and forecasting of interval time series, it is necessary to determine the validity of the forecasting method by means of the estimated errors between the forecast results. Chatfield [9] declared that the error made by an inappropriate interval prediction method is more severe than the error made by a simple point prediction. In order to perform the efficiency of interval forecasting, we define the mean squared error of interval and the mean relative interval error by combining the two factors of the center and the radius of interval. While considering a good forecast interval, it is the most important whether the forecast interval does cover the actual interval.

In this paper, we denote an interval by the central point and the interval radius instead of the traditional interval expression. Moreover we define mean ratio of exclusive-or which is more sufficient to show the efficiency of interval forecasting. By proposing the forecasting performance evaluation for interval data, we will demonstrate the validation of the interval forecasting effect which will be helpful for the study and judgment on the choice of the interval forecasting models.

2. Interval time series analysis

In traditional analysis of time series, the data of time series is sampled from the values present at discrete points of time. However time is a continuous variable, the data variation between two consecutive samples cannot be known. Besides, the forecasting result of a time series is merely a single value. Therefore, the forecasting by a set of discrete numerical data may be too subjective. In order to allow more latitude of forecasting result, the concept of interval time series is to represent the time series data in the form of interval. Then the centers and radii of interval time series are used to make analysis of forecasting. Thus, the result of interval time series forecasting is also in the form of interval obtained by the forecasting center and radius. The interval time series forecasting is explained in the following sections.

2.1. The operation of interval data

While we consider the data to be of interval type, we must encounter the various problems of interval operations as well as the realistic meanings. Dwyer [10] called intervals as "range numbers" and defined the relevant operations. The subsequent studies relative to the interval operations continually quote such definitions. Nevertheless, it is still unable to give the standard rules of interval operations on the computer hardware. Hayes [11] pointed out that the rules of interval operations seem simple, but there often appears a trap of miscalculation in the practical calculations. Especially, Comparisons between intervals are more complicated than those of point-like numbers. Fig. 1 shows 15 meaningful relations between intervals. It's unclear even how to name all these comparisons.

Furthermore, when we process a set of dynamic data represent by the interval form, we often encounter certain realistic dilemma. For instance, does the value increase or decrease from [2, 8] to [3, 5]? We may consider the location variation between intervals. But if we take the interval scale into considerations, this case becomes more complicated. Hence, in this paper, we will propose the bivariate parameters, which are (i) the interval radius to express the interval length and (ii) the interval center to express the interval position, to demonstrate the variations of intervals. For the above example, the interval location is decreased from 5 to 4, whereas the interval radius is also decreased from 3 to 1. Such a new interval expression, which integrates the interval center with the interval radius, is apt to show the location shift and the length variation of interval data. By the proposed interval expression, we are able to make an appropriate interval forecasting for the interval time series. And under the assistance of computer programs, even more complicated calculation can be easily solved.

2.2. Some definitions of interval time series

The interval time series is an analytical method to apply intervals to the analysis of time series, incorporating with the interval operations, so as to solve the uncertainty of the data. As a result, before forecasting the model of interval time series and determining the validity of the forecasting method, several definitions relevant to interval time series must be given first. The definition of a random interval defined by Nguyen and Wu [8] is given in Definition 2.1, and the other relevant definitions are given as follows.

Definition 2.1 (*A Random Interval*). Let $X = [a, b]$. If a and b are random variables, then the interval $X = [a, b]$ is called a random interval.

Definition 2.2 (*A Random Interval of Alternate Notation, $X = (c, r)$*). Suppose $X = [a, b]$ be a random interval over the real numbers $\Re$, $c = (a + b)/2$ be the center of the interval X, and $r = (b - a)/2 \geq 0$ be the radius of the interval X, then the interval X can be expressed as $X = (c, r)$.

Definition 2.3 (*The Interval Length*). Let $X = [a, b] = (c, r)$ be a random interval, the interval length of X is $2r = b - a$, and denoted as $\|X\| = 2r$.

Definition 2.4 (*The Operation of Random Interval*). Let $X_1 = [a_1, b_1] = (c_1, r_1)$ and $X_2 = [a_2, b_2] = (c_2, r_2)$ be random intervals. The interval addition, scalar multiplication and interval subtraction are defined as follows:
Interval addition: $X_1 \oplus X_2 = (c_1, r_1) \oplus (c_2, r_2) = (c_1 + c_2, r_1 + r_2)$.
Scalar multiplication: $kX = k(c, r) = (kc, |k| r)$, where k is a scalar.
Interval subtraction: $X_1 \ominus X_2 = X_1 \oplus (-X_2) = (c_1, r_1) \oplus (-c_2, r_2) = (c_1 - c_2, r_1 + r_2)$.

The set difference $A - B$ is defined by $A - B = \{x \,|x \in A \text{ and } x \notin B\}$ (Smith [12]). While revising the definition of the set difference on the closed intervals, it should be a half-closed interval. But the closed-intervals are used to express the boundary of the data in this paper, so we make interval difference little diverse.

