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Asymptotic Behavior for a Version of Directed Percolation on the Triangular Lattice

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Abstract We consider a version of directed bond percolation on the triangular lattice such that vertical edges are directed upward with probability y, diagonal edges are directed from lower-left to upper-right or lower-right to upper-left with probability d, and horizontal edges are directed rightward with probabilities x and one in alternate rows. Let $\tau(M, N)$ be the probability that there is at least one connected-directed path of occupied edges from (0, 0) to (M, N). For each $x \in [0, 1]$, $y \in [0, 1)$, $d \in [0, 1)$ but $(1-y)(1-d) \neq 1$ and aspect ratio $\alpha = M/N$ fixed for the triangular lattice with diagonal edges from lower-left to upper-right, we show that there is an $\alpha_c = (d-y-dy)/[2(d+y-dy)] + [1-(1-d)^2(1-y)^2 x]/[2(d+y-dy)^2]$ such that as $N \to \infty$, $\tau(M, N)$ is 1, 0 and 1/2 for $\alpha > \alpha_c$, $\alpha < \alpha_c$ and $\alpha = \alpha_c$, respectively. A corresponding result is obtained for the triangular lattice with diagonal edges from lower-right to upper-left. We also investigate the rate of convergence of $\tau(M, N)$ and the asymptotic behavior of $\tau(M_N^-, N)$ and $\tau(M_N^+, N)$ where $M_N^-/N \uparrow \alpha_c$ and $M_N^+/N \downarrow \alpha_c$ as $N \uparrow \infty$.

Keywords Domany–Kinzel model · Directed percolation · Random walk · Asymptotic behavior · Berry–Esseen theorem · Large deviation

1 Introduction

Directed percolation, or oriented percolation, can be thought of simply as a percolation process on a directed lattice in which connections are allowed only in a preferred direction.

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It was first studied by Broadbend and Hammersley [1] and it has remained to this day as one of the most outstanding interesting problems in probability and statistical mechanics. Furthermore, directed percolation is closely related to the Reggeon field theory in high-energy physics and the Markov processes with branching, recombination and absorption that occur in chemistry and biology [2,3], etc. Various properties, results and conjectures of directed percolation can be found in [4,5] and the references therein. However very little is known in the way of exact solutions for the directed percolation problem.

Domany and Kinzel [6] defined a solvable version of compact directed percolation on the square lattice in 1981 as follows. For a fixed $p \in (0, 1)$, each vertical bond is directed upward with occupation probability p (independently of the other bonds) and each horizontal bond is directed rightward with occupation probability 1. Furthermore, it is known that the boundary of the Domany–Kinzel model has the same distribution as the one-dimensional last passage percolation model [7]. A three-dimensional version of Domany–Kinzel model with occupation probability 1 along two spatial directions was considered in Ref. [8]. Recently, one of the authors considered a version of directed percolation on the square lattice whose vertical edges occupied with a probability p_v and horizontal edges in the n-th row occupied with a probability 1 if n is even and p_h if n is odd [9]. Particularly for $p_h = 0$ or 1, that model reduces to the Domany–Kinzel model. In this article, we generalize further to consider a triangular lattice as follows. Instead of using regular triangles, it is easier to start from a square lattice with vertical probability y and horizontal probabilities 1 and x alternatively, then add diagonal edges from lower-left to upper-right or from lower-right to upper-left with probability d as shown in Fig. 1. Notice that the model we study is not a compact directed percolation as holes may exist.

The vertices (sites) of the triangular lattice are now located at a two-dimensional rectangular net $\{(m, n) \in \mathbb{Z} \times \mathbb{Z}_+ : -M \le m \le M \text{ and } 0 \le n \le N\}$. Consider the probabilities $x \in [0, 1]$, $y \in [0, 1)$ and $d \in [0, 1)$ but $(1 - y)(1 - d) \ne 1$, i.e., d and y should not be zero simultaneously, throughout this article, and the percolation always starts from the origin $(0, 0)$. We say that the vertex (m, n) is percolating if there is at least one connected-directed path of occupied edges from $(0, 0)$ to (m, n). Given any $\alpha \in \mathbb{R}$, let $N_\alpha = \lfloor \alpha N \rfloor = \sup\{m \in \mathbb{Z} : m \le \alpha N\}$ with $N \in \mathbb{Z}_+$. It is clear that $\alpha \ge 0$ for the triangular lattice with diagonal edges from lower-left to upper-right and $\alpha \ge -1$ with diagonal edges from lower-right to upper-left. Let us define

$$\alpha_{\min} = \begin{cases} 0 & \text{if diagonal edges from lower-left to upper-right ,} \\ -1 & \text{if diagonal edges from lower-right to upper-left .} \end{cases}$$

Denote $\mathbb{P}$ as the probability distribution of the bond variables, and define the two point correlation function

$$\tau(N_\alpha, N) = \mathbb{P}((N_\alpha, N) \text{ is percolating}) .$$

It is appropriate to define some of the standard critical exponents and to sketch the phenomenological scaling theory of $\tau(N_\alpha, N)$. A critical value of α exists, that is denoted as α_c. For $\alpha < \alpha_c$ and α close to α_c, the scaling theory of critical behavior asserts that the singular part of $\tau(N_\alpha, N)$ varies asymptotically as (c.f. [10])

$$\tau(N_\alpha, N) \approx \exp\left(\frac{-BN}{(\alpha_c - \alpha)^{-\nu}}\right) , \tag{1.1}$$

where the notation $f_{1,\alpha}(N) \approx f_{2,\alpha}(N)$ means that $\lim_{N\to\infty} \log f_{1,\alpha}(N)/\log f_{2,\alpha}(N) = 1$. The critical exponent $\nu \in (0, \infty)$ is a universal constant [11]. The constant B will be derived explicitly below, that does not depend on α but does depend on x, y and d. Note that there

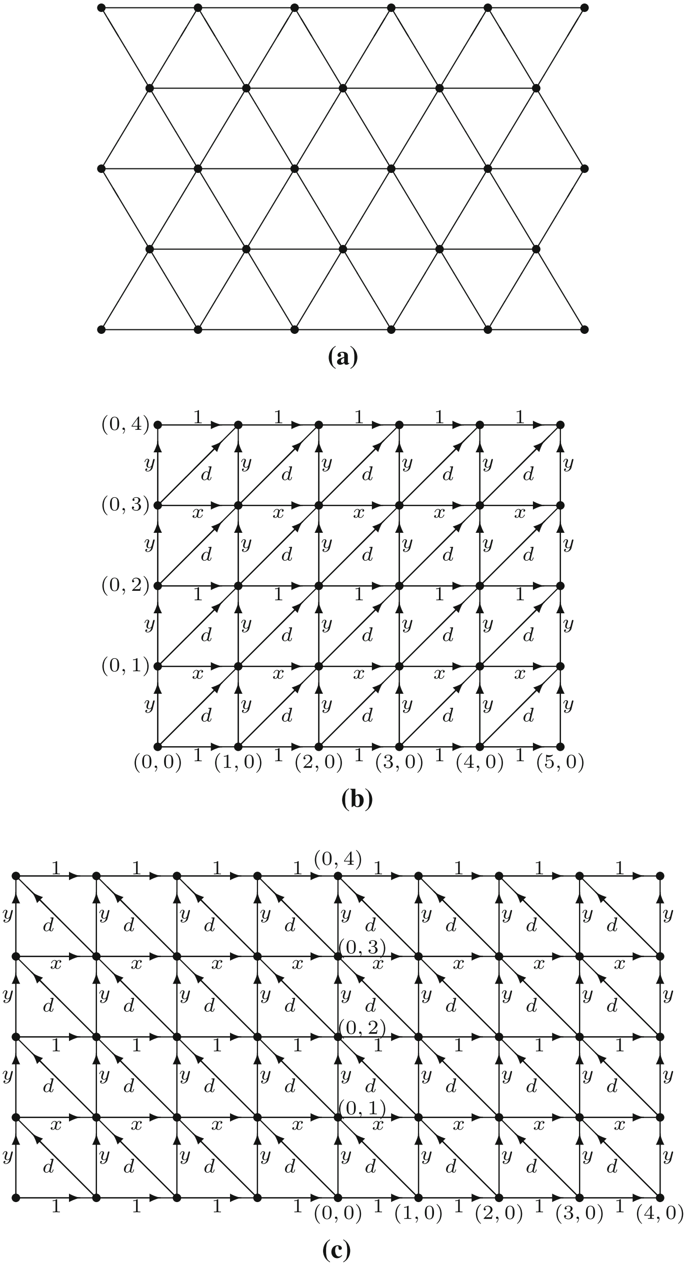

Fig. 1 **a** Triangular lattice with regular triangles; **b** square lattice plus diagonal edges from lower-left to upper-right; **c** square lattice plus diagonal edges from lower-right to upper-left. The probabilities for the edges to be occupied are shown in **b** and **c**

has been no general proof of the existence of the critical exponents. For $\alpha < \alpha_c$, the critical exponent of the correlation length $\nu = 2$ as shown below is the same as what was found in the Domany–Kinzel model [6,12–14]. The consideration here generalizes and amends the corresponding results for the square lattice in Ref. [9] by one of the authors (Chen).

The main purpose of this article is to find the critical value

$$\alpha_c = \frac{d-y-dy}{2(d+y-dy)} + \frac{1-(1-d)^2(1-y)^2x}{2(d+y-dy)^2} \tag{1.2}$$

for the triangular lattice with diagonal edges from lower-left to upper-right, and the critical value

$$\alpha_c = -\frac{3d+y-dy}{2(d+y-dy)} + \frac{1-(1-d)^2(1-y)^2x}{2(d+y-dy)^2} \tag{1.3}$$

for the triangular lattice with diagonal edges from lower-right to upper-left, such that

$$\lim_{N\to\infty} \tau(2N_\alpha, 2N) = \begin{cases} 1 & \text{if } \alpha > \alpha_c\ , \\ 0 & \text{if } \alpha < \alpha_c\ , \\ \frac{1}{2} & \text{if } \alpha = \alpha_c\ . \end{cases} \tag{1.4}$$

Notice that we use the same symbol α_c to denote the critical value for the triangular lattice with diagonal edges either from lower-left to upper-right or from lower-right to upper-left, because (1.4) and the following theorems apply to both cases. The meaning will be clear from context. We also obtain the values of ν and B for the triangle lattice. We use large derivation argument and the Berry–Esseen theorem to quantify the rate.

