Multistability for Delayed Neural Networks via Sequential Contracting

Chang-Yuan Cheng, Kuang-Hui Lin, Chih-Wen Shih, and Jui-Pin Tseng

***Abstract*— In this paper, we explore a variety of new multistability scenarios in the general delayed neural network system. Geometric structure embedded in equations is exploited and incorporated into the analysis to elucidate the underlying dynamics. Criteria derived from different geometric configurations lead to disparate numbers of equilibria. A new approach named sequential contracting is applied to conclude the global convergence to multiple equilibrium points of the system. The formulation accommodates both smooth sigmoidal and piecewise-linear activation functions. Several numerical examples illustrate the present analytic theory.**

***Index Terms*— Complete stability, delay equations, multistability, neural network.**

I. Introduction

In recent decades, various neural network models have been proposed and employed in diverse applied sciences and engineering successfully. Most of the neural networks developed are based on Hopfield model [12] or Cohen–Grossberg model [8]. Delays have been considered in the neural network modeling, as time lags occur in transmitting signal among real neurons and artificial neurons. Hopfield-type neural network with delays was introduced in [25]. Later, delay has been considered and extensively studied in neural networks [1], [2], [13], [20], [29], [31], [36]. Concerning the mathematical modeling, delays can modify the collective dynamics for neural networks. Thus, in neural network models, it is appealing to investigate how the collective dynamics are determined by the connection strength, nonlinear coupling functions, and transmission delays. Although the accomplishments in those investigations have advanced the theories on these delayed network systems, effective approaches combined with powerful mathematical techniques are still in demand in tackling the unsolved problems, for example, to elucidate the variations of collective dynamics of the network system with respect to the size of the network, connection strength, and delay magnitude.

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Multistability is a notion to describe the coexistence of multiple stable equilibria or cycles. Such dynamics is essential in several applications of neural networks, including pattern recognition and associative memory storage [6], [9], [11], [12]. Hopfield-type neural networks, with or without consideration of transmission delays, have been the primary models that attract a great deal of research interests on multistability.

In this paper, we consider the general delayed neural network

$$\frac{dx_i(t)}{dt} = -\mu_i x_i(t) + \sum_{j=1}^{n}[\alpha_{ij} g_j(x_j(t)) + \beta_{ij} g_j(x_j(t-\tau_{ij}))] + I_i \quad (1)$$

where $i = 1, \ldots, n$, $\mu_i > 0$, α_{ij}, β_{ij} are connection weights from neuron j to neuron i, $g_j(\cdot)$ are activation functions, $0 \le \tau_{ij}$ are time lags that are bounded above by τ_M, and I_i stand for independent bias current sources. A typical class of activation functions is

$$\text{class } \mathcal{A}: \begin{cases} g_i \in \mathcal{C}^2(\mathbb{R}),\ g_i'(\xi) > 0, \text{ for all } \xi \in \mathbb{R} \\ (\xi - \sigma_i) g_i''(\xi) < 0, \text{ for all } \xi \in \mathbb{R},\ \xi \neq \sigma_i \\ \lim_{\xi \to +\infty} g_i(\xi) = v_i,\ \lim_{\xi \to -\infty} g_i(\xi) = u_i \end{cases}$$

where v_i, u_i, and σ_i are constants with $u_i < v_i$, $i = 1, \ldots, n$. Class $\mathcal{A}$ contains the general bounded sigmoidal functions; a typical example is $g_i(\xi) = \tanh \xi$. The present theory also applies to (1) with piecewise-linear activation functions, such as the three-segment standard output function in cellular neural networks, $g_i(\xi) = [|\xi + 1| - |\xi - 1|]/2$. Let $\rho_i := \max\{|u_i|, |v_i|\}$, $i = 1, \ldots, n$.

System (1) reduces to the classical and delayed Hopfield neural networks in [12] and [25], as $\beta_{ij} = 0$ and $\alpha_{ij} = 0$ for all i, j, respectively. It also represents the cellular neural networks without delays [6] and with delays [29].

Multistability for (1) has been studied in several papers. The existence of 3^n equilibria for (1) with activation functions in class $\mathcal{A}$ was established in [4] and [5]. Therein, it was also shown that the 2^n equilibria out of those 3^n equilibria are stable, and each of them is located in a positively invariant region. Later, different criteria for stability were obtained in [14] using the Lyapunov functional and matrix inequality techniques. Several works are strongly restricted to the class of piecewise-linear activation functions. For example, for (1) with standard piecewise-linear output function, it was shown in [38] that the system cannot have more than 3^n equilibria, and can have 2^n locally exponentially stable equilibria. Hopfield neural network with nondecreasing piecewise-linear activation functions with $2r$ corner points ($2r + 1$ saturation regions), without time delays, was investigated in [34]. It was asserted that the n-neuron system can have and only have $(2r + 1)^n$ equilibria under some conditions, of which $(r+1)^n$ are locally exponentially stable and the others are unstable. In [40], the existence of $(2k + 2m - 1)^n$ equilibria was concluded for (1) with piecewise-linear activation functions with $k+m$ saturation regions. Other related works include [19], [21], and [26]–[28].