Definition 2.5 (*The Interval Difference $X_1 - X_2$*). Let $X_1 = [a_1, b_1] = (c_1, r_1)$ and $X_2 = [a_2, b_2] = (c_2, r_2)$ be random intervals. And $a_1 \leq a_2 \leq b_1 \leq b_2$, then the interval difference $X_1 - X_2$ is defined as follows:

$$\begin{aligned} X_1 - X_2 &= [a_1, b_1] - [a_2, b_2] = [a_1, a_2] \\ &= (\bar{c} - \bar{r}, (\partial c - \partial r)/2) \end{aligned}$$

where $\bar{c} = (c_1 + c_2)/2$, $\bar{r} = (r_1 + r_2)/2$, $\partial c = c_2 - c_1$, and $\partial r = r_2 - r_1$.

Definition 2.6 (*Exclusive-OR, XOR*). Let $X_1 = [a_1, b_1] = (c_1, r_1)$ and $X_2 = [a_2, b_2] = (c_2, r_2)$ be random intervals. And $a_1 \leq a_2 \leq b_1 \leq b_2$, then the exclusive-or denoted $X_1 \,\Delta\, X_2$, is defined as follows:

$$\begin{aligned} X_1 \,\Delta\, X_2 &= (X_1 - X_2) \cup (X_2 - X_1) = [a_1, a_2] \cup [b_1, b_2] \\ &= (\bar{c} - \bar{r}, (\partial c - \partial r)/2) \cup (\bar{c} + \bar{r}, (\partial c + \partial r)/2). \end{aligned}$$

Definition 2.7 (*Interval Time Series*). An interval time series is a sequence of random intervals $X_t = [a_t, b_t] = (c_t, r_t)$, $t = 1, 2, 3, \ldots$, denoted as $\{X_t\} = \{X_t = [a_t, b_t] = (c_t, r_t) \,|t = 1, 2, 3, \ldots\}$.

Example 2.1. Let $A = [1, 3] = (2, 1)$, $B = [2, 6] = (4, 2)$, then

$$\begin{aligned} &A \oplus B = (2, 1) \oplus (4, 2) = (2 + 4, 1 + 2) = (6, 3), \\ &A \ominus B = (2, 1) \ominus (4, 2) = (2 - 4, 1 + 2) = (-2, 3), \\ &A - B = (2, 1) - (4, 2) = ((2 + 4)/2 - (1 + 2)/2, (4 - 2)/2 - (2 - 1)/2) \\ &\qquad = (1.5, 0.5), \\ &A \,\Delta\, B = [1, 2] \cup [3, 6] = (1.5, 0.5) \cup (4.5, 1.5). \end{aligned}$$

2.3. Properties of interval time series

Let $\{X_t = [a_t, b_t] = (c_t, r_t)\}$ be an interval time series and $\hat{X}_t = [\hat{a}_t, \hat{b}_t] = \left(\hat{c}_t, \hat{r}_t\right)$ be the forecast interval with respect to $X_t = [a_t, b_t]$. In the analysis and forecasting of interval time series, there are four forecasting situations:

(1) If $\hat{a}_t \leq a_t \leq b_t \leq \hat{b}_t$, then the forecast interval is too wide, and denoted by *FIW*.
(2) If $a_t \leq \hat{a}_t \leq \hat{b}_t \leq b_t$, then the forecast interval is too narrow, and referred to as *FIN*.
(3) If $a_t \leq \hat{a}_t \leq b_t \leq \hat{b}_t$, then the forecast interval inclines to the right, and indicated as *FIR*.
(4) If $\hat{a}_t \leq a_t \leq \hat{b}_t \leq b_t$, then the forecast interval inclines to the left, which is expressed as *FIL*.

It is difficult to know which forecasting situation is better than the others. By calculating the length of the exclusive-or, it can help us to find which forecast interval is better for forecasting. So we will present some properties for forecasting situations.

Property 2.1 (*The Interval Length of XOR for FIW*). *If $\hat{X}$ is FIW, then the interval length of XOR, denoted by $\left\|X \triangle \hat{X}\right\|$, is $\left\|X \triangle \hat{X}\right\| = -2\partial r$, where $\partial r = r - \hat{r}$.*

Proof. Since $\hat{X}$ is *FIW*, $X \triangle \hat{X} = [\hat{a}, a] \cup [b, \hat{b}]$. By $a = c - r, \hat{a} = \hat{c} - \hat{r}, b = c + r$, and $\hat{b} = \hat{c} + \hat{r}$, we have $a - \hat{a} = \left(c - \hat{c}\right) - \left(r - \hat{r}\right)$ and $\hat{b} - b = \left(\hat{c} - c\right) + \left(\hat{r} - r\right)$. Therefore, $\left\|X \triangle \hat{X}\right\| = \left\|[\hat{a}, a]\right\| + \left\|[b, \hat{b}]\right\| = \left(a - \hat{a}\right) + (\hat{b} - b) = -2\left(r - \hat{r}\right) = -2\partial r$. ■