The rest of this paper is organized as follows. In Sect. 2, we state the main results (Theorem 2.1, Theorem 2.2 and Theorem 2.4) of this paper. In Sect. 3, we derive the critical value α_c and the variance σ^2. Theorem 2.1 is proved in Sect. 4 while Theorem 2.2 and Theorem 2.4 are proved in Sect. 5.

2 Main Results

First we study the rate of convergence of $\tau(2N_\alpha, 2N)$ for a fixed α. For notation convenience, define

$$a = 1 + b - (d+y-dy)^2\ , \quad b = (1-d)^2(1-y)^2x \tag{2.1}$$

from now on, thus (1.2) and (1.3) can be written as

$$\alpha_c = \frac{1}{2} - \frac{y}{d+y-dy} + \frac{1-b}{2(1-a+b)}$$

for the triangular lattice with diagonal edges from lower-left to upper-right, and the critical value

$$\alpha_c = -\frac{1}{2} - \frac{d}{d+y-dy} + \frac{1-b}{2(1-a+b)}$$

for the triangular lattice with diagonal edges from lower-right to upper-left. Moreover, we define

$$\underline{\alpha} = \begin{cases} \alpha_{\min} & \text{if} \quad \alpha_c < -\frac{3}{4} + \frac{\sigma}{2}\ , \\ \frac{-3+\sqrt{(4\alpha_c+3)^2 - 4\sigma^2}}{4} & \text{if} \quad \alpha_c \geq -\frac{3}{4} + \frac{\sigma}{2}\ , \end{cases}$$

where the variance is given by

$$\sigma^2 = \frac{2(1-y)dy - 1 - b}{1-a+b} + \frac{(1-b)^2}{(1-a+b)^2} \tag{2.2}$$

for the triangular lattice with diagonal edges from lower-left to upper-right, and

$$\sigma^2 = \frac{2(1-d)dy - 1 - b}{1-a+b} + \frac{(1-b)^2}{(1-a+b)^2} \tag{2.3}$$

for the triangular lattice with diagonal edges from lower-right to upper-left. Here we again use the same symbol σ^2 to denote the variance for the two cases.

Theorem 2.1 *Given $x \in [0, 1]$, $y \in [0, 1)$, $d \in [0, 1)$ with $(1-y)(1-d) \neq 1$ and the critical aspect ratio α_c in (1.2) or (1.3), the asymptotic behavior of the two point correlation function is*

$$\begin{cases} \tau(2N_\alpha, 2N) & \approx \exp\left(-2NI(\alpha)\right) \quad \text{for} \quad \alpha < \alpha_c\,, \\ \tau(2N_\alpha, 2N) & = \frac{1}{2} + O(\frac{1}{\sqrt{N}}) \quad \text{for} \quad \alpha = \alpha_c\,, \\ 1 - \tau(2N_\alpha, 2N) & \approx \exp\left(-2NI(\alpha)\right) \quad \text{for} \quad \alpha > \alpha_c\,, \end{cases} \tag{2.4}$$

where

$$\frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \le I(\alpha) \le -\ln y \quad \text{for} \quad \alpha \in (\alpha_{\min}, \alpha_c) \tag{2.5}$$

$$\frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \le I(\alpha) \le \frac{\frac{1}{\sigma^2}(\alpha_c-\alpha)^2}{1 - \left(\frac{4(\alpha+\alpha_c)+6}{\sigma^2}\right)(\alpha_c - \alpha)} \quad \text{for} \quad \alpha \in (\underline{\alpha}, \alpha_c) \tag{2.6}$$

$$\frac{\frac{1}{\sigma^2}(\alpha_c-\alpha)^2}{1 + \left(\frac{4(\alpha+\alpha_c)+6}{\sigma^2}\right)(\alpha - \alpha_c)} \le I(\alpha) \le \frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \quad \text{for} \quad \alpha > \alpha_c\,. \tag{2.7}$$

Furthermore,

$$\begin{cases} \tau(2N_\alpha, 2N) \le \exp\left(\frac{-2N}{\sigma^2}(\alpha_c-\alpha)^2\right) & \text{for} \quad \alpha \in (\underline{\alpha}, \alpha_c)\,, \\ 1 - \tau(2N_\alpha, 2N) \le \exp\left(\frac{2(\alpha-\alpha_c)}{\sigma^2}\right) \exp\left(\frac{\frac{-2N}{\sigma^2}(\alpha_c-\alpha)^2}{1+\left(\frac{4(\alpha+\alpha_c)+6}{\sigma^2}\right)(\alpha-\alpha_c)}\right) & \text{for} \quad \alpha > \alpha_c\,. \end{cases} \tag{2.8}$$

According to the definition of $\underline{\alpha}$, it is easy to see that $\underline{\alpha} < \alpha_c$, and for $\alpha \in (\underline{\alpha}, \alpha_c)$ we have

$$\left(\frac{4(\alpha+\alpha_c)+6}{\sigma^2}\right)(\alpha_c - \alpha) < 1,$$

such that the upper bound of $I(\alpha)$ in (2.6) remains positive and finite. By Theorem 2.1, we have the following theorem:

Theorem 2.2 *Given $x \in [0, 1]$, $y \in [0, 1)$, $d \in [0, 1)$ with $(1-y)(1-d) \neq 1$ and the critical aspect ratio α_c in (1.2) or (1.3), inequalities*

$$\frac{\tau(2N_\alpha, 2N+1)}{\tau(2N_\alpha, 2N)} \le 1\,, \qquad \frac{\tau(2N_\alpha, 2N+2)}{\tau(2N_\alpha, 2N)} \le 1$$

hold and the asymptotic behavior of the two point correlation function is

$$\begin{cases} \tau(N_\alpha, N) & \approx \exp\left(-NI(\alpha)\right) \quad \text{for} \quad \alpha < \alpha_c\,, \\ \tau(N_\alpha, N) & = \frac{1}{2} + O(\frac{1}{\sqrt{N}}) \quad \text{for} \quad \alpha = \alpha_c\,, \\ 1 - \tau(N_\alpha, N) & \approx \exp\left(-NI(\alpha)\right) \quad \text{for} \quad \alpha > \alpha_c\,. \end{cases}$$

Remark 2.3 Theorem 2.1 and Theorem 2.2 lead to the following information:

1. The function $I(\alpha)$ for $\alpha \neq \alpha_c$ does not have a simple expression. However, for the original Domany–Kinzel model on the square lattice (i.e., $d = 0,\ x = 1$), it is given by (see Remark 4.1)

$$I(\alpha) = \alpha \ln \left(\frac{\alpha}{(1-y)(1+\alpha)} \right) - \ln (y(1+\alpha)) \ . \tag{2.9}$$

2. For $d = 0$, the expressions of α_c in (1.2), (1.3) and the expressions of σ^2 in (2.2), (2.3) reduce to those for the square lattice in [9].
3. For $x = 1$, our model corresponds to a Domany–Kinzel model on the $2N_\alpha \times 2N$ triangular lattice. (1.2) and (2.2) lead to $\alpha_c = (1-y)/(d+y-dy), \sigma^2 = 2(1-y)(1-d+dy)/(d+y-dy)^2$ for the triangular lattice with diagonal edges from lower-left to upper-right. (1.3) and (2.3) lead to $\alpha_c = (1-2d-y+dy)/(d+y-dy), \sigma^2 = 2(1-d)(1-y+dy)/(d+y-dy)^2$ for the triangular lattice with diagonal edges from lower-right to upper-left.
4. Our result gives that $\tau(N_\alpha, N)$ with $\alpha < \alpha_c$ and $1 - \tau(N_\alpha, N)$ with $\alpha > \alpha_c$ both decay exponentially to zero. Furthermore, we obtain $B = 1/\sigma^2$ and the critical exponent $\nu = 2$ in (1.1) for $\alpha < \alpha_c$.

Finally, we investigate the asymptotic phenomena of $\tau(N_{\alpha_N^-}, N)$ and $\tau(N_{\alpha_N^+}, N)$ where $\alpha_N^+ \downarrow \alpha_c$ and $\alpha_N^- \uparrow \alpha_c$ as $N \uparrow \infty$. A sequence $\{\ell_n\}_{n=1}^\infty$ is called a regularly varying sequence if for any $\lambda \in (0,\infty)$, $\lim_{n\to\infty} \ell_{\lfloor \lambda n \rfloor}/\ell_n = 1$. For example, $\ell_n = \log n$ or $\ell_n = c \in (0,\infty)$ for all n. For convenience, we denote $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}}\, du$ as the standard cumulative distribution function of *Gaussian* distribution with mean 0, variance 1 and let $\Psi(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}}\, du$. It is not difficult to see that

$$\Psi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \left(1 + O(x^{-2})\right) \quad \text{when } x \text{ is large.}$$

Theorem 2.4 *Given $x \in [0,1]$, $y \in [0,1)$, $d \in [0,1)$ with $(1-y)(1-d) \neq 1$, $\rho \in (0,\infty)$ and a positive regularly varying sequence $\{\ell_n\}_{n=1}^\infty$. Denote $\alpha_N^- = \alpha_c - \sigma N^{-\rho}\ell_N/\sqrt{2}$ and $\alpha_N^+ = \alpha_c + \sigma N^{-\rho}\ell_N/\sqrt{2}$, then both*

$$\tau(N_{\alpha_N^-}, N)\ , \qquad 1 - \tau(N_{\alpha_N^+}, N)$$
$$\begin{cases} \approx \exp(-N^{-2\rho+1}\ell_N^2) & \text{if } \ \rho \in (0, \frac{1}{2}) \\ \approx \exp(-\ell_N^2) & \text{if } \ \rho = \frac{1}{2},\ \ell_N \to \infty \\ = \Psi(\ell) + O(1) \max\{\frac{1}{\sqrt{N}}, |\ell - \ell_N|\} & \text{if } \ \rho = \frac{1}{2},\ \ell_N \to \ell \in [0,\infty) \\ = \frac{1}{2} + O(1)N^{-\rho+\frac{1}{2}}\ell_N & \text{if } \ \rho \in (\frac{1}{2}, 1) \\ = \frac{1}{2} + O(\frac{1}{\sqrt{N}}) & \text{if } \ \rho \in [1,\infty) \end{cases}$$

Note that $\rho = \frac{1}{2}$ is a critical value and we have the following corollary.

