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Abstract

Purpose – This paper sets out to consider the problem that the initial value of the American option is less than its fair price; this implies that the replication portfolio does not exist in the market.

Design/methodology/approach – The paper develops an optimization model whose solution provides an optimal strategy for the writer to minimize the expected loss for this problem.

Findings – The numerical results reveal that loaning money to construct a replication portfolio may not be an optimal strategy for the writer.

Practical implications – The solution of the minimum expected loss model provides an optimal strategy to construct a lower expected loss portfolio.

Originality/value – The numerical results reveal that loaning money to construct a replication portfolio may not be an optimal strategy for the writer.

Keywords Pricing, Options markets, Hedging, United States of America

Paper type Research paper

1. Introduction

In the no-arbitrage pricing model, the fair price of the American option is equal to the cost of its replication portfolio. The writer who publishes an American option always constructs a replication portfolio with the option's initial value. When the initial value of the American option is less than its fairs price, the writer does not have enough money to construct a replication portfolio. Therefore, to develop a self-finance model whose solution provides an optimal hedging strategy for writer to minimize the expected loss becomes an important problem for the writer.

To develop this self-finance model, the optimal exercise strategy of the buyer is the necessary input parameter. The problem of finding the optimal exercise strategy as well as the value of the American option can usually be formulated as a free boundary problem (FBP). To this day, a considerable number of researchers have studied the solution of the FBP by using the technique of mathematical programming. Dempster and Hutton (1999) investigated a linear programming formulation for formulating the numerical solution of the finite difference approximations to the FBP. Dempster and Richards (2000) proposed a special simplex solver for tridiagonal constraint matrices, exploiting the rapid LU decomposition algorithms for such matrices, which produces dramatic speed-ups.

On the other hand, King (2002) modelled the asset price process as a scenario tree and proposed a stochastic programming model for the hedging of contingent claim in the discrete time, discrete state case on this tree. Pennanen and King (2004) proposed a convex programming model, that extends King's model to analyze the American

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contingent claim in the incomplete markets and obtained the martingale-expressions for seller's and buyer's prices.

However, the binomial method which solves a linear system recursively provides a simple and intuitive numerical method for valuing American option. In this paper, we first formulates the procedure of the binomial method as a mixed-integer nonlinear programming (MINLP) model. To investigate the exercising strategy of buyer, we add a 0-1 variable in both models to represent the decision of an option buyer. The solution of the MINLP model provide a perfect hedging portfolio for writers, an optimal exercising strategy for buyers, and a fair price for both writers and buyers.

The methods to solve a MINLP problem require dramatically more mathematical computation. We show that this model can be solved by their non-linear relaxation. This provides a far more efficient approach for computing the value, the replication portfolio and the optimal exercising strategy.

Now the optimal exercise strategy of the buyer has been obtained by solving this non-linear relaxation. When the market price is less than the fair price, the buyer still have a right to exercise the American option based on this exercising strategy. Regarding the solution of the MINLP model as an input parameter, we shall propose an optimization model whose solution will provide an optimal hedging strategy for the writer to minimize the expected loss.

In the computational results, we find that the use of the non-linear relaxations reduces the computation time. When the fair price is less than the market price, the computational results reveal that making a loan of money to construct a perfect hedging portfolio may not be the optimal strategy for the writer. We find that the solution of the minimum expected loss model provides an optimal strategy to construct a lower expected loss portfolio.

The rest of this paper is organized as follows. Sections 2.1 and 2.2 introduce the notations and the binomial valuation approach, respectively. Section 2.3 proposes and analyzes the MINLP models for European option and for American option. In Section 3, we extend the MINLP model to investigate writer's problems. The computational results are displayed in Section 4. Section 5 provides a concise conclusion and a direction of the future studies.

2. The binomial option pricing method

The set of parameters and variables used in the model includes the forecasts of the stock price, the allocation of money market account and stock position, and the value of option over the investment time horizon.

2.1 Notations

Parameters:

T = investment time horizon.

n = number of time step.