Property 2.2 (*The Interval Length of XOR for FIN*). *If $\hat{X}$ is FIN, then the interval length of XOR is $\left\|X \triangle \hat{X}\right\| = 2\partial r$, where $\partial r = r - \hat{r}$.*

Proof. For $\hat{X}$ is *FIN*, $X \triangle \hat{X} = [a, \hat{a}] \cup [\hat{b}, b]$. From $\hat{a} - a = \left(\hat{c} - c\right) - \left(\hat{r} - r\right)$ and $b - \hat{b} = \left(c - \hat{c}\right) + \left(r - \hat{r}\right)$, $\left\|X \triangle \hat{X}\right\| = \left\|[a, \hat{a}]\right\| + \left\|[\hat{b}, b]\right\| = \left(\hat{a} - a\right) + (b - \hat{b}) = 2\partial r$. ■

Property 2.3 (*The Interval Length of XOR for FIR*). *If $\hat{X}$ is FIR, then the interval length of XOR is $\left\|X \triangle \hat{X}\right\| = -2\partial c$, where $\partial c = c - \hat{c}$.*

Proof. If $\hat{X}$ is *FIR*, $X \triangle \hat{X} = [a, \hat{a}] \cup [b, \hat{b}]$ By $\hat{a} - a = \left(\hat{c} - c\right) - \left(\hat{r} - r\right)$ and $\hat{b} - b = \left(\hat{c} - c\right) + \left(\hat{r} - r\right)$, $\left\|X \triangle \hat{X}\right\| = \left\|[a, \hat{a}]\right\| + \left\|[b, \hat{b}]\right\| = \left(\hat{a} - a\right) + (\hat{b} - b) = -2\partial c$. ■

Property 2.4 (*The Interval Length of XOR for FIL*). *If $\hat{X}$ is FIL, then the interval length of XOR is $\left\|X \triangle \hat{X}\right\| = 2\partial c$, where $\partial c = c - \hat{c}$.*

Proof. When $\hat{X}$ is *FIL*, $X \triangle \hat{X} = [\hat{a}, a] \cup [\hat{b}, b]$. By $a - \hat{a} = \left(c - \hat{c}\right) - \left(r - \hat{r}\right)$ and $b - \hat{b} = \left(c - \hat{c}\right) + \left(r - \hat{r}\right)$, $\left\|X \triangle \hat{X}\right\| = \left\|[\hat{a}, a]\right\| + \left\|[\hat{b}, b]\right\| = \left(a - \hat{a}\right) + (b - \hat{b}) = 2\partial c$. ■

Example 2.2. Let $X = [1, 3] = (2, 1), \hat{X} = [2, 6] = (4, 2)$. Then $\hat{X}$ is *FIR*.

From Property 2.3, we get $\left\|X \triangle \hat{X}\right\| = 2\,(4 - 2) = 4$.

Example 2.3. Let $X = [3, 5] = (4, 1), \hat{X} = [2, 6] = (4, 2)$. Then $\hat{X}$ is *FIW*.

From Property 2.1, we obtain $\left\|X \triangle \hat{X}\right\| = 2\,(2 - 1) = 2$.

3. Efficiency evaluation for interval time series forecasting

The quality of the forecast result is the most concern of the analysts after we use interval to proceed forecasting. In a traditional forecasting of time series, it is to compare the distances between the actual values and the predicted values to assess the quality of forecasting. With regard to the interval forecasting, not only the forecasting of interval length, we are also concerned with the location disparity between the predicted interval and the actual interval. Therefore, traditional methods to evaluate the forecasting efficiency of time series are unable to analyze the forecasting performance for interval time series. The following will define the criteria for analyzing the efficiency of interval forecasting.

3.1. The mean squared error of interval

Suppose an interval time series can be represented by $\{X_t = (c_t, r_t)\}$, and the forecast interval time series will be specified with $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right)\right\}$. The definition of the mean squared error of interval is giving as follows:

Definition 3.1 (*Mean Squared Error of Interval with respect to Position and Length, MSEP and MSEL*). Let $\delta_{c_t} = c_t - \hat{c}_t$ is the position error between $\hat{X}_t$ and X_t, then the mean squared error of interval position (*MSEP*) is given by

$$MSEP = \frac{\sum_{t=1}^{s} \delta^2_{c_{n+t}}}{s} = \frac{\sum_{t=1}^{s} \left(c_{n+t} - \hat{c}_{n+t}\right)^2}{s}.$$

Let $\varepsilon_{r_t} = r_t - \hat{r}_t$ is the error between the length of the forecast interval $\hat{X}_t$ and that of the actual interval X_t, then the mean squared error of interval length (*MSEL*) is given by

$$MSEL = \frac{\sum_{t=1}^{s} \varepsilon_{r_{n+t}}^2}{s} = \frac{\sum_{t=1}^{s} \left(r_{n+t} - \hat{r}_{n+t}\right)^2}{s},$$

where n denotes the current time, and s is the number of the preceding intervals.