Corollary 2.5 *Under the same assumptions of Theorem 2.2, we have*

$$\lim_{N\to\infty} \tau(N_{\alpha_N^-}, N) = \lim_{N\to\infty} \left(1 - \tau(N_{\alpha_N^+}, N)\right) = \begin{cases} 0 & \text{if } \ \rho \in (0, \frac{1}{2})\ , \\ \frac{1}{2} & \text{if } \ \rho \in (\frac{1}{2}, \infty)\ . \end{cases}$$

When $\rho = 1/2$ and $\ell_N \to \ell \in [0,\infty]$, we have

$$\lim_{N\to\infty} \tau(N_{\alpha_N^-}, N) = \exp(-\ell^2)\ , \quad \lim_{N\to\infty} \tau(N_{\alpha_N^+}, N) = 1 - \exp(-\ell^2)\ .$$

3 Derivation of α_c and σ^2

For any $N \in \mathbb{N}$, we say that an occupied vertical or diagonal edge in a bond configuration is wet if it lies on a percolating path where $(2N_\alpha, 2N)$ is percolating. For a certain occupied vertical or diagonal edge ending at (m, n), we say that it is *primary* wet if it is the wet edge with smallest m value for that n. In a percolating configuration where $(2N_\alpha, 2N)$ is percolating, there is one *primary wet* edge for each $n \in \{1, 2, \cdots, 2N\}$. Define $P_N(m)$ as the probability that the *primary wet* edge for $n = 2N$ ends at $(m, 2N)$, and formally define $P_0(m) = \delta_{0,m}$ where δ is the Kronecker delta. Since the primary wet edge can occur at any value of $m \le 2N_\alpha$, we have

$$\tau(2N_\alpha, 2N) = \sum_{m \le 2N_\alpha} P_N(m)$$

for $N \in \mathbb{N}$.

In terms of one-dimensional independent and identically distributed random variables $w(m) = P_1(m)$ for $m \in \mathbb{Z}$, $P_N(m)$ can be written as

$$P_N(m) = \sum_{k \in \mathbb{Z}} P_1(k) P_{N-1}(m-k) = w^{*N}(m) ,$$

where w^{*N} is the N-fold convolution. We can define a N-step random walk S_N with the probability *Prob.* such that for $N \in \mathbb{N}$

$$S_N = X_1 + X_2 + \cdots + X_N ,$$

where $Prob.(X_j = m) = w(m)$ for $j \in \{1, 2, ..., N\}$, $Prob.(S_N = m) = P_N(m)$ with $m \in \mathbb{Z}$ and $Prob.(S_0 = m) = \delta_{0,m}$. The expectation for *Prob.* is denoted by *Exp*.

In this section, we shall obtain that the mean and the variance of w are $2\alpha_c$ and σ^2, and $Exp.(|X_1|^3) < \infty$ for the triangular lattice with diagonal edges from lower-left to upper-right in Sect. 3.1 and that with diagonal edges from lower-right to upper-left in Sect. 3.2.

3.1 Diagonal Edges from Lower-Left to Upper-Right

Let us first consider the triangular lattice with diagonal edges from lower-left to upper-right (c.f. Fig. 1b) in this subsection. We shall derive the generating function

$$W(t) = \sum_{m=0}^{\infty} w(m) t^m .$$

As aforementioned $w(m)$ is the probability that $(m, 2)$ is percolating with the primary wet edge in the top row ending at $(m, 2)$. However the primary wet edge in the bottom row can be ending at $(k, 1)$ for any k in $0 \le k \le m$. Therefore, we have

$$w(m) = \sum_{k=0}^{m} u(k) v(m-k) ,$$

where $u(k)$ is the probability that the primary wet edge in the bottom row is ending at $(k, 1)$ and $v(m-k)$ is the probability that starting from $(k, 1)$ the primary wet edge in the upper row is ending at $(m, 2)$. Now since the primary wet edge can be either vertical or diagonal, both $u(m)$ and $v(m)$ should be further divided into two cases:

$$u(m) = u_1(m) + u_2(m) , \qquad v(m) = v_1(m) + v_2(m) ,$$

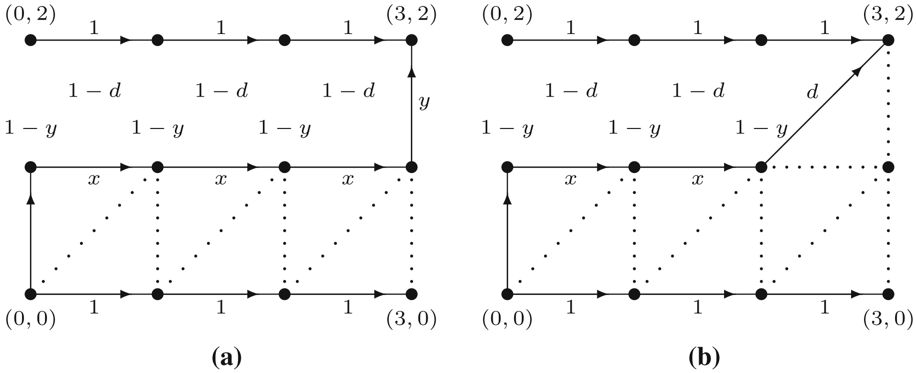

Fig. 2 Constructions of **a** $v_1(m)$ and **b** $v_2(m)$ for m = 3 in (3.1). Occupied edges are shown as oriented edges; while unoccupied edges are not shown. *Dotted edges* can be either occupied or vacant

where we use the subscript 1 when the primary wet edge is vertical and 2 when the primary wet edge is diagonal. It is easy to see that

$$v_1(m) = (1-d)^m(1-y)^m x^m y \qquad \text{for} \quad m \geq 0\,,$$

$$v_2(m) = \begin{cases} 0 & \text{for} \quad m = 0\,, \\ (1-d)^{m-1}(1-y)^m x^{m-1} d & \text{for} \quad m \geq 1\,, \end{cases} \tag{3.1}$$

as shown in Fig. 2, and the generating functions are

$$V_1(t) = \sum_{m=0}^{\infty} v_1(m)t^m = \frac{y}{1-(1-d)(1-y)xt},$$

$$V_2(t) = \sum_{m=0}^{\infty} v_2(m)t^m = \frac{(1-y)dt}{1-(1-d)(1-y)xt},$$

$$V(t) = \sum_{m=0}^{\infty} v(m)t^m = \frac{y+(1-y)dt}{1-(1-d)(1-y)xt}. \tag{3.2}$$

The computation of the factor $u(m)$ with $m \geq 1$ is more complicated. We first notice that $u_1(0) = y$ and $u_2(0) = 0$, and decompose both $u_1(m)$ and $u_2(m)$ into two terms for $m \geq 1$ as follows:

$$u_1(m) = \delta_1(m) + \theta_1(m)\,, \qquad u_2(m) = \delta_2(m) + \theta_2(m)\,,$$

where $\delta_j(m)$ with $j \in \{1, 2\}$ is the probability that the first primary wet edge in the bottom row is ending at $(m, 1)$ such that the site $(m-1, 1)$ is disconnected from $(0, 0)$, while $\theta_j(m)$ is the probability such that the site $(m-1, 1)$ is connected with $(0, 0)$. For $\delta_1(m)$ with $m > 0$, one can start from $u_1(m-1)$, then convert the vertical edge connecting $(m-1, 0)$ and $(m-1, 1)$ into a vacant edge, so that

$$\delta_1(m) = \left[\frac{1-y}{y}u_1(m-1)\right](1-d)y = (1-d)(1-y)u_1(m-1)\,. \tag{3.3}$$

For $\theta_1(m)$ with $m > 0$, one can start from $w_1(m-1) = \sum_{k=0}^{m-1} u(k)v_1(m-1-k)$ whose primary wet edge in the upper row is a vertical edge connecting $(m-1, 1)$ and $(m-1, 2)$,

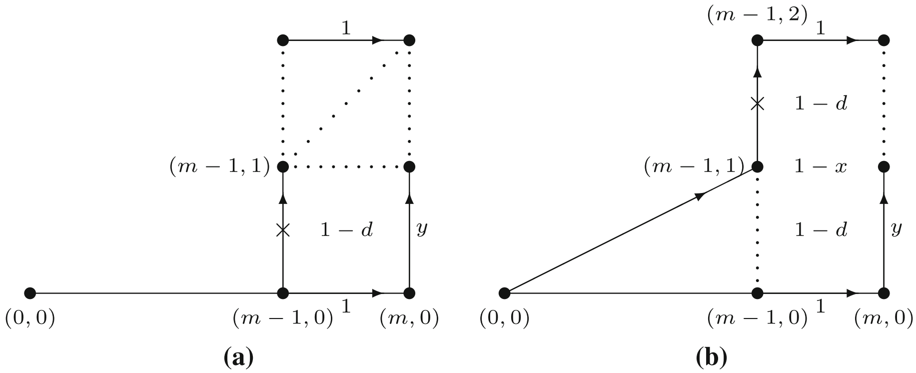

Fig. 3 Constructions of **a** $\delta_1(m)$ and **b** $\theta_1(m)$ in (3.3) and (3.4), respectively. Occupied edges are shown as oriented edges; while unoccupied edges and most edges between (0, 0) and $(m-1, 2)$ are not shown. *Dotted edges* can be either occupied or vacant. The *line*between (0, 0) and $(m-1, 1)$ in **b** indicates that they are connected