S_0 = initial value of the stock.

K = the exercise price of the option.

r = r is one plus risk-free rate.

$u(d)$ = size of upward (downward) movement.

Indices:

t = time step; and

$i \leq t$ = state at each time step t.

Variables:

x_t^i, y_t^i = allocation of money market account and stock, respectively, in the portfolio, at time t and state i;

z_t^i = decision variable of the option buyer at time t and state i; if buyer exercise the option set $z_t^i = 1$, otherwise $z_t^i = 0$;

c_t^i = Option value at time t and state i; and

v_t^i = value of the portfolio consisted of x_t^i market account and y_t^i stock.

2.2 Binomial pricing approach

Consider the one period binomial tree model: there are two states, up and down, at time 1. The asset price and the final payoff of the European call are uS_0 and $\max\{uS_0 - K, 0\}$ for the "up" state. For the "down" state, the asset price and the final payoff are dS_0 and $\max\{udS_0 - K, 0\}$. The increment size, u and d, are selected to fit the asset's dynamics. There are several possible selection methods provided by Cox *et al.* (1979) and Jarrow and Rudd (1983) based on the assumption of the asset price dynamics.

Extending one period model to multi-period model, the time interval $[0, T]$ is divided equally into n time periods. It is convenient to label the nodes in the binomial tree by (t, i) which indicates the node at time step t and state i. Hence, the asset price on node (t, i) is denoted by $S_t^i = S_0 u^i d^{t-i} = S_0 u^{2i-t}, \quad i = 0, 1, \ldots, t$ and $t = 0, 1, \ldots, n$. The last equality holds for the selection of $d = 1/u$. By applying the no-arbitrage condition, the value of the European call at node (t, i), denoted as c_t^i, is:

$$c_t^i = x_t^i + S_t^i y_t^i, \tag{1}$$

where (x_t^i, y_t^i) is the solution of following linear system:

$$r x_t^i + S_{t+1}^{i+1} y_t^i = c_{t+1}^{i+1}, \tag{2}$$

$$r x_t^i + S_{t+1}^{i} y_t^i = c_{t+1}^{i}, \tag{3}$$

for all $i = 0, 1, \ldots, t$ and $t = 0, 1, \ldots, n-1$. At the expiration date, namely $t = n$, the value of a call option is given as $c_n^i = \max(S_n^i - K, 0)$. To avoid an arbitrage opportunity, we should make an assumption as follows (Cox *et al.*, 1979): $d < 1 \leq r > u$. By working backwards from node n to node 1, we solve these linear systems recursively.

The solution of these linear systems provides a dynamic replication trading strategy for the entire binomial tree which is called hedging portfolio by the practitioners. The number of shares in the hedging portfolio is called delta. In summary, we have devised a self-financing trading strategy which costs c_0 at the initialization and without adding any sources along the way as it generates. Note that the self-financing trading strategy relates only to the interim time step $t = 1, 2, \ldots, n-1$. At $t = n$, we will collect a random amount of final payoff depending on the terminal asset price, namely $\max(S_n^i - K, 0)$ for an European call option.

The American call, however, gives the buyer a right to exercise the option before the expiration. The value of the American call at each node (t, i) is higher than the immediate exercise price. Thus, the value of American call at each interim node satisfies the following equation:

$$c_t^i = \max\left\{v_t^i, \left(S_t^i - K\right)\right\} \tag{4}$$

at node (t, i), where v_t^i is defined as:

$$v_t^i = x_t^i + S_t^i y_t^i. \tag{5}$$

Note that, for the case of European call, c_t^i is equal to v_t^i for all $t < n$.

2.3 MINLP valuation models

A rational buyer does not exercise the call when the final payoff $S_n - K$ is negative. In this section, we introduce a decision variable z_n^i which is a 0-1 variable to represent the decision of an option buyer. Therefore, the final payoff of a call can be rewritten as:

$$c_n^i = \left(S_n^i - K\right)z_n^i. \tag{6}$$

If $z_n^i = 1$, the buyer exercises the call option and gets the final payoff $S_n^i - K$; otherwise, the buyer disclaims the right to exercise and gets nothing. Hence, the rational option buyer's task is to find c_0 which maximize the expected utility of his final payoff.