Definition 3.2 (*Mean Squared Error of Interval, MSEI*). The error between the forecast intervals $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right)\right\}$ and the actual intervals $\{X_t = (c_t, r_t)\}$ consists of two parts: the position error and the length error. The mean squared error of interval (*MSEI*) is given by

$$MSEI = \frac{\sum_{t=1}^{s} \left(c_{n+t} - \hat{c}_{n+t}\right)^2}{s} + \frac{\sum_{t=1}^{s} \left(r_{n+t} - \hat{r}_{n+t}\right)^2}{s} = MSEP + MSEL,$$

where n represents the current time, and s is the number of the preceding interval.

Example 3.1. Let the interval time series be $X_1 = [4, 6] = (5, 1)$, $X_2 = [5, 8] = (6.5, 1.5)$, the forecast intervals are $\hat{X}_1 = [2.8, 5.4] = (4.1, 1.3)$ and $\hat{X}_2 = [3.8, 7.8] = (5.8, 2)$. Then the mean squared error of interval position is given by

$$MSEP = \frac{(5 - 4.1)^2}{2} + \frac{(6.5 - 5.8)^2}{2} = 0.65,$$

and the mean squared error of interval length is given by

$$MSEL = \frac{(1 - 1.3)^2}{2} + \frac{(1.5 - 2)^2}{2} = 0.17.$$

Thus $MSEI = MSEP + MSEL = 0.65 + 0.17 = 0.82$.

3.2. The mean relative interval error

Consider the interval $X = [4, 7] = (5.5, 1.5)$, and the forecast intervals $\hat{X}_1 = [1, 8] = (4.5, 3.5)$ and $\hat{X}_2 = [6, 8] = (7, 1)$ obtained by two different forecasting methods. The *MSEI* of $\hat{X}_1$ (denoted as $MSEI_1$) is 5. The *MSEI* of $\hat{X}_2$ (denoted as $MSEI_2$) is 2.5. Then $\hat{X}_2$ is a better forecast interval than $\hat{X}_1$ by comparing $MSEI_1$ and $MSEI_2$. Actually, it is not true. Although the radius of $\hat{X}_1$ is larger than that of $\hat{X}_2$, the central point of $\hat{X}_1$ is closer to the central point of X. Since the range of $\hat{X}_1$ covers the range of the actual interval X is more than the range of $\hat{X}_2$ does. As a result, we still regard $\hat{X}_1$ as the better forecast interval.

Therefore, while considering the efficiency of the interval forecasting, it is the most important whether the forecast interval does cover the actual interval. Explicitly speaking, a forecast result is better if the center $\hat{c}$ is closer to the center c and their interval overlap is larger. By combining the two factors of the center and the radius of interval, we have three decision conditions; (1) when $\frac{|c-\hat{c}|}{r+\hat{r}} < 1$, there is an overlap of the forecast and the actual intervals, it means that the interval forecasting is better; (2) when $\frac{|c-\hat{c}|}{r+\hat{r}} << 1$, it means that there is more overlap so that the interval forecasting is much better; (3) while $\frac{|c-\hat{c}|}{r+\hat{r}} > 1$, the forecast interval and the actual interval are completely separated, so the interval forecasting is undesirable. Because $\left\|X_t \ominus \hat{X}_t\right\| = 2\left(r + \hat{r}\right)$, $\frac{|c_t-\hat{c}_t|}{\left\|X_t \ominus \hat{X}_t\right\|}$ can be a criterion for the evaluating the forecasting. Therefore, we propose the following definition to be another criterion for analyzing the integrated efficiency of interval forecasting.

Definition 3.3 (*Mean Relative Interval Error, MRIE*). Let $\varepsilon_t = \frac{2|c_t-\hat{c}_t|}{\left\|X_t \ominus \hat{X}_t\right\|}$ is the relative error between the forecast interval $\hat{X}_t$ and the actual interval X_t, then the mean relative interval error (*MRIE*) is given by

$$MRIE = \frac{1}{s}\sum_{t=1}^{s} \varepsilon_{n+t} = \frac{1}{s}\sum_{t=1}^{s} \frac{2\left|c_t - \hat{c}_t\right|}{\left\|X_t \ominus \hat{X}_t\right\|},$$

where n denotes the current of time, and s is the number of the preceding intervals.

Example 3.2. Assume as in Example 3.1, then the mean relative interval error is given by

$$MRIE = \frac{1}{2}\left(\frac{2\,|5 - 4.1|}{2\,(1 + 1.3)} + \frac{2\,|6.5 - 5.8|}{2\,(1.5 + 2)}\right) = 0.34.$$

Example 3.3. Consider the interval $X = [4, 7] = (5.5, 1.5)$, the forecast intervals $\hat{X}_1 = [1, 8] = (4.5, 3.5)$ and $\hat{X}_2 = [6, 8] = (7, 1)$. Assume the *MRIE* of $\hat{X}_1$ be denoted by $MRIE_1$ and the *MRIE* of $\hat{X}_2$ be denoted by $MRIE_2$. Then $MRIE_1 = 0.2$ and $MRIE_2 = 0.6$.