Convergence of dynamics (also called complete stability), which means that every solution of the system converges to one of the equilibrium points, is a key ingredient in multistability. Quasi-convergence of dynamics for (1) was established in [5]. It indicates that almost every orbit converges to one of the 3^n equilibria, and 2^n among these 3^n equilibria are stable, under a delay-dependent criterion. Therein, (1) is shown to be a strongly order-preserving under a special order on the space of continuous functions $C([-\tau_M, 0])$. System (1) is said to be cooperative (competitive) if $\alpha_{ij}, \beta_{ij} \geq 0$ (≤ 0) for all $j \neq i$. Chua and Roska [7] have demonstrated that if the interconnection matrix is irreducible, and the neuron activations are modeled by sigmoidal functions (C^1, bounded and strictly increasing), then the solution flow generated by a cooperative cellular neural network without delay is eventually strongly monotone. According to the standard theory of cooperative dynamical systems, the flow enjoys the so-called limit set dichotomy, and generically, the solution converges to the set of equilibria. Marco *et al.* extended the limit set dichotomy to (1) with piecewise-linear activation functions, nonsymmetric cooperative interconnection matrix, and without delays [22] and with delays [23]. There were other studies on complete stability that consider only piecewise-linear activation functions, such as [35] and [39]. Some investigations on convergence of neural networks with discontinuous activation functions can be found in [16] and [24].

Multistability has also been studied for Cohen–Grossberg neural networks with a class of piecewise-smooth activation functions with two saturated segments in [3]. It was shown that under some conditions, the n-neuron networks can have 2^n locally exponentially stable equilibrium points, each located in a saturated region.

The purpose of this paper is two folds. The first is to explore more multistability scenarios in the delayed neural networks (1). The second is to introduce new analytic methodologies into the study of multistability in delayed systems. For locating an equilibrium, we develop an approach that combines the Brouwer's fixed-point theorem or contracting mapping theorem with the geometric configurations induced from the structure of the equations. With such an approach, we shall exploit a variety of multiple equilibria for (1). For global convergence of dynamics (complete stability), we introduce a new approach named sequential contracting. We start by constructing a preliminary upper and lower dynamics for each component of the system. The upper and lower dynamics are designed to have their own solutions contracted to some compact intervals. We then construct finer upper and lower dynamics successively, so that the original dynamics are attracted to more concentrated regions. A criterion for contraction is then formulated so that these nested intervals collapse into points, as the iterative constructions of upper and lower dynamics carry on. Under different formulations of lower and upper dynamics, delay-dependent criteria and delay-independent criteria for multistability of (1) can be derived, respectively. This approach can also lead to a network-scale-dependent criterion for asymptotic behaviors and synchronization in network systems [31], [32].

In previous works, as mentioned above, mathematical studies on multistability in neural networks centered around analyzing the existence of multiple equilibria using standard method or utilizing the piecewise-linear structure of the activation functions. Stability of the equilibrium and local dynamics were studied by the linearization theory or Lyapunov functional and matrix inequality techniques. On the other hand, monotone dynamics theory, which requires the interconnection matrix to be cooperative, has been adopted to conclude the global dynamics. To explore further dynamical scenarios for (1), which are embedded in the equations, new ideas are required. The sequential contracting has been applied to establish global convergence of dynamics for (1) in [30]. The dynamical scenario concluded therein is that every orbit converges to one of the 3^n equilibria. As only 2^n out of those 3^n equilibria are stable, it was further concluded that almost every orbit converges to one of the 2^n stable equilibria. In this paper, we improve the methodology to exploit further fruitful dynamics in (1). One of the main results in this paper is the existence of 3^k equilibria, among which 2^k equilibria are attracting, for any $k \leq n$, in the n-neuron network (1).

Almost all the existing results on the number of multiple equilibria are in terms of n-power of the number of saturated (or near saturated) regions in a n-neuron system, as mentioned above. Herein, the numbers of equilibria we derive are not in power of n. These new multistability scenarios demonstrate the strength of the present methodology, as they are inaccessible by other treatments. Our approach applies to both smooth sigmoidal and nondecreasing piecewise-linear activation functions. Thus, the formulation also covers the case of piecewise-linear activation functions with two saturated segments, but with smooth corners.