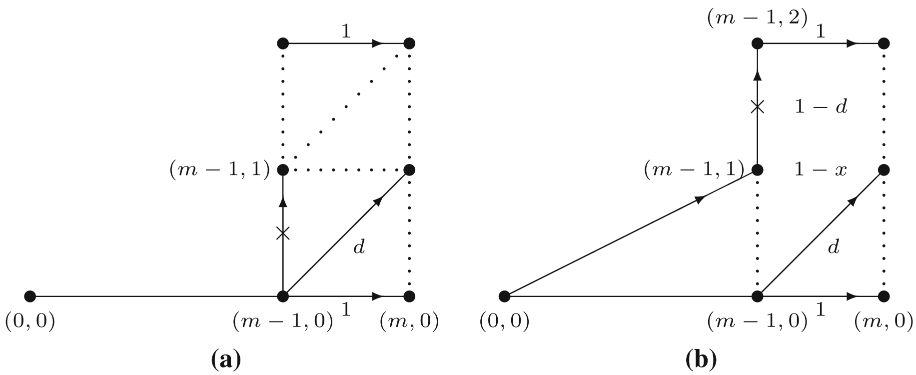

Fig. 4 Constructions of **a** $\delta_2(m)$ and **b** $\theta_2(m)$ in (3.5). Occupied edges are shown as oriented edges; while unoccupied edges and most edges between (0, 0) and $(m-1, 2)$ are not shown. *Dotted edges* can be either occupied or vacant. The *line* between (0, 0) and $(m-1, 1)$ in **b** indicates that they are connected

then convert that edge into a vacant edge, and impose the conditions that the edges from $(m-1, 1)$ to $(m, 1)$ and $(m, 2)$ are vacant to ensures that the site $(m-1, 1)$ is not on a percolating path, so that

$$\theta_1(m) = \left[\frac{1-y}{y} w_1(m-1)\right](1-d)^2(1-x)y = (1-d)^2(1-x)(1-y)w_1(m-1)\ . \tag{3.4}$$

(3.3) and (3.4) are illustrated in Fig. 3. Similarly, we have

$$\begin{aligned} \delta_2(m) &= \frac{(1-y)d}{y} u_1(m-1)\ , \\ \theta_2(m) &= \frac{(1-d)(1-x)(1-y)d}{y} w_1(m-1) \end{aligned} \tag{3.5}$$

for $m > 0$ as illustrated in Fig. 4.

Because the generating function of $w_1(m)$ is given by $W_1(t) = \sum_{m=0}^{\infty} w_1(m)t^m = U(t)V_1(t)$, using (3.2), (3.4) and (3.5) we have

$$\begin{aligned}\Theta_1(t) &= \sum_{m=1}^{\infty} \theta_1(m)t^m = (1-d)^2(1-x)(1-y)tW_1(t) \\ &= \frac{(1-d)^2(1-x)(1-y)yt}{1-(1-d)(1-y)xt}U(t)\ , \\ \Theta_2(t) &= \sum_{m=1}^{\infty} \theta_2(m)t^m = \frac{(1-d)(1-x)(1-y)dt}{1-(1-d)(1-y)xt}U(t) = \frac{d}{(1-d)y}\Theta_1(t)\ .\end{aligned}$$

Using (3.3) and (3.5) to have

$$\begin{aligned}\Delta_1(t) &= \sum_{m=1}^{\infty} \delta_1(m)t^m = (1-d)(1-y)tU_1(t) \\ &= (1-d)(1-y)t\left[y + \Delta_1(t) + \Theta_1(t)\right]\ , \\ \Delta_2(t) &= \sum_{m=1}^{\infty} \delta_2(m)t^m = \frac{(1-y)dt}{y}U_1(t) = \frac{d}{(1-d)y}\Delta_1(t)\ ,\end{aligned}$$

we solve

$$\Delta_1(t) = \frac{(1-d)^3(1-x)(1-y)^2yt^2}{[1-(1-d)(1-y)t][1-(1-d)(1-y)xt]}U(t) + \frac{(1-d)(1-y)yt}{1-(1-d)(1-y)t}\ .$$

Therefore,

$$\begin{aligned}U(t) &= y + \Delta_1(t) + \Theta_1(t) + \Delta_2(t) + \Theta_2(t) = y + \frac{d+y-dy}{(1-d)y}\Big[\Delta_1(t) + \Theta_1(t)\Big] \\ &= \frac{(1-d)(1-x)(1-y)(d+y-dy)t}{[1-(1-d)(1-y)t][1-(1-d)(1-y)xt]}U(t) + \frac{y+(1-y)dt}{1-(1-d)(1-y)t}\end{aligned}$$

which leads to

$$U(t) = \frac{[1-(1-d)(1-y)xt][y+(1-y)dt]}{[1-(1-d)(1-y)t][1-(1-d)(1-y)xt]-(1-d)(1-x)(1-y)(d+y-dy)t}\ . \tag{3.6}$$

Combining (3.2) and (3.6), we finally have

$$\begin{aligned}W(t) &= U(t)V(t) \\ &= \frac{[y+(1-y)dt]^2}{1-[1+(1-d)^2(1-y)^2x-(d+y-dy)^2]t+(1-d)^2(1-y)^2xt^2} \\ &= \frac{[y+(1-y)dt]^2}{1-at+bt^2}\ .\end{aligned} \tag{3.7}$$

Furthermore, it follows that the mean of $w(m)$ is given by

$$\mu = \sum_{m=0}^{\infty} mw(m) = \frac{dW(t)}{dt}\Big|_{t=1} = \frac{d-y-dy}{d+y-dy} + \frac{1-b}{1-a+b} = 2\alpha_c\ , \tag{3.8}$$

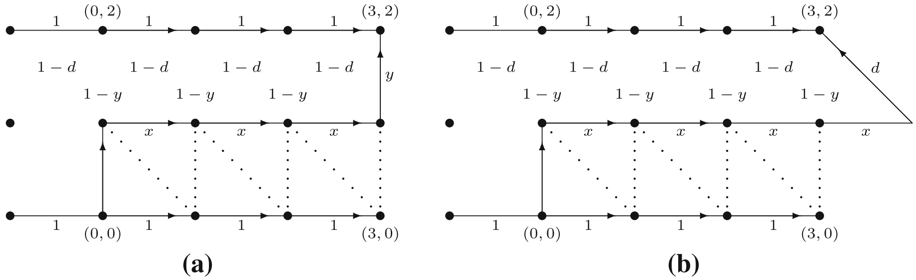

Fig. 5 Constructions of **a** $v_1(m)$ and **b** $v_2(m)$ for m = 3 in (3.10). Occupied edges are shown as oriented edges; while unoccupied edges are not shown. *Dotted edges* can be either occupied or vacant

and the variance of $w(m)$ is given by

$$\sigma^2 = \sum_{m=0}^{\infty} m^2 w(m) - \mu^2 = \frac{d^2 W(t)}{dt^2}\Big|_{t=1} + \mu - \mu^2$$
$$= \frac{2(1-y)dy - 1 - b}{1-a+b} + \frac{(1-b)^2}{(1-a+b)^2} \tag{3.9}$$

for the triangular lattice with diagonal edges from lower-left to upper-right. It is easy to check that $Exp.(|X_1|^3) = \sum_{m=0}^{\infty} m^3 w(m) = \frac{d^3 W(t)}{dt^3}\Big|_{t=1} + 3\frac{d^2 W(t)}{dt^2}\Big|_{t=1} + \mu < \infty$.

3.2 Diagonal Edges from Lower-Right to Upper-Left

Let us consider the triangular lattice with diagonal edges from lower-right to upper-left (c.f. Fig. 1c) in this subsection. No confusion should result from our use of the same notations as in the previous subsection. $w(m)$ again is the probability that $(m, 2)$ is percolating with the primary wet edge in the top row ending at $(m, 2)$. However now the primary wet edge in the bottom row can be ending at $(k, 1)$ for any k in $-1 \le k \le m+1$. Therefore, we have

$$w(m) = \sum_{k=-1}^{m+1} u(k)v(m-k) .$$

It is easy to see that for the probability $v(m) = v_1(m) + v_2(m)$ in the upper row,

$$v_1(m) = \begin{cases} 0 & \text{for } \ m = -1 , \\ (1-d)^{m+1}(1-y)^m x^m y & \text{for } \ m \ge 0 , \end{cases}$$
$$v_2(m) = (1-d)^{m+1}(1-y)^{m+1}x^{m+1}d \qquad \text{for } \ m \ge -1 , \tag{3.10}$$

as shown in Fig. 5, and the generating functions are

$$V_1(t) = \sum_{m=-1}^{\infty} v_1(m)t^m = \frac{(1-d)y}{1-(1-d)(1-y)xt} ,$$
$$V_2(t) = \sum_{m=-1}^{\infty} v_2(m)t^m = \frac{d/t}{1-(1-d)(1-y)xt} ,$$
$$V(t) = \sum_{m=-1}^{\infty} v(m)t^m = \frac{(1-d)y + d/t}{1-(1-d)(1-y)xt} . \tag{3.11}$$