4. The mean ratio of exclusive-or

4.1. Are MSEI and MRIE good enough for the efficiency analysis of forecasting?

As described in Section 3.1, the error between the forecast interval $\hat{X}_t$ and the actual interval X_t contains two parts; the position error and the length error. The former is the distance between the central points of two intervals, while the latter is the difference between the radii of two intervals. If the mean squared errors of the position and the length are always summed up, it will be hard to discern the efficiencies of the forecasting method between the position and the length.

While trying to use *MRIE*, it seems to be superior to *MSEI*. But there are some questionable problems in the four forecasting situations. For instance, the interval $X = [4, 7] = (5.5, 1.5)$, and the forecast intervals $\hat{X}_1 = [1, 8] = (4.5, 3.5)$ and $\hat{X}_2 = [0, 10] = (5, 5)$ obtained by two different forecasting methods. Then the *MRIE* of $\hat{X}_1$ (denoted as $MRIE_1$) is 0.1. The *MRIE* of $\hat{X}_2$ (denoted as $MRIE_2$) is 0.08. And $\hat{X}_2$ looks like better than $\hat{X}_1$ by evaluating $MRIE_1$ and $MRIE_2$. Is it right? Since $\hat{X}_1$ and $\hat{X}_2$ are *FIW*s, the forecast radius is longer, the *MRIE* will be smaller. Hence the *MRIE* is not an ideal method especially when the forecast interval is too wide.

How do we know which one is better interval forecasting in the four forecasting situations? For example, the actual interval is $X = [4, 7] = (5.5, 1.5)$, and the forecast intervals are $\hat{X}_1 = [2.2, 8.4] = (5.3, 3.1)$ and $\hat{X}_2 = [4.2, 6] = (5.1, 0.9)$. The *MSEI* of $\hat{X}_1$ is 2.6 and the *MSEI* of $\hat{X}_2$ is 0.52. The *MRIE* of $\hat{X}_1$ is 0.04 and the *MRIE* of $\hat{X}_2$ is 0.17. Is $\hat{X}_1$ better than $\hat{X}_2$ by observing their *MRIE*s? Or is $\hat{X}_2$ finer than $\hat{X}_1$ by examining their *MSEI*s? It is very difficult to describe which one is superior among them.

Since $\hat{X}_1$ is *FIW* that means it can cover all range of the actual interval X. $\hat{X}_2$ is *FIN* which is enclosed by the actual interval X.Because the forecast interval is too wide, it could be forced to include some 'noisy message'. In consequence it will disturb our decision. On the contrary, while the forecast interval is too narrow such as $\hat{X}_2$, it maybe lose some 'important message'. Thus, it will mislead the executive's judgment. The similar question always happens when the forecast interval is *FIL* or *FIR*. They conclude some noisy message and lose some important message at the same time. It is unfair to compare the forecasting efficiency with the different forecasting situations. Sometimes it depends on policymaker's requirement. If we try to clarify how better in the same forecasting situation, the *XOR* can offer a good explanation in the forecasting efficiency. We will present another technique for forecasting efficiency analysis.

4.2. The mean ratio of XOR

Generally speaking, if the center and radius of the forecast interval are almost matched the center and radius of the actual interval, respectively, then it is a better interval forecasting. Therefore, when the length of *XOR* showing non-overlap of the actual interval and the forecast interval is small, it appears the forecast interval covers more the actual interval. Using the character of *XOR*, we offer another technique of the efficiency analysis for the interval time series forecasting.

Definition 4.1 (*Mean Ratio of XOR, MRXOR*). Let $\{X_t = (c_t, r_t)\}$ be an interval time series, and $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right)\right\}$ is the forecast interval time series. The mean ratio of exclusive-or is denoted *MRXOR*, and the definition of the *MRXOR* is given as follows:

$$MRXOR = \frac{1}{s}\sum_{t=1}^{s}\frac{\left\|X_{n+t}\,\Delta\,\hat{X}_{n+t}\right\|}{\|X_{n+t}\|},$$

where n denotes the current of time, and s is the number of the preceding intervals.

Definition 4.2 (*The Efficiency of MRXOR*). Let $\{X_t = (c_t, r_t)\}$ be an interval time series, and the forecast interval time series $\left\{\hat{X}_{1t} = \left(\hat{c}_{1t}, \hat{r}_{1t}\right)\right\}$ and $\left\{\hat{X}_{2t} = \left(\hat{c}_{2t}, \hat{r}_{2t}\right)\right\}$ be obtained by two different forecasting methods. If the *MRXOR* of $\left\{\hat{X}_{1t}\right\}$ (denoted as $MRXOR_1$) is smaller than the *MRXOR* of $\left\{\hat{X}_{2t}\right\}$ (denoted as $MRXOR_2$), then we say the forecast interval $\left\{\hat{X}_{1t}\right\}$ is efficient as compared to the forecast interval $\left\{\hat{X}_{2t}\right\}$ *.i.e.* $\left\{\hat{X}_{1t}\right\}$ is more efficient than $\left\{\hat{X}_{2t}\right\}$, if $MRIXOR_1 < MRIXOR_2$.