Now, we consider the case of an American call option. The current value of an American call option in the internal of binomial tree can be derived as equations (4) and (5). This implies that the buyer exercises the call option if the exercising value is greater than the replication portfolio value; otherwise the buyer keeps holding the call option. We introduce a binary variable z_t^i not only for the ending node but also for every interim node to represent the option buyer whether exercise the option or not. Therefore, the value of American call option can be defined as:

$$c_t^i = \left(S_t^i - K\right)z_t^i + v_t^i(1 - z_t^i). \tag{7}$$

Then a MINLP model for valuation an American call option is formulated as follows:
Model A:

$$\max \sum_{t=0}^{n} \sum_{i=0}^{t} \frac{p_t^i}{r^t} U\left(c_t^i\right) \tag{8}$$

$$\text{s.t. } rx_t^i + S_{t+1}^{i+1} y_t^i = c_{t+1}^{i+1}, \quad i = 0, 1, \ldots t, \ \ t = 0, 1, \ldots, n-1, \tag{9}$$

$$rx_t^i + S_{t+1}^{i} y_t^i = c_{t+1}^{i}, \quad i = 0, 1, \ldots, t, \ \ t = 0, 1, \ldots, n-1, \tag{10}$$

$$x_t^i + S_t^i y_t^i = v_t^i, \quad i = 0, 1, \ldots, t, \ \ t = 0, 1, \ldots, n-1, \tag{11}$$

$$\left(S_n^i - K\right) z_n^i = c_n^i, \quad i = 0, 1, \ldots, n \tag{12}$$

$$\left(S_t^i - K\right) z_t^i + v_t^i (1 - z_t^i) = c_t^i, \quad i = 0, 1, \ldots, t, \ \ t = 0, 1, \ldots, n-1, \tag{13}$$

$$x_t^i, y_t^i \in \mathbb{R}, \quad i = 0, 1, \ldots, t, \ \ t = 0, 1, \ldots, n,$$

$$z_t^i \in \{0, 1\}, \quad i = 0, 1, \ldots, t, \ \ t = 0, 1, \ldots, n, \tag{14}$$

$$v_t^i, c_t^i \geq 0, \quad i = 0, 1, \ldots, t, \ \ t = 0, 1, \ldots, n.$$

where $p_t^i, \quad i = 0, \ldots, t, \ t = 0, \ldots, n$ is the objective probability of option buyer and U is option buyer's utility function which is strictly increasing and concave.

Since z_t^i is an integer decision variable, Model A is an MINLP model. Solving an MINLP program is always much harder than a similarly sized pure non-linear program. The non-linear relaxation of Model A is obtained by replacing equation (14) by:

$$0 \leq z_t^i \leq 1,$$

We will analytical investigate the non-linear relaxation solution for Model A and show that Model A and its non-linear relaxation have the same optimal solution.

Let $x_t = (x_t^1, x_t^2, \ldots, x_t^t)'$ denote the allocation of the money market account over all the state at time t. Then $x = (x_1, x_2, \ldots, x_n)$ denote the allocation of the money market account over all the binomial tree. The same definition is also applied to y, z, c and v.

Suppose that (x, y, z, c) satisfies equations (2) and (3), we have:

$$\begin{bmatrix} x_t^i \\ y_t^i \end{bmatrix} = \frac{1}{r\left(S_{t+1}^i - S_{t+1}^{i+1}\right)} \begin{bmatrix} S_{t+1}^i & -S_{t+1}^{i+1} \\ -r & r \end{bmatrix} \begin{bmatrix} c_{t+1}^{i+1} \\ c_{t+1}^i \end{bmatrix} \tag{15}$$

for all $i \leq t, t \leq n$.