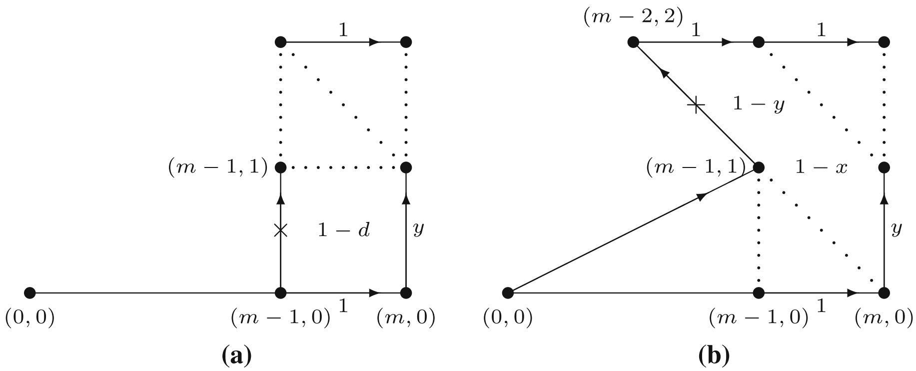

Fig. 6 Constructions of **a** $\delta_1(m)$ with $m \geq 1$ and **b** $\theta_1(m)$ in (3.12). Occupied edges are shown as oriented edges; while unoccupied edges and most edges between (0, 0) and $(m-1, 2)$ are not shown. *Dotted edges* can be either occupied or vacant. The *line* between (0, 0) and $(m-1, 1)$ in **b** indicates that they are connected

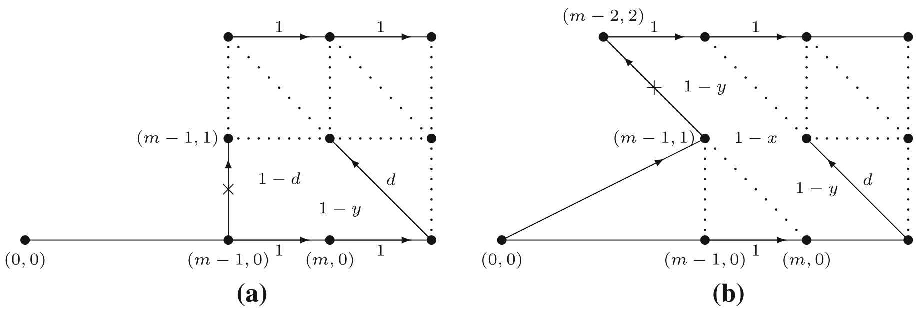

Fig. 7 Constructions of **a** $\delta_2(m)$ with $m \geq 1$ and **b** $\theta_2(m)$ in (3.13). Occupied edges are shown as oriented edges; while unoccupied edges and most edges between (0, 0) and $(m-1, 2)$ are not shown. *Dotted edges* can be either occupied or vacant. The *line* between (0, 0) and $(m-1, 1)$ in **b** indicates that they are connected

For the probability $u(m) = u_1(m) + u_2(m)$ in the bottom row, we again have the decomposition $u_1(m) = \delta_1(m) + \theta_1(m)$ and $u_2(m) = \delta_2(m) + \theta_2(m)$. We first notice that $u_1(-1) = 0$ and $u_2(-1) = d$. By the arguments similar to those in the previous subsection, we have

$$\begin{aligned} \delta_1(m) &= \begin{cases} (1-d)y & \text{for} \quad m = 0\,, \\ (1-d)(1-y)u_1(m-1) & \text{for} \quad m \geq 1\,, \end{cases} \\ \theta_1(m) &= \frac{(1-d)(1-x)(1-y)y}{d} w_2(m-2) \qquad \text{for} \quad m \geq 0\,, \end{aligned} \tag{3.12}$$

as illustrated in Fig. 6, and

$$\begin{aligned} \delta_2(m) &= \begin{cases} (1-d)(1-y)d & \text{for} \quad m = 0\,, \\ \frac{(1-d)(1-y)^2 d}{y} u_1(m-1) & \text{for} \quad m \geq 1\,, \end{cases} \\ \theta_2(m) &= (1-d)(1-x)(1-y)^2 w_2(m-2) \qquad \text{for} \quad m \geq 0 \end{aligned} \tag{3.13}$$

as illustrated in Fig. 7.

Because the generating function of $w_2(m)$ is given by $W_2(t) = \sum_{m=-2}^{\infty} w_2(m)t^m = U(t)V_2(t)$, using (3.11), (3.12) and (3.13) we have

$$\begin{aligned}
\Theta_1(t) &= \sum_{m=0}^{\infty} \theta_1(m)t^m = \frac{(1-d)(1-x)(1-y)yt^2}{d} W_2(t) \\
&= \frac{(1-d)(1-x)(1-y)yt}{1-(1-d)(1-y)xt} U(t) \ , \\
\Theta_2(t) &= \sum_{m=0}^{\infty} \theta_2(m)t^m = \frac{(1-d)(1-x)(1-y)^2 dt}{1-(1-d)(1-y)xt} U(t) = \frac{(1-y)d}{y} \Theta_1(t) \ .
\end{aligned}$$

Using (3.12) and (3.13) to have

$$\begin{aligned}
\Delta_1(t) = \sum_{m=0}^{\infty} \delta_1(m)t^m &= (1-d)y + (1-d)(1-y)tU_1(t) \\
&= (1-d)y + (1-d)(1-y)t\left[\Delta_1(t) + \Theta_1(t)\right] \ , \\
\Delta_2(t) = \sum_{m=0}^{\infty} \delta_2(m)t^m &= (1-d)(1-y)d + \frac{(1-d)(1-y)^2 dt}{y} U_1(t) \\
&= \frac{(1-y)d}{y} \Delta_1(t) \ ,
\end{aligned}$$

we solve

$$\Delta_1(t) = \frac{(1-d)^2(1-x)(1-y)^2 yt^2}{[1-(1-d)(1-y)t][1-(1-d)(1-y)xt]} U(t) + \frac{(1-d)y}{1-(1-d)(1-y)t} \ .$$

Therefore,

$$\begin{aligned}
&U(t) \\
&\quad = \Delta_1(t) + \Theta_1(t) + \frac{d}{t} + \Delta_2(t) + \Theta_2(t) = \frac{d}{t} + \frac{d+y-dy}{y}\Big[\Delta_1(t) + \Theta_1(t)\Big] \\
&\quad = \frac{(1-d)(1-x)(1-y)(d+y-dy)t}{[1-(1-d)(1-y)t][1-(1-d)(1-y)xt]} U(t) + \frac{d/t + (1-d)y}{1-(1-d)(1-y)t}
\end{aligned}$$

which leads to

$$U(t) = \frac{[1-(1-d)(1-y)xt][d/t+(1-d)y]}{[1-(1-d)(1-y)t][1-(1-d)(1-y)xt] - (1-d)(1-x)(1-y)(d+y-dy)t} . \tag{3.14}$$

Combining (3.11) and (3.14), we finally have

$$W(t) = U(t)V(t) = \frac{[d/t + (1-d)y]^2}{1-at+bt^2} \ . \tag{3.15}$$

The mean of $w(m)$ is given by

$$\mu = \sum_{m=-2}^{\infty} mw(m) = \frac{dW(t)}{dt}\bigg|_{t=1} = -\frac{3d+y-dy}{d+y-dy} + \frac{1-b}{1-a+b} = 2\alpha_c \ , \tag{3.16}$$

and the variance of $w(m)$ is given by

$$\sigma^2 = \sum_{m=-2}^{\infty} m^2 w(m) - \mu^2 = \frac{d^2 W(t)}{dt^2}\Big|_{t=1} + \mu - \mu^2$$
$$= \frac{2(1-d)dy - 1 - b}{1 - a + b} + \frac{(1-b)^2}{(1-a+b)^2} \tag{3.17}$$

for the triangular lattice with diagonal edges from lower-right to upper-left. It is easy to check that $Exp.(|X_1|^3) = \sum_{m=-2}^{\infty} m^3 w(m) = \frac{d^3 W(t)}{dt^3}\Big|_{t=1} + 3\frac{d^2 W(t)}{dt^2}\Big|_{t=1} + \mu < \infty$.

Notice that if d and y are switched to each other, then the μ in (3.16) plus two is equal to the mean we obtain in the previous subsection in (3.8). This is expectable since the triangular lattice with diagonal edges from lower-right to upper-left (c.f. Fig. 1c) is equivalent to the triangular lattice with diagonal edges from lower-left to upper-right (c.f. Fig. 1b) if each vertex (m, n) is moved to $(m + n, n)$ and the probabilities d and y are switched. Therefore, the σ^2 in (3.17) is equal to the variance we obtain in the previous subsection in (3.9) with d and y switched.

Let us remark that we only need the set of vertices (m, n) with $m \geq 0$ and $n \geq 0$ for the triangular lattice with diagonal edges from lower-left to upper-right as the percolation directions are chosen to be directed upward and rightward. However, if we only consider the set of vertices (m, n) with $m \geq 0$ and $n \geq 0$ for the triangular lattice with diagonal edges from lower-right to upper-left, then the problem is non-trivial and the current method does not apply. The reason is because the boundary condition for the percolation from the origin $(0, 0)$ to $(m_1, 2)$ for a certain m_1 is different from that for the percolation from $(m_1, 2)$ to $(m_2, 4)$ for a certain m_2, etc. That is, they are not independent and identically distributed.

4 Proof of Theorem 2.1

When $\alpha > \alpha_{\min}$, by the definition of N_α given in the introduction, we have

$$\tau(2N_\alpha, 2N) = Prob.(S_N \leq 2N_\alpha) = Prob.\,(S_N \leq 2\alpha N - c_\alpha)$$

for some $c_\alpha \in [0, 1)$ depending on α. Berry–Esseen theorem (c.f. [15]) asserts that

$$\tau(2N_\alpha, 2N) = Prob.\left(\frac{S_N - 2\alpha_c N}{\sqrt{2N}\sigma} \leq \frac{2N(\alpha - \alpha_c) - c_\alpha}{\sqrt{2N}\sigma}\right) + O\left(\frac{1}{\sqrt{N}}\right)$$
$$= \int_{-\infty}^{\frac{2N(\alpha-\alpha_c)-c_\alpha}{\sqrt{2N}\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O\left(\frac{1}{\sqrt{N}}\right). \tag{4.1}$$

Setting $\alpha = \alpha_c$, we have

$$\tau(2N_{\alpha_c}, 2N) = \int_{-\infty}^{\frac{-c_\alpha}{\sqrt{2N}\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O\left(\frac{1}{\sqrt{N}}\right)$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du - \int_{\frac{-c_\alpha}{\sqrt{2N}\sigma}}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du + O\left(\frac{1}{\sqrt{N}}\right)$$

$$= \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right), \tag{4.2}$$

which gives the second line in (2.4).