Example 4.1. Assume as in Example 3.1.

Since $\hat{X}_1$ and $\hat{X}_2$ are *FIL*s and from Property 2.4, $\left\|X_1\,\Delta\,\hat{X}_1\right\| = 2\,(5 - 4.1) = 1.8$ and $\left\|X_2\,\Delta\,\hat{X}_2\right\| = 2\,(6.5 - 5.8) = 1.4$. Thus the mean ratio of exclusive-or is given by

$$MRXOR = \frac{1}{2}\left(\frac{1.8}{2} + \frac{1.4}{3}\right) = 0.68.$$

Example 4.2. Let the random interval $X = [4, 7] = (5.5, 1.5)$, and the forecast intervals be $\hat{X}_1 = [2.2, 8.4] = (5.3, 3.1)$ and $\hat{X}_2 = [4.2, 6] = (5.1, 0.9)$. Since $\hat{X}_1$ is *FIW*, we have $\left\| X_1 \,\Delta\, \hat{X}_1 \right\| = 2\,(3.1 - 1.5) = 3.2$. By Property 2.1, $MRXOR_1 = \frac{3.2}{3} = 1.07$. Similarly, $\hat{X}_2$ is *FIN*, then we have $\left\| X_2 \,\Delta\, \hat{X}_2 \right\| = 2\,(1.5 - 0.9) = 1.2$ by Property 2.2.

Therefore, $MRXOR_2 = \frac{1.2}{3} = 0.40$. Because $MRXOR_1 > MRXOR_2$, $\hat{X}_2$ is more efficient than $\hat{X}_1$.

4.3. Discussion of MRXOR in different forecasting situations

If we consider two sets of the forecast intervals having the same forecasting situation, *MRXOR* will be a good method of efficiency analysis. What information can be revealed by *MRXOR* in the forecast solutions? Assume $\hat{X}_1 = \left(\hat{c}_1, \hat{r}_1\right)$ and $\hat{X}_2 = \left(\hat{c}_2, \hat{r}_2\right)$ attained by different forecasting methods be the forecast solutions of the actual interval. Their mean ratios of *XOR* are $MRXOR_1$ and $MRXOR_2$, respectively. *MRXOR* is discussed according to four forecasting situations as follows.

Case 1. When $\hat{X}_1$ and $\hat{X}_2$ are *FIW*s.

If $\left\|\hat{X}_1\right\| < \left\|\hat{X}_2\right\|$, then $MRXOR_1 < MRXOR_2$. It means $\hat{X}_2$ has more noisy message than $\hat{X}_1$. Therefore $\hat{X}_1$ is more efficient than $\hat{X}_2$.

When the forecasting interval time series $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right) | t = 1, 2, \ldots, s\right\}$ are all *FIW*s, what should we do for this state? Because the interval radius influences the length of *XOR* from Property 2.1, we will correct the forecasting method of the interval radius first.

Case 2. When $\hat{X}_1$ and $\hat{X}_2$ are *FIN*s.

If $\left\|\hat{X}_1\right\| < \left\|\hat{X}_2\right\|$, then $MRXOR_1 > MRXOR_2$. It means $\hat{X}_1$ lose more message than $\hat{X}_2$. Therefore $\hat{X}_2$ is more efficient than $\hat{X}_1$.

Considering the forecast interval time series $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right) | t = 1, 2, \ldots, s\right\}$ are all *FIN*s. From Property 2.2, the interval radius dominates the length of *XOR*. Then the forecasting method of the interval radius will be properly corrected.

Case 3. When $\hat{X}_1$ and $\hat{X}_2$ are *FIR*s.

The interval center can manipulate the *XOR* through Property 2.3. As the center of $\hat{X}_1$ is closer to the center of X than that of $\hat{X}_2$, it presents $\hat{X}_1$ covers more vital message and contain less boisterous message than $\hat{X}_2$. That is, if $c < \hat{c}_1 < \hat{c}_2$, then $MRXOR_1 < MRXOR_2$. Therefore $\hat{X}_1$ is more efficient than $\hat{X}_2$.

When the proceeding forecasting interval time series $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right) | t = 1, 2, \ldots, s\right\}$ are all *FIR*s, the forecasting method of the interval center will be modified.

Case 4. When $\hat{X}_1$ and $\hat{X}_2$ are *FIL*s.

As the same argument in Case 3, *XOR* can be operated by the interval center through Property 2.4. When the center of $\hat{X}_1$ is nearer to the center of X than that of $\hat{X}_2$, $\hat{X}_1$ encloses more essential message and has fewer confusing message than $\hat{X}_2$ does. That is, if $c > \hat{c}_1 > \hat{c}_2$, then $MRXOR_1 < MRXOR_2$. Therefore, $\hat{X}_1$ is more efficient than $\hat{X}_2$.