Lemma 2.1. Let (x, y, z, v, c) and $(\bar{x}, \bar{y}, \bar{z}, \bar{v}, \bar{c})$ be any two feasible solutions of the non-linear relaxation of Model A. Suppose that there is a node (τ, k) with $z_\tau^k \neq \bar{z}_\tau^k$ and $z_t^i = \bar{z}_t^i$ for $(t, i) \neq (\tau, k)$. If $c_\tau^k > \bar{c}_\tau^k$, then $v_t^i \geq \bar{v}_t^i$ and $c_t^i \geq \bar{c}_t^i$ for all the predecessor node (t, i) of (τ, k).

Proof. Suppose both (x, y, z, v, c) and $(\bar{x}, \bar{y}, \bar{z}, \bar{v}, \bar{c})$ are feasible solutions of the non-linear relaxation of Model A. We have, for any interim node (t, i):

$$x_t^i = \frac{uc_{t+1}^i - dc_{t+1}^{i+1}}{r(u-d)}, \bar{x}_{ti} = \frac{u\bar{c}_{t+1}^i - d\bar{c}_{t+1}^{i+1}}{r(u-d)}, \quad S_t^i y_t^i = \frac{c_{t+1}^{i+1} - c_{t+1}^i}{u-d}, S_t^i \bar{y}_t^i = \frac{\bar{c}_{t+1}^{i+1} - \bar{c}_{t+1}^i}{u-d}$$

by solving equation (15).

By equation (5), we calculate:

$$v_t^i - \bar{v}_t^i = \left(x_t^i + S_t^i y_t^i\right) - \left(\bar{x}_t^i + S_t^i \bar{y}_t^i\right)$$

from $\tau - 1$ to 0 iteratively. Let $(\tau - 1, i)$ be a predecessor node of (τ, k). We consider the following two cases:

(1) $i = k$; or

(2) $i = k - 1$.

For case (1), we have:

$$x_{\tau-1}^k + S_{\tau-1}^k y_{\tau-1}^k - \bar{x}_{\tau-1}^k - S_{\tau-1}^k \bar{y}_{\tau-1}^{-k} \geq \left(\frac{u}{r} - 1\right)\left(\frac{c_\tau^k - \bar{c}_\tau^k}{u - d}\right) > 0.$$

For case (2), we have:

$$x_{\tau-1}^{k-1} + S_{\tau-1}^{k-1} y_{\tau-1}^{k-1} - \bar{x}_{\tau-1}^{k-1} - S_{\tau-1}^{k-1} \bar{y}_{\tau-1}^{k-1} \geq \left(1 - \frac{d}{r}\right)\left(\frac{c_\tau^k - \bar{c}_\tau^k}{u - d}\right) > 0.$$

Therefore, we have $c_t^i \geq \bar{c}_t^i$ by equation (7). Replacing (τ, k) by $(\tau - 1, i)$, we can show that this lemma holds for the predecessor nodes at time step $\tau - 2$. Hence, applying the same method iteratively, we prove this lemma. □

Theorem 2.2. Model A has the same solution of its non-linear relaxation.

Proof. Let $(x^*, y^*, z^*, v^*, c^*)$ be an optimal solution of the non-linear relaxation of Model A. It is sufficient to show that $z_t^{*i} = 1$ when $S_t^i - K > 0$ and $z_t^{*i} = 0$ when $S_t^i - K < 0$ for all i and t. If there is (τ, k) such that $z_\tau^{*k} < 1$ when $S_\tau^k - K > v_\tau^{*k}$, we claim that there exists a feasible solution with which the object value is greater than the optimal value.

Let:

$$\mathcal{P} = \{\text{Any path from } (\tau, k) \text{ backtrack to } (0, 0)\}$$

and C be the set of (t, i) in $\mathcal{P}$ which are the node in the path from (τ, k) to (0, 0). Such feasible solution is constructed as follows:

- let $\bar{z}_\tau^k = 1$ and $\bar{z}_t^i = z_t^{*i}$ if $(t, i) \neq (\tau, k)$;
- if $(t, i) \notin C$, set $\left(\bar{x}_t^i, \bar{y}_t^i, \bar{v}_t^i, \bar{c}_t^i\right) = \left(x_t^{*i}, y_t^{*i}, v_t^{*i}, c_t^{*i}\right)$; and
- if $(t, i) \in C$, $(\bar{x}_t^i, \bar{y}_t^i, \bar{v}_t^i, \bar{c}_t^i)$ are selected by solving the system of equations (2)-(5) with respect to $\bar{z}$.