In the rest part of this section, we shall use the argument of large deviation [16] to consider a general $\alpha \neq \alpha_c$. Since X_1 is a finite logarithmic moment generating function, we set $\lambda = -\ln t$ for $\alpha < \alpha_c$ and use the Chebyshev inequality and Markov's property to have

$$\begin{aligned} Prob.(S_N \leq 2N_\alpha) &= \inf_{\lambda>0} Prob.(e^{-\lambda S_N} \geq e^{-2\lambda\alpha N + \lambda c_\alpha}) \\ &\leq \inf_{\lambda>0} Prob.(e^{-\lambda S_N} \geq e^{-2\lambda\alpha N}) \\ &\leq \inf_{\lambda>0} \left(\frac{Exp.(e^{-\lambda X_1})}{e^{-2\lambda\alpha}}\right)^N \\ &= e^{-2NI(\alpha)}, \end{aligned}$$

where we define

$$I(\alpha) = \sup_{t \in \left(0, \frac{a-\sqrt{a^2-4b}}{2b}\right)} \left(\alpha \ln t - \frac{1}{2} \ln W(t)\right) := \alpha \ln t_\alpha - \frac{1}{2} \ln W(t_\alpha),$$

and t_α is a function of α. Notice that t should be less than $\frac{a-\sqrt{a^2-4b}}{2b}$ such that the denominator of $W(t)$ in (3.7) or (3.15), and hence $W(t)$ itself remains positive. Similarly for $\alpha > \alpha_c$, we set $\lambda = \ln t$ to have

$$\begin{aligned} Prob.(S_N > 2N_\alpha) &= \inf_{\lambda>0} Prob.(e^{\lambda S_N} > e^{2\lambda\alpha N - \lambda c_\alpha}) \\ &\leq \inf_{\lambda>0} \left(\frac{Exp.(e^{\lambda X_1})}{e^{2\lambda\alpha}}\right)^N e^{\lambda} \\ &\leq e^{\ln t_\alpha} e^{-2NI(\alpha)}. \end{aligned}$$

By Cramér's theorem, the first and third lines of (2.4) are established.

Remark 4.1

1. We shall show that $t_\alpha \in (0, 1)$ for $\alpha < \alpha_c$ and $t_\alpha > 1$ for $\alpha \in \left(\alpha_c, \frac{a-\sqrt{a^2-4b}}{2b}\right)$.
2. $I(\alpha)$ does not have a simple expression. However, for the original Domany–Kinzel model on the square lattice ($d = 0,\ x = 1$), both (3.7) and (3.15) reduce to $W(t) = \left(\frac{y}{1-(1-y)t}\right)^2$. For each $\alpha \neq \alpha_c = \frac{1-y}{y}$, we solve

$$\frac{d}{dt}\left(\alpha \ln t_\alpha - \frac{1}{2} \ln W(t_\alpha)\right) = 0$$

   to obtain $t_\alpha = \frac{\alpha}{(1-y)(1+\alpha)}$, and (2.9) follows.

The following discussion is devoted to the derivation of upper and lower bounds of $I(\alpha)$ given in (2.5)–(2.7). Let us first consider the case with diagonal edges from lower-left to upper-right, so that α is positive. By definition,

$$\frac{\alpha}{t_\alpha} = \frac{1}{2}\left(\frac{\frac{dW(t_\alpha)}{dt}}{W(t_\alpha)}\right) \tag{4.3}$$

such that

$$\frac{d}{d\alpha}I(\alpha) := I'(\alpha) = \ln t_\alpha - \frac{1}{2}\left(\frac{\frac{d}{dt}W(t_\alpha)}{W(t_\alpha)} - \frac{2\alpha}{t_\alpha}\right){t_\alpha}' = \ln t_\alpha \ , \tag{4.4}$$

where $t' = \frac{dt_\alpha}{d\alpha}$ and

$$I''(\alpha) = \frac{{t_\alpha}'}{t_\alpha} \ , \qquad I'''(\alpha) = \frac{{t_\alpha}'' t_\alpha - ({t_\alpha}')^2}{t_\alpha^2} \ . \tag{4.5}$$

As $W(t)|_{t=1} = 1$ by definition and (3.8), (3.16) give $dW/dt|_{t=1} = 2\alpha_c$, setting $t_\alpha = 1$ in (4.3) leads to $t_{\alpha_c} = W(t_{\alpha_c}) = 1$ and hence $I(\alpha_c) = I'(\alpha_c) = 0$. To obtain (2.5)–(2.7), we need to show the following properties:

$$\begin{cases} I'(\alpha) \le 0 & \text{for} \quad \alpha \in (0, \alpha_c) \\ I''(\alpha_c) = \frac{2}{\sigma^2} & \\ -(4\alpha + 3)I''(\alpha)^2 \le I'''(\alpha) \le 0 & \text{for} \quad \alpha > 0 \end{cases} \ . \tag{4.6}$$

Assuming that (4.6) holds, by Taylor formula we have

$$\begin{aligned} I(\alpha) &\ge \frac{I''(\alpha_c)}{2}(\alpha - \alpha_c)^2 = \frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \quad \text{for} \quad \alpha \in (\underline{\alpha}, \alpha_c) \ , \\ I(\alpha) &\le \frac{I''(\alpha_c)}{2}(\alpha - \alpha_c)^2 = \frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \quad \text{for} \quad \alpha > \alpha_c \ , \end{aligned}$$

which yields the lower bound of (2.6) and the upper bound of (2.7).

On the other hand, consider $\alpha \in (0, \alpha_c)$ and integrate α in $-(4\alpha+3)I''(\alpha)^2 \le I'''(\alpha) \le 0$ from α to α_c, we obtain

$$-(2\alpha_c + 2\alpha + 3)(\alpha_c - \alpha) \le \frac{1}{I''(\alpha)} - \frac{1}{I''(\alpha_c)} \le 0 \ ,$$

such that

$$I''(\alpha) \le \frac{I''(\alpha_c)}{1 - (2\alpha_c + 2\alpha + 3)(\alpha_c - \alpha)I''(\alpha_c)} \quad \text{for any} \quad \alpha \in (\underline{\alpha}, \alpha_c) \ ,$$

where α must be larger than $\underline{\alpha}$ so that the denominator on the right-hand-side is positive. Therefore by Taylor formula, for any $\alpha \in (\underline{\alpha}, \alpha_c)$, there exists a $\xi \in (\alpha, \alpha_c)$ such that

$$\begin{aligned} I(\alpha) &= I(\alpha_c) + I'(\alpha_c)(\alpha - \alpha_c) + \frac{I''(\xi)}{2}(\alpha - \alpha_c)^2 \\ &\le \frac{I''(\alpha_c)}{2\{1 - (2\xi + 2\alpha_c + 3)(\alpha_c - \xi)I''(\alpha_c)\}}(\alpha_c - \alpha)^2 \\ &\le \frac{I''(\alpha_c)}{2\big\{1 - (2\alpha + 2\alpha_c + 3\big)(\alpha_c - \alpha)I''(\alpha_c)\big\}}(\alpha_c - \alpha)^2 \\ &\le \frac{\frac{1}{\sigma^2}(\alpha_c - \alpha)^2}{1 - \left(\frac{4(\alpha + \alpha_c) + 6}{\sigma^2}\right)(\alpha_c - \alpha)} \ . \end{aligned}$$

Similarly for any $\alpha > \alpha_c$, we have

$$I''(\alpha) \geq \frac{I''(\alpha_c)}{1 + (2\alpha_c + 2\alpha + 3)\,(\alpha - \alpha_c) I''(\alpha_c)}$$

and use Taylor formula again,

$$I(\alpha) \geq \frac{\frac{1}{\sigma^2}(\alpha_c - \alpha)^2}{1 + \left(\frac{4(\alpha+\alpha_c)+6}{\sigma^2}\right)(\alpha - \alpha_c)} \quad \text{for} \quad \alpha > \alpha_c \,.$$

These give the upper bound of (2.6) and the lower bound of (2.7). To establish the second line of (2.8) for $\alpha > \alpha_c$, we use (4.4) and $I'''(t) < 0$ for all $t > 0$ to have

$$\ln t_\alpha = I'(\alpha) \leq I'(\alpha_c) + I''(\alpha_c)(\alpha - \alpha_c) = \frac{2(\alpha - \alpha_c)}{\sigma^2} \,.$$

So the remaining task is to claim that (4.6) holds. First we show the second identity of (4.6) as follows. The first equality of (4.5) at $\alpha = \alpha_c$ gives

$$I''(\alpha_c) = \frac{t_{\alpha_c}{}'}{t_{\alpha_c}} = t_{\alpha_c}{}' \,.$$

Taking derivative with respect to α on both sides of (4.3) and setting $\alpha = \alpha_c$, we have

$$\frac{1}{t_{\alpha_c}} - \frac{\alpha_c t'{}_{\alpha_c}}{t_{\alpha_c}^2} = \frac{1}{2}\left(\frac{\frac{d^2}{dt^2}W(t_{\alpha_c})}{W(t_{\alpha_c})} - \left(\frac{\frac{d}{dt}W(t_{\alpha_c})}{W(t_{\alpha_c})}\right)^2\right) t_{\alpha_c}{}' \,.$$

Since $t_{\alpha_c} = 1$ and $2\alpha_c = \mu = \frac{d}{dt}W(t_{\alpha_c})$, it follows that

$$2 - 2\alpha_c t'{}_{\alpha_c} = (\sigma^2 - 2\alpha_c) t_{\alpha_c}{}' \,,$$

such that $I''(\alpha_c) = \frac{2}{\sigma^2}$.