Once the proceeding forecast interval time series $\left\{\hat{X}_t = \left(\hat{c}_t, \hat{r}_t\right) | t = 1, 2, \ldots, s\right\}$ are all *FIR*s, we will modify the forecasting method of the interval center.

5. Empirical studies

In this section we use an example to illustrate the efficiency analysis of forecasting techniques. Table 1 lists the actual intervals and three sets of forecasting values obtained, respectively, by three forecasting methods. Fig. 2 shows the actual intervals and the forecast intervals obtained by Method 1. The dark solid line represents the actual interval and the gray dash line symbolizes the forecast interval. It illustrates the forecast intervals are *FIW*s. The forecast intervals of Method 2 are *FIR*s in Fig. 3. In Fig. 4, the forecast intervals attained form Methods 3 are *FIN*s. Table 2 demonstrates their *MSEI*, *MRIE* and *MRXOR*.

As described in Section 4.1, the forecast results have too wide interval lengths and cover actual data completely in Fig. 1. The forecasting method 1 presents the minimum *MRIE* than other methods. But the forecast intervals contain noisier message in Method 1, it has the worst *MSEI* and *MRXOR*. Method 2 performs better *MSEI* than the others. The reason is the lengths of the forecast intervals almost equal to the actual interval lengths. Owing to the forecast results are *FIR*s in Fig. 3, the centers of forecast intervals deviate to the centers of actual intervals badly. Then *MRIE* of Method 2 is made larger than the others.

When we evaluate their *MRXOR*s, data in Method 3 get the smallest amount of *MRXOR*. Not because their centers are near to the actual centers, but also the relative length of non-overlap between forecast intervals and the actual intervals are less than the others. The radii of intervals by Method 3 are small so that *MSEI* of Method 3 is larger than *MSEI* of Method 2. But it is still better than that of Method 1. The Method 3 is a good forecasting technique by means of surveying among those *MESI*s, *MRIE*s and *MRXOR*s. As shown in Table 2, if the value of *MRXOR* is small, then *MSEI* and *MRIE* are not too large.

Table 1
The actual intervals and three sets of forecasting intervals

Actual interval		Method 1		Method 2		Method 3	
$[a_t, b_t]$	(c_t, r_t)	$[\hat{a}_{1t}, \hat{b}_{1t}]$	$(\hat{c}_{1t}, \hat{r}_{1t})$	$[\hat{a}_{2t}, \hat{b}_{2t}]$	$(\hat{c}_{2t}, \hat{r}_{2t})$	$[\hat{a}_{3t}, \hat{b}_{3t}]$	$(\hat{c}_{3t}, \hat{r}_{3t})$
[32.25, 34.95]	(33.60, 1.35)	[31.10, 35.55]	(33.33, 2.22)	[33.69, 36.35]	(35.02, 1.33)	[32.88, 34.20]	(33.54, 0.66)
[28.10, 32.00]	(30.05, 1.95)	[27.75, 34.25]	(31.00, 3.25)	[29.34, 33.10]	(31.22, 1.88)	[29.94, 31.66]	(30.80, 0.86)
[26.85, 31.75]	(29.30, 2.45)	[26.00, 33.40]	(29.70, 3.70)	[27.95, 32.73]	(30.34, 2.39)	[27.85, 29.95]	(28.90, 1.05)
[27.10, 30.00]	(28.55, 1.45)	[25.95, 31.35]	(28.65, 2.70)	[28.20, 31.28]	(29.74, 1.54)	[28.04, 29.44]	(28.74, 0.70)
[26.00, 27.35]	(26.68, 0.68)	[24.85, 31.20]	(28.02, 3.17)	[26.44, 27.76]	(27.10, 0.66)	[26.80, 27.30]	(27.05, 0.25)
[26.20, 28.85]	(27.52, 1.33)	[24.50, 30.60]	(27.55, 3.05)	[26.45, 29.05]	(27.75, 1.30)	[26.35, 27.15]	(26.75, 0.40)

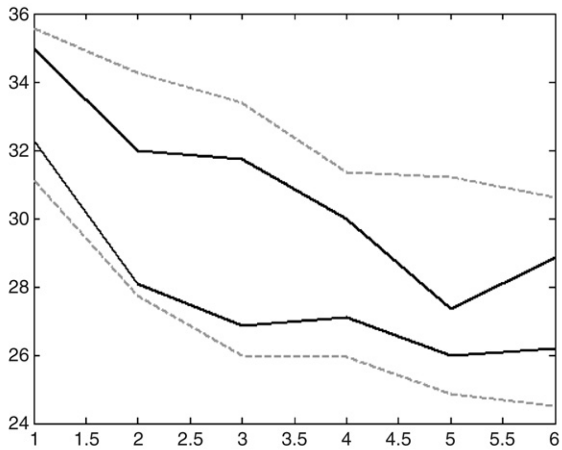

Fig. 2. The forecast intervals are too wide.