Then $(\bar{x}, \bar{y}, \bar{z}, \bar{v}, \bar{c})$ is a feasible solution of the non-linear relaxation of Model A with $\bar{c}_\tau^k = (S_\tau^k - K) > (S_\tau^k - K)z_\tau^{*k} + v_\tau^{*k}(1 - z_\tau^{*k}) = c_\tau^{*k}$.

Now, by applying Lemma 2.1, we have $\bar{c}_t^i \geq c_t^{*i}$ for $(t, i) \in C$ since $\bar{c}_\tau^k > c_\tau^{*k}$. This implies that:

$$\sum_{t=0}^{n} \sum_{i=0}^{t} \frac{p_t^i}{r^t} U\left(\bar{c}_t^i\right) > \sum_{t=0}^{n} \sum_{i=0}^{t} \frac{p_t^i}{r^t} U\left(c_t^{*i}\right).$$

Finally, the same method can be applied to show the case that the optimal solution of the non-linear relaxation has a node (τ, k) with $0 < z_{\tau}^{*k} < 1$ and $S_n^k - K < 0$. □

Solving equation (15) we obtain the following corollary.

Corollary 2.3. Suppose that (x^*, y^*, z^*, v^*, c^*) is an optimal solution of the non-linear relaxation of Model A. Then:

- $$c_t^{*i} = \max(v_t^{*i}, S_t^i - K), \text{ where } v_t^{*i} = \frac{1}{r}\left(qc_{t+1}^{*i+1} + (1-q)c_{t+1}^{*i}\right)$$ and $$q = \frac{u-r}{u-d}.$$
- $$y_t^{*i} = \frac{c_{t+1}^{*i+1} - c_{t+1}^{*i}}{S_{t+1}^{i+1} - S_{t+1}^{i}}.$$

By solving Model A, we obtain the following information:

- optimal exercising strategy for buyer;
- the perfect hedge strategy for writer; and
- the cost for constructing the perfect hedge portfolio.

Under the no-arbitrage condition, the cost must be the "fair" price of the American option for both the writers and the buyers.

3. Writer's problems

When the market price is less than the fair price, the buyer still have a right to exercise the American option by the optimal exercising strategy. However, the writer do not have enough initial value to construct a perfect hedging portfolio. Therefore, knowing the optimal exercising strategy of buyer and the observed market price, we shall propose a self-finance model to minimize the expected loss for the writer.

The optimal exercising strategy obtained from Model A and the observed market price are given as $z_t^i, i = 0, 1, \ldots, t, \quad t = 0, 1, \ldots, n$ and M_0, respectively. Though the option should be traded with its market price, the writer can make a loan of money l for constructing a hedging portfolio. Therefore, we have:

$$c_0 \leq M_0 + l. \tag{16}$$

If the American option is not exercised, the hedging portfolio satisfies the self-finance trading strategy, which is formulated as equations (2), (3), (5), and:

$$v_t^i(1 - z_t^i) = c_t^i(1 - z_t^i), \quad i = 0, \ldots, t, \; t = 0, 1, \ldots, n-1, \tag{17}$$

in the internal of binomial tree. By analyzing equation (12), we find that $v_t^i = c_t^i$ if $z_t^i = 0$ and $1 - z_t^i = 0$ if $z_t^i = 1$. This implies that the self-finance trading strategy only holds for hedging the alive American option.