Next we show that $I'(\alpha) \leq 0$, i.e., $t_\alpha \in (0, 1)$ by (4.4), for $\alpha \in (0, \alpha_c)$. Using $W(t) = [y + (1-y)dt]^2/(1 - at + bt^2)$ given in (3.7), where a and b are defined in (2.1), (4.3) multiplied by 2 becomes

$$\frac{2\alpha}{t_\alpha} = \frac{2(1-y)d}{y + (1-y)dt_\alpha} + \frac{a - 2bt_\alpha}{1 - at_\alpha + bt_\alpha^2} \,, \tag{4.7}$$

so that

$$2\alpha = -\frac{2y}{y + (1-y)dt_\alpha} + \frac{2 - at_\alpha}{1 - at_\alpha + bt_\alpha^2} \,. \tag{4.8}$$

Taking derivative with respect to α on both sides of (4.8), we have

$$2 = \left(\frac{2yd(1-y)}{(y + (1-y)dt_\alpha)^2} + \frac{a - 4bt_\alpha + abt_\alpha^2}{(1 - at_\alpha + bt_\alpha^2)^2}\right) t_\alpha{}' \,. \tag{4.9}$$

Since $a - 4bt_\alpha + abt_\alpha^2 = a(1 - at_\alpha + bt_\alpha^2) + (a^2 - 4b)t_\alpha$ where $1 - at_\alpha + bt_\alpha^2 > 0$ and $a^2 - 4b \geq 0$ (the equality $a^2 - 4b = 0$ holds when $x = 1$), we have

$$\frac{a - 4bt_\alpha + abt_\alpha^2}{(1 - at_\alpha + bt_\alpha^2)^2} \geq 0 \,.$$

(4.9) leads to $t_\alpha{}' > 0$ for all $\alpha > 0$. Hence $t_\alpha \in (0, 1)$ for $\alpha \in (0, \alpha_c)$ and $t_\alpha \in \left(1, \frac{a-\sqrt{a^2-4b}}{2b}\right)$ for $\alpha > \alpha_c$. This yields $I(\alpha) \le I(0) = -\ln y$ for $\alpha \in (0, \alpha_c)$.

Finally we show that $-(4\alpha+3)I''(\alpha)^2 \le I'''(\alpha) \le 0$ for all $\alpha > 0$. Taking derivative with respect to α on both sides of (4.9), we have

$$0 = \left(\frac{2yd(1-y)}{(y+(1-y)dt_\alpha)^2} + \frac{a-4bt_\alpha+abt_\alpha^2}{(1-at_\alpha+bt_\alpha^2)^2}\right) t_\alpha{}'' \\ + \left[\frac{d}{d\,t_\alpha}\left(\frac{2yd(1-y)}{(y+(1-y)dt_\alpha)^2} + \frac{a-4bt_\alpha+abt_\alpha^2}{(1-at_\alpha+bt_\alpha^2)^2}\right)\right](t_\alpha{}')^2 . \tag{4.10}$$

To obtain $I'''(\alpha) \le 0$ for all $\alpha > 0$, it is sufficient to show that

$$\frac{d}{d\,t_\alpha}\left(\frac{2yd(1-y)}{(y+(1-y)dt_\alpha)^2} + \frac{a-4bt_\alpha+abt_\alpha^2}{(1-at_\alpha+bt_\alpha^2)^2}\right) \ge 0.$$

Since $1+at_\alpha-3bt_\alpha^2 = 1-at_\alpha+bt_\alpha^2+2t_\alpha(a-2bt_\alpha) \ge 0$ for $t_\alpha \in \left(0, \frac{a-\sqrt{a^2-4b}}{2b}\right)$, we have

$$\begin{aligned}
&\frac{d}{d\,t_\alpha}\left(\frac{2yd(1-y)}{(y+(1-y)dt_\alpha)^2} + \frac{a-4bt_\alpha+abt_\alpha^2}{(1-at_\alpha+bt_\alpha^2)^2}\right) \\
&= \frac{-4yd^2(1-y)^2}{(y+(1-y)dt_\alpha)^3} + \frac{a(a-2bt_\alpha)}{(1-at_\alpha+bt_\alpha^2)^2} + \underbrace{\frac{(a^2-4b)}{(1-at_\alpha+bt_\alpha^2)^3}(1+at_\alpha-3bt_\alpha^2)}_{\ge 0} \\
&\ge \frac{-4yd^2(1-y)^2}{(y+(1-y)dt_\alpha)^3} + \frac{a(a-2bt_\alpha)}{(1-at_\alpha+bt_\alpha^2)^2} .
\end{aligned} \tag{4.11}$$

By (4.8), we have

$$\begin{aligned}
\frac{(a-2bt_\alpha)t_\alpha}{1-at_\alpha+bt_\alpha^2} &= -2 + \frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2} \\
&\ge -2 + \frac{2y}{y+(1-y)dt_\alpha} = \frac{2(1-y)dt_\alpha}{y+(1-y)dt_\alpha} .
\end{aligned}$$

This yields

$$\frac{(a-2bt_\alpha)}{1-at_\alpha+bt_\alpha^2} \ge \frac{2(1-y)d}{y+(1-y)dt_\alpha} ,$$

such that

$$\begin{aligned}
&\frac{-4yd^2(1-y)^2}{(y+(1-y)dt_\alpha)^3} + \frac{a(a-2bt_\alpha)}{(1-at_\alpha+bt_\alpha^2)^2} \\
&\ge \frac{-4yd^2(1-y)^2}{(y+(1-y)dt_\alpha)^3} + \frac{(a-2bt_\alpha)^2}{(1-at_\alpha+bt_\alpha^2)^2} \\
&\ge \frac{4d^2(1-y)^2}{(y+(1-y)dt_\alpha)^2}\left(\frac{-y}{y+(1-y)dt_\alpha} + 1\right) \ge 0 ,
\end{aligned}$$

and $I'''(\alpha) \le 0$ for all $\alpha > 0$ is proved.

To show the lower bound $-(4\alpha+3)I''(\alpha)^2 \le I'''(\alpha)$, we use (4.5), (4.10) and (4.11) to have

$$I'''(\alpha) = -I''(\alpha)^2 \left(\frac{\frac{-4yd^2(1-y)^2 t_\alpha}{[y+(1-y)dt_\alpha]^3} + \frac{2t_\alpha(6b-a^2-abt_\alpha)}{[1-at_\alpha+bt_\alpha^2]^2} + \frac{2(a^2-4b)t_\alpha(2-at_\alpha)}{[1-at_\alpha+bt_\alpha^2]^3}}{\frac{2yd(1-y)}{(y+(1-y)dt_\alpha)^2} + \frac{a}{1-at_\alpha+bt_\alpha^2} + \frac{t_\alpha(a^2-4b)}{(1-at_\alpha+bt_\alpha^2)^2}} + 1 \right)$$

$$\ge -I''(\alpha)^2 \left(\frac{\frac{2t_\alpha(6b-a^2-abt_\alpha)}{[1-at_\alpha+bt_\alpha^2]^2} + \frac{2t_\alpha(a^2-4b)(2-at_\alpha)}{[1-at_\alpha+bt_\alpha^2]^3}}{\frac{a}{1-at_\alpha+bt_\alpha^2} + \frac{t_\alpha(a^2-4b)}{(1-at_\alpha+bt_\alpha^2)^2}} + 1 \right). \qquad (4.12)$$

It is easy to see that $2 - at_\alpha > 0$ for all $t_\alpha \in (0, \left(a - \sqrt{a^2-4b}\right)/(2b))$, we have

$$\begin{aligned} &\frac{2t_\alpha(6b-a^2-abt_\alpha)}{[1-at_\alpha+bt_\alpha^2]^2} \\ &= \frac{-2t_\alpha(a^2-4b)}{[1-at_\alpha+bt_\alpha^2]^2} + \frac{2bt_\alpha(2-at_\alpha)}{[1-at_\alpha+bt_\alpha^2]^2} \\ &= \frac{-2t_\alpha(a^2-4b)}{[1-at_\alpha+bt_\alpha^2]^2} + \frac{a}{1-at_\alpha+bt_\alpha^2} \times \frac{2bt_\alpha}{a} \times \frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2} \\ &\le \frac{-2t_\alpha(a^2-4b)}{[1-at_\alpha+bt_\alpha^2]^2} + \frac{a}{1-at_\alpha+bt_\alpha^2} \times \frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2}, \end{aligned}$$

so that

$$\begin{aligned} &\frac{2t_\alpha(6b-a^2-abt_\alpha)}{[1-at_\alpha+bt_\alpha^2]^2} + \frac{2(a^2-4b)t_\alpha(2-at_\alpha)}{[1-at_\alpha+bt_\alpha^2]^3} \\ &\le \left(\frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2}\right)\left(\frac{a}{1-at_\alpha+bt_\alpha^2} + \frac{t_\alpha(a^2-4b)}{(1-at_\alpha+bt_\alpha^2)^2}\right) \\ &\quad + \frac{t_\alpha^2(a^2-4b)(a-2bt_\alpha)}{(1-at_\alpha+bt_\alpha^2)^3}. \end{aligned}$$

(4.12) becomes

$$\begin{aligned} I'''(\alpha) &\ge -I''(\alpha)^2 \left(\frac{\left(\frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2}\right)\left(\frac{a}{1-at_\alpha+bt_\alpha^2} + \frac{t_\alpha(a^2-4b)}{(1-at_\alpha+bt_\alpha^2)^2}\right) + \frac{t_\alpha^2(a^2-4b)(a-2bt_\alpha)}{(1-at_\alpha+bt_\alpha^2)^3}}{\frac{a}{1-at_\alpha+bt_\alpha^2} + \frac{t_\alpha(a^2-4b)}{(1-at_\alpha+bt_\alpha^2)^2}} + 1 \right) \\ &\ge -I''(\alpha)^2 \left(\underbrace{\frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2}}_{\in(0,2\alpha+2)} + \underbrace{\frac{t_\alpha(a-2bt_\alpha)}{1-at_\alpha+bt_\alpha^2}}_{\in(0,2\alpha)} + 1 \right) \\ &\ge -(4\alpha+3)I''(\alpha)^2, \end{aligned}$$

where the last inequality holds by (4.7) and (4.8), and the proof for the case with diagonal edges from lower-left to upper-right is completed.