The existence of 3^k equilibria for (1) is discussed in Section II. The global convergence of dynamics to 3^k equilibria via sequential contracting is presented in Section III. More varieties of multiple equilibria are discussed in Section IV. The extension of the results in Sections II–IV to piecewise-linear activation functions is mentioned in Section V. We provide two numerical examples in Section VI. Finally, the conclusion is drawn in Section VII.

II. Multiple Equilibria in (1)

Set $\mathcal{N} = \{1, 2, \ldots, n\}$. The stationary equation for (1) is

$$F_i(\mathbf{x}) := -\mu_i x_i + \sum_{j=1}^{n}(\alpha_{ij} + \beta_{ij})g_j(x_j) + I_i = 0 \qquad (2)$$

for $i \in \mathcal{N}$, and we denote $\mathbf{F} = (F_1, \ldots, F_n)$.

There are various ways to find the equilibrium for an ordinary differential equation system or delay equations. In this paper, we present an approach that combines a geometric formulation on $F_i(\mathbf{x})$ in (2) and the Brouwer's fixed-point theorem to study the existence of equilibrium for (1). Brouwer's fixed-point theorem states as: every continuous function from a convex compact subset K of a Euclidean space to K itself has a fixed point. Our idea is to locate a region (box) $K := K_1 \times K_2 \times \cdots \times K_n$, with each K_i an interval in $\mathbb{R}$, so that for an arbitrary $(\zeta_1, \ldots, \zeta_n) \in K$, there exists a solution $x_i \in K_i$ to $F_i(\zeta_1, \ldots, \zeta_{i-1}, x_i, \zeta_{i+1}, \ldots, \zeta_n) = 0$, for every $i \in \mathcal{N}$. If this holds and the corresponding mapping $\Phi(\zeta_1, \ldots, \zeta_n) = (x_1, \ldots, x_n)$ is continuous, then there exists a $\bar{\mathbf{x}}$ that satisfies $F_i(\bar{\mathbf{x}}) = 0$ for all $i \in \mathcal{N}$. $\bar{\mathbf{x}}$ is then an equilibrium of (1). To locate such a region K, we develop a geometric formulation based on the structure of F_i.

Let us define the single-variable upper and lower functions

$$\hat{f}_i(\xi) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + k_i^+$$
$$\check{f}_i(\xi) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + k_i^-$$

for $i \in \mathcal{N}$, where $k_i^+ := \sum_{j=1,j\neq i}^{n} \rho_j|\alpha_{ij} + \beta_{ij}| + I_i$, $k_i^- := -\sum_{j=1,j\neq i}^{n} \rho_j|\alpha_{ij} + \beta_{ij}| + I_i$. It follows that

$$\check{f}_i(x_i) \le F_i(\mathbf{x}) \le \hat{f}_i(x_i)$$

for all $\mathbf{x} = (x_1, \ldots, x_n)$ and $i \in \mathcal{N}$. For J_i with $k_i^- \le J_i \le k_i^+$, $i \in \mathcal{N}$, we introduce a family of single-neuron equations

$$\frac{d\xi}{dt} = f_i(\xi) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + J_i \qquad (3)$$

where $\xi \in \mathbb{R}$.

For given parameters μ_i, α_{ii}, β_{ii}, and $i \in \mathcal{N}$, we denote

$$\mathcal{M} := \left\{ i \in \mathcal{N} \,|\, \max_{\xi\in\mathbb{R}} g_i'(\xi) \le \frac{\mu_i}{\alpha_{ii} + \beta_{ii}} \right\}$$
$$\mathcal{B} := \left\{ i \in \mathcal{N} \,|\, \inf_{\xi\in\mathbb{R}} g_i'(\xi) < \frac{\mu_i}{\alpha_{ii} + \beta_{ii}} < \max_{\xi\in\mathbb{R}} g_i'(\xi) \right\}.$$

The notation $\mathcal{M}$ and $\mathcal{B}$ are related to the sense of monostable and (potentially) bistable scenarios, respectively.

Lemma 1: If $i \in \mathcal{B}$, there exist two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \sigma_i < \tilde{q}_i$ such that $f_i'(\tilde{p}_i) = f_i'(\tilde{q}_i) = 0$, or equivalently, $g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = \mu_i/(\alpha_{ii} + \beta_{ii})$.