To measure the loss between the value of the exercise payoff and the hedging portfolio, we add two nonnegative deviation variables D_t^i, and $d_t^i, i = 0, 1, \ldots, t$, $t = 1, \ldots, n$, and have:

$$\left(S_t^i - K\right)z_t^i - c_t^i = D_t^i - d_t^i, \quad i = 0, 1, \ldots, t, \; t = 1, 2, \ldots, n. \tag{18}$$

Note that D_t^i is the possible loss of writers at (t, i). The goal of this model is to minimize the expected loss, which can be formulated as:

$$U(l) + \sum_{t=1}^{n}\sum_{i=0}^{t}\frac{p_t^i}{r^t}\,U\left(D_t^i\right),$$

where U(·) and $p_t^i, i = 0, 1, \ldots, t, \quad t = 1, \ldots, n$ are the utility function and the objective probability of the writer, respectively.

Therefore, we obtain the following minimum loss hedging model.

3.1 Minimum loss model

$$\min U(l) + \sum_{t=1}^{n}\sum_{i=0}^{t}\frac{p_t^i}{r^t}\,U\left(D_t^i\right)$$

s.t. Equations (2), (3), (5), (16), (17) and (18):

$$x_t^i, y_t^i \in \mathbb{R}, \quad i = 0, 1, \ldots, t, \;\; t = 0, 1, \ldots, n,$$

$$D_t^i, d_t^i, l, c_n^i \geq 0, \quad i = 0, 1, \ldots, t, \;\; t = 0, 1, \ldots, n.$$

Let $\bar{c}_0$ be the fair price of the American put. Making a loan of $\bar{c}_0 - M_0$ to construct a perfect hedging portfolio is a feasible solution of the minimum loss model. So the minimum loss model always has an optimal solution, that is less than or equal to: $\bar{c}_0 - M_0$.

Note that the results in previous two sections are developed for the call option. By replacing $S_t^i - K$ with $K - S_t^i$, the same results can be extended to hedge and valuate the put option.

4. Numerical results

In this section, we will find that the relaxation model reduces the computational time rapidly and verify that Model A has the same optimal solution of its relaxation for American put. When the market price is less than the market price, Example 4 reveals that the optimal value of the minimum loss model has a lower expected loss. The objective probability of upward movement and the utility function of the writer is given as 1/2 and $U(x) = x$, respectively. Here, all models are coded by GAMS (Brooke *et al.*, 1988). Note that Model A, which is an MINLP problem, is solved by BARON solver, and its non-linear relaxation is solved by MINOS solver.

Example 1. We assume that the asset dynamics satisfies the geometric Brownian motion and price a one-year maturity, at-the-money American call option with the current price at 100. The continuous compounded interest rate and the volatility of the asset are assumed to be 6 and 16 percent, respectively. That is, $K = 100,\ T = 1$, $S = 100$, $r = 0.06$. The increment size is given as:

$$u = \exp\left(\sigma\sqrt{\frac{T}{n}}\right)$$

and $d = 1/u$.

In Table I, we display the American call's fair price and the execution times for solving Model A and its relaxation model from columns 2 to 4. We find that both models have the same optimal solution. The execution time for Model A is 45.47 and 3,568.66 seconds for the 20 and 40 time steps, respectively. However, the execution time for its relaxation model are 0.219 and 1.105 seconds for the 20 and 40 time steps, respectively. Moreover, when the time steps are greater than 60, the execution time for solving Model A raises to several hours. Comparing the execution time with both two models, we find that the relaxation model reduces the computation time rapidly.

In the following example, we replace $(S_t^i - K)^+$ in Model A with $(K - S_t^i)^+$ to valuate the American put.

Example 2. Under the same situation, we price a one-year maturity, at-the-money American put option with the current price at 100. The value and the execution time for solving the relaxation model are displayed in Table II. For $N = 20$, the optimal exercising strategy for buyers is displayed in Table III, which is used in Example 4. The execution time for solving Model A is greater than 1 hour when the time steps are greater than 20. So we do not display the execution time for solving Model A. Table III reveals that the optimal solution of the relaxation model is a 0-1 variable. This implies that Model A has the same optimal solution of its relaxation for valuating American put.