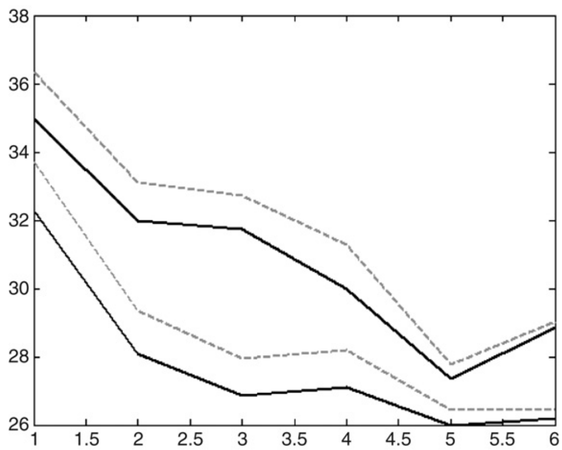

Fig. 3. The forecast intervals incline to right.

Table 2
The comparison of evaluating forecasting performance for interval data

	MSEI	*MRIE*	*MRXOR*
Method 1	2.96	0.12*	1.28
Method 2	1.02*	0.31	0.62
Method 3	1.12	0.23	0.58*

6. Conclusions

In the progress of scientific research and analysis, the uncertainty in the statistical numerical data is the crux of the problem that the traditional mathematical models are hard to be established. Manski [13] has pointed out that the numerical data are over-demanded and over-explained. If we exploit this artificial accuracy to do causal analysis or measurement,

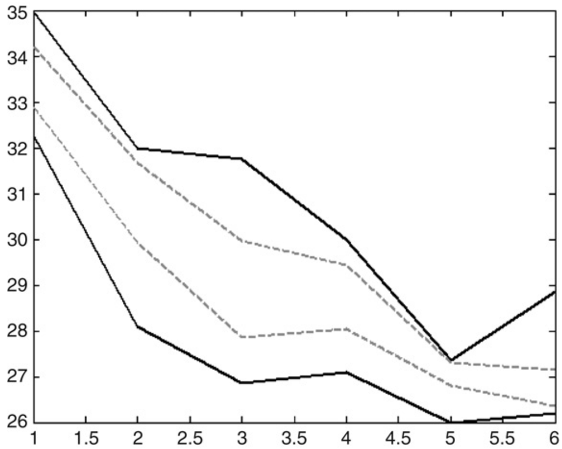

Fig. 4. The forecast intervals are too narrow.

it may lead to the deviation of the causal judgment, the misleading of the decision model, or the exaggerated difference between the predicted result and the actual data. This paper proposes to use the interval data to avoid such risks happening. In fact, using interval data to establish a model and to predict, we can find that the forecasting in each step is carried out by means of intervals, so as to increase the objectiveness of the forecast results. In the general aspect, the "intervalization" seems to be a very normal phenomenon too.

This paper discusses the quality of the forecast result through evaluating forecasting performance, such as *MSEI*, *MRIE* and *MRXOR*. They had advantages and disadvantages as illustrated in Sections 3 and 4. From the example in Section 5, we find *MRXOR* provides an important efficiency analysis for interval forecasting. Based on the value of *MRXOR* in different forecasting situations, such as *FIW*, *FIN*, *FIR* and *FIL*, it may modify the forecasting method of the center and radius, respectively. It is noteworthy that if we can establish a good efficiency process, we can make a superior interval forecasting for the interval time series.

Although the approaches in this paper proposed the efficiency evaluations of interval forecasting, there are some problems still remaining to be solved and some improvement can be done for further research, which is described, respectively, as follows.

1. There are so many factors associated with interval data. Consequently, we only consider the boundaries of the intervals and their centers and the radiuses caused by all factors of efficiency analysis in this paper. If it needs to make the result more accurate, it can consider finding out the key factors of influencing the interval data.
2. Besides FIW, FIN, FIR and FIL, a forecasting situation was not discussed in this paper. That is the forecast interval and the actual interval not overlapping at all. There are two cases: the forecast interval is certainly greater than the actual interval. And the forecast interval is certainly smaller than the actual interval (Interval FAQ from Dominque Faudot [14]). They are not good forecast outcomes at all. We don't like such forecasting result happened certainly. Once it occurs. Computing their *MRXOR*s may reveal what drawbacks of the forecast system does? And how is the forecast scheme made improvements?
3. What is a good forecast? When the forecast results have the same forecasting situations, they are easily judged which one is better forecast among those forecasting methods. While the forecast consequences are not in the same situation such as FIW and FIN, it is hard to choose between them. Especially their *MRXOR*s are equal; they always make us in confusion. Is the interval containing entire actual data and extra noisy message superior? Or is the interval which is not disturbed by the boisterous message but losing some data fit? It should be defined a criterion which is more sufficient to show the efficiency of interval forecasting.

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