For the case with diagonal edges from lower-right to upper-left, (3.15) reads $W(t) = [(d/t + (1-d)y)^2]/(1 - at + bt^2)$, where a and b are defined in (2.1). Multiply both sides of (4.3) by 2 again, we have

$$\frac{2(\alpha+1)}{t_\alpha} = \frac{2(1-d)y}{d+(1-d)yt_\alpha} + \frac{a-2bt_\alpha}{1-at_\alpha+bt_\alpha^2},$$

so that

$$2\alpha+2 = -\frac{2d}{d+(1-d)yt_\alpha} + \frac{2-at_\alpha}{1-at_\alpha+bt_\alpha^2}. \tag{4.13}$$

Note that $2\alpha+2>0$ since we only consider $\alpha>-1$. Taking derivative with respect to α on both sides of (4.13), we have

$$2 = \left(\frac{2yd(1-d)}{(d+(1-d)yt_\alpha)^2} + \frac{a-4bt_\alpha+abt_\alpha^2}{(1-at_\alpha+bt_\alpha^2)^2}\right) t_\alpha{}'.$$

By the same argument, (4.6) can be shown. This completes the proof.

5 Proof of Theorem 2.2 and Theorem 2.4

5.1 Proof of Theorem 2.2

To show Theorem 2.2, it is easy to see the case for $\alpha=\alpha_c$ by the same argument of (4.2). For the case of $\alpha\neq\alpha_c$, we consider diagonal edges from lower-left to upper-right with $y>0$ first. Since $\tau(m_1,2N)\le\tau(m_2,2N)$ for any $m_1<m_2$ and $N\in\mathbb{N}$, by the definition of $u_1(m)$ and $u_2(m)$, we have

$$\begin{aligned}
\tau(2N_\alpha,2N+1) &= \sum_{m=0}^{2N_\alpha} P_N(m)\,(u_1(2N_\alpha-m)+u_2(2N_\alpha-m))\\
&\le \tau(2N_\alpha,2N)\sum_{m=0}^{2N_\alpha}(u_1(2N_\alpha-m)+u_2(2N_\alpha-m))\\
&\le \tau(2N_\alpha,2N),
\end{aligned}$$

such that

$$\frac{\tau(2N_\alpha,2N+1)}{\tau(2N_\alpha,2N)}\le 1 \quad \text{for all } N\in\mathbb{N}. \tag{5.1}$$

On the other hand, we also have

$$\begin{aligned}
&\tau(2N_\alpha,2N+2)\\
&= \sum_{m=0}^{2N_\alpha}\tau(m,2N+1)\,(v_1(2N_\alpha-m)+v_2(2N_\alpha-m))\\
&= \sum_{m=0}^{2N_\alpha-1}\tau(m,2N+1)v_1(2N_\alpha-m)+\sum_{m=0}^{2N_\alpha-1}\tau(m,2N+1)v_2(2N_\alpha-m)\\
&\quad +\tau(2N_\alpha,2N+1)v_1(0)\\
&\le \sum_{m=0}^{2N_\alpha-1}\tau(m,2N)v_1(2N_\alpha-m)+\sum_{m=0}^{2N_\alpha-1}\tau(m,2N)v_2(2N_\alpha-m)\\
&\quad + y\tau(2N_\alpha,2N+1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \tau(2N_\alpha, 2N)\left(\frac{(1-y)(1-d)xy}{1-(1-y)(1-d)x} + \frac{(1-y)d}{1-(1-y)(1-d)x}\right) \\
&\quad + y\tau(2N_\alpha, 2N+1) \\
&\leq (1-y)\tau(2N_\alpha, 2N) + y\tau(2N_\alpha, 2N+1) \leq \tau(2N_\alpha, 2N).
\end{aligned} \tag{5.2}
$$

By (5.1) and (5.2) and using $\tau(m,n) \in (0,1]$ for all $m, n \in \mathbb{N}$, we have

$$
\frac{\log\tau(2N_\alpha, 2N+2)}{\log\tau(2N_\alpha, 2N)} \leq \frac{\log\left((1-y)\tau(2N_\alpha, 2N) + y\tau(2N_\alpha, 2N+1)\right)}{\log\tau(2N_\alpha, 2N)} \leq 1\,.
$$

Taking the limit of $N \to \infty$, Theorem 2.1 yields

$$
\tau(2N_\alpha, 2N+1) \approx \tau(2N_\alpha, 2N) \approx e^{-2NI(\alpha)}\,. \tag{5.3}
$$

Using similar argument for the case with $y = 0$, we have

$$
\begin{aligned}
\tau(2N_\alpha + 1, 2N+1) &\leq \tau(2N_\alpha, 2N)\,, \\
\tau(2N_\alpha + 2, 2N+2) &\leq (1-d)\tau(2N_\alpha, 2N) + d\tau(2N_\alpha + 1, 2N+1) \\
&\leq \tau(2N_\alpha, 2N)\,.
\end{aligned}
$$

Taking the limit of $N \to \infty$, we also obtain (5.3). The proof for the case with diagonal edges from lower-right to upper-left can be shown by the same method.

5.2 Proof of Theorem 2.4

First consider $\rho \in (0, \frac{1}{2})$ or $\rho = \frac{1}{2}$ with $\lim_{N\to\infty} \ell_N = \infty$. By Theorem 2.1, we have

$$
\begin{aligned}
N^{-2\rho}\ell_N^2 \leq I(\alpha_N^-) \leq N^{-2\rho}\ell_N^2(1 + cN^{-\rho}\ell_N)\,, \\
N^{-2\rho}\ell_N^2(1 - cN^{-\rho}\ell_N) \leq I(\alpha_N^+) \leq N^{-2\rho}\ell_N^2
\end{aligned}
$$

for some $c > 0$, so both

$$
\tau(N_{\alpha_N^-}, N) \quad \text{and} \quad 1 - \tau(N_{\alpha_N^+}, N) \approx \exp(-N^{1-2\rho}\ell_N^2)\,.
$$

For $\rho > \frac{1}{2}$ or $\rho = \frac{1}{2}$ with $\lim_{N\to\infty} \ell_N = \ell \in [0, \infty)$, we use (4.1) to have

$$
\tau(N_{\alpha_N^-}, N) = \Psi\left(N^{\frac{1}{2}-\rho}\ell_N + O(\frac{1}{\sqrt{N}})\right) + O(\frac{1}{\sqrt{N}})\,.
$$

For the first case $\rho > \frac{1}{2}$, we have

$$
\begin{aligned}
\Psi\left(N^{-\rho+\frac{1}{2}}\ell_N + O(\frac{1}{\sqrt{N}})\right) &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du - \int_0^{N^{-\rho+\frac{1}{2}}\ell_N + O(\frac{1}{\sqrt{N}})} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du \\
&= \begin{cases} \frac{1}{2} + O(1)N^{-\rho+\frac{1}{2}}\ell_N & \rho \in (\frac{1}{2}, 1)\,, \\ \frac{1}{2} + O(1/\sqrt{N}) & \rho \geq 1\,, \end{cases}
\end{aligned}
$$

and for the second case $\rho = \frac{1}{2}$ with $\lim_{n\to\infty} \ell_N = \ell \in [0, \infty)$

$$
\Psi(N^{-\rho+\frac{1}{2}}\ell_N) = \int_\ell^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du + \int_{\ell_N}^\ell \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du = \Psi(\ell) + O(|\ell - \ell_N|)\,.
$$

Therefore,

$$\tau(N_{\alpha_N^-}, N) = \begin{cases} \frac{1}{2} + O(1)\max\{\frac{1}{\sqrt{N}}, N^{-\rho+\frac{1}{2}}\ell_N\} & \rho > \frac{1}{2}\ , \\ \Psi(\ell) + O(1)\max\{\frac{1}{\sqrt{N}}, |\ell - \ell_N|\} & \rho = \frac{1}{2}\ ,\ \lim_{n\to\infty}\ell_N = \ell \in [1,\infty). \end{cases}$$

Using the same argument, it is easy to obtain the corresponding result for $\alpha = \alpha_N^+$ for $\rho > \frac{1}{2}$ or $\rho = \frac{1}{2}$ with $\lim_{N\to\infty}\ell_N = \ell \in [0,\infty)$, and the proof is omitted here. This completes the proof of Theorem 2.4.

6 Future Works

In conclusion, we have considered a version of directed percolation model on the triangular lattice, and investigated the critical behavior of the probability that there is at least one connected-directed path of occupied edges from (0, 0) to (M, N). The main result given as Theorem 2.2 includes the lower and upper bounds of the exponential rate of convergence for the two point correlation function when $\alpha \neq \alpha_c$. These bounds are by no means optimal and may be tightened. A more general model would be considering all the horizontal probability are directed rightward with probability $x \in (0, 1)$. This reduces to the original directed percolation model when $x = y$ and $d = 0$. The real challenging work is the directed percolation on the $\mathbb{Z}^d$ lattice with nontrivial occupation probabilities in all directions. As the limiting behavior of the three-dimensional version of Domany–Kinzel model has been considered in Ref. [8], it would be interesting to investigate the rate of convergence for this model.

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