If we assume that the asset process does not satisfy the geometric Brownian motion, we give an example that assume the asset process satisfies pure Poisson process. Model A and its relaxation can also provide the optimal exercising strategy and the perfect hedging portfolio.

Example 3. We consider the same put of the previous example but assume that the asset price satisfies the pure poisson process. Since the pure poisson process does not drop down, the rational buyer does not exercise the American option prior the expiration date. Therefore, we find that the decision variables $z_t^i = 0$, for all i and t. In Table IV, we display the American put's fair price and the execution times for

Time steps	Fair price	Model A (seconds)	Relaxation (seconds)
20	9.780	45.47	0.219
40	9.745	3,568	1.105
60	9.734	>1 hour	5.563
80	9.728	>1 hour	17.917
100	9.725	>1 hour	44.188
150	9.720	>1 hour	246.922

Table I.
American call

Time steps	Fair price	Relaxation (seconds)
20	4.636	0.344
40	4.593	2.484
60	4.557	10.750
80	4.547	37.375
100	4.542	148.094
150	4.534	1,174.680

Table II.
American put

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
																				0
																			0	0
																		0	0	0
																	0	0	0	0
																0	0	0	0	0
															0	0	0	0	0	0
														0	0	0	0	0	0	0
													0	0	0	0	0	0	0	0
z												0	0	0	0	0	0	0	0	0
											0	0	0	0	0	0	0	0	0	0
										0	0	0	0	0	0	0	0	0	0	1
									0	0	0	0	0	0	0	0	0	0	1	1
								0	0	0	0	0	0	0	0	0	0	1	1	1
							0	0	0	0	0	0	0	0	0	1	1	1	1	1
						0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
					0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
				0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
			0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
		0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table III. Optimal decision variable of buyer

Time steps	Fair price	Model A	Relaxation (seconds)
20	5.541	32.220	0.313
40	5.682	>1 hour	2.734
60	5.635	>1 hour	15.539
80	5.603	>1 hour	81.859
100	5.541	>1 hour	129.734

Table IV. American put under pure poission process

solving Model A and its relaxation model from columns 2 to 4. When the time steps are greater than 40, the execution time of the MINLP model rises to several hours. Comparing columns 3 and 4, we also find that the relaxation model reduces the computation time rapidly.

Example 4. For the writer's problem, we consider the same put option in Example 2. In this case, the initial cost of the perfect hedging portfolio is 4.636. When the market price is 4.6, the writer will lose 0.036 to construct a perfect hedging portfolio. However, by solving the minimum loss model, the expected loss is 0.0175, which is less than 0.036. Therefore, the solution of the minimum loss portfolio provides a better strategy for constructing a minimum loss portfolio.

In summary, these examples reveal the following three results:

(1) the relaxation of Model A reduces the computational time rapidly;

(2) by replacing $(S_t^i - K)^+$ with $(K - S_t^i)^+$, Model A has the same optimal solution of its relaxation for valuating the American put; and

(3) making a loan to construct a perfect hedging portfolio may not be the optimal strategy for writer.

5. Conclusions and future studies

We have proposed a MINLP model for valuating an American option and shown that this model can be solved by their non-linear relaxations. For a modest size of model, we can easily get a solution from non-linear programming software package. The numerical results reveal that the non-linear model provide a far more efficient approach for computing the fair price of options, its associated hedging portfolio and the optimal exercising strategy. The use of mathematical programming framework can easily extend to valuate the option in the real markets. Moreover, we develop a self-finance model whose solution provides an optimal strategy for the writer to minimize the expected loss when the market price is less than the fair price. In the computational results, we find that the solution of this model provides an optimal portfolio for the writer to minimize the expected loss.

One of our future studies is to analyze the duality of its non-linear relaxation. By the martingale replication theorem and (King, 2002) results, there may exist a martingale probability that makes the asset price is a martingale. Furthermore, we will investigate the theorem of that the value of the American option at each state is equal to the conditional expectation of all the possible future payoff under the synthesis probability measures in my future studies.

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Further reading

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