Proof: For each $i \in \mathcal{N}$, with $f_i'(\xi) = -\mu_i + (\alpha_{ii} + \beta_{ii})$ $g_i'(\xi)$, we have $f_i'(\xi) = 0$ if and only if $g_i'(\xi) = \mu_i/\alpha_{ii} + \beta_{ii}$. The graph of function $g_i'(\xi)$ has a global maximum at σ_i and $\lim_{\xi\to\pm\infty} g_i'(\xi) = 0$. Hence, by continuity of g_i', if

$$0 = \inf_{\xi\in\mathbb{R}} g_i'(\xi) < \frac{\mu_i}{\alpha_{ii} + \beta_{ii}} < \max_{\xi\in\mathbb{R}} g_i'(\xi) = g_i'(\sigma_i)$$

that is, and $i \in \mathcal{B}$, there exist two points $\tilde{p}_i$ and $\tilde{q}_i$, with $\tilde{p}_i < \sigma_i < \tilde{q}_i$, such that $g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = \mu_i/\alpha_{ii} + \beta_{ii}$. ∎

Note that $\check{f}_i, \hat{f}_i$, and f_i are vertical shifts of one another, and they attain local minimum at $\tilde{p}_i$ and local maximum at $\tilde{q}_i$.

We consider six disjoint subsets of $\mathcal{B}$

$$\mathcal{B}_r^r = \{i \in \mathcal{N} | i \in \mathcal{B},\ \check{f}_i(\tilde{p}_i) > 0\}$$
$$\mathcal{B}_l^l = \{i \in N | i \in \mathcal{B},\ \hat{f}_i(\tilde{q}_i) < 0\}$$
$$\mathcal{B}_3^3 = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) < 0,\ \check{f}_i(\tilde{q}_i) > 0\}$$
$$\mathcal{B}_3^r = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) > 0,\ \check{f}_i(\tilde{p}_i) < 0,\ \check{f}_i(\tilde{q}_i) > 0\}$$
$$\mathcal{B}_l^3 = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) < 0,\ \hat{f}_i(\tilde{q}_i) > 0,\ \check{f}_i(\tilde{q}_i) < 0\}$$
$$\mathcal{B}_l^r = \{i \in \mathcal{N} | i \in \mathcal{B},\ \hat{f}_i(\tilde{p}_i) > 0,\ \check{f}_i(\tilde{q}_i) < 0\}.$$

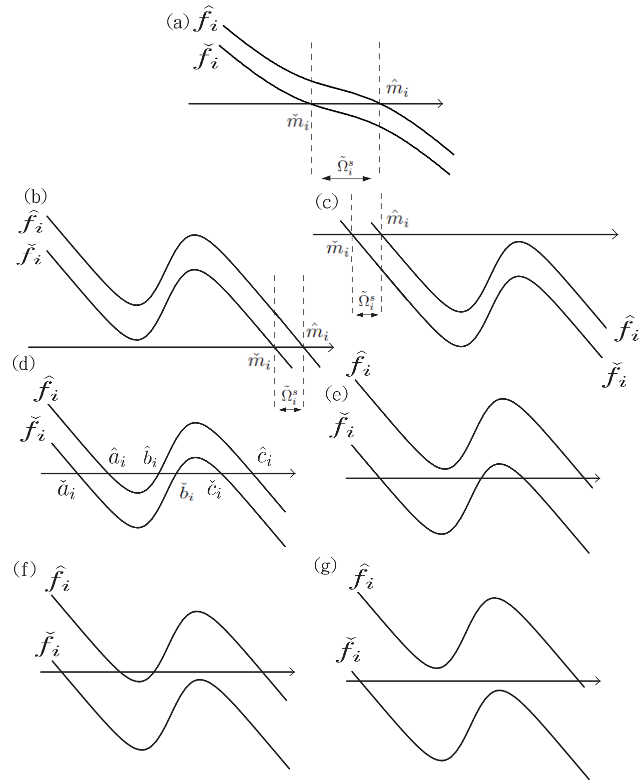

Fig. 1. (a)–(g) Type $\mathcal{M}$, $\mathcal{B}_r^r$, $\mathcal{B}_l^l$, $\mathcal{B}_3^3$, $\mathcal{B}_3^r$, $\mathcal{B}_l^3$, and $\mathcal{B}_l^r$, respectively.

Herein, l and r stand for left and right; the superscript $\star$ and subscript $\bullet$ in $\mathcal{B}_\bullet^\star$, $\star, \bullet \in \{l, r, 3\}$ show the configurations for upper function $\hat{f}_i$ and lower function $\check{f}_i$, respectively. More precisely, if $\star = l$ (resp., r), then $\hat{f}_i$ has a unique zero at the left (resp., right) arm of its graph; if $\star = 3$, then $\hat{f}_i$ has exactly three zeros. Same interpretation applies to $\bullet$ in $\mathcal{B}_\bullet^\star$ and $\check{f}_i$. The configurations corresponding to each $\mathcal{B}_\bullet^\star$ are shown in Fig. 1. In particular, for $i \in \mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l$, there exist points $\check{m}_i$, $\hat{m}_i$ with $\check{m}_i < \hat{m}_i$ such that $\hat{f}_i(\hat{m}_i) = \check{f}_i(\check{m}_i) = 0$. For $i \in \mathcal{B}_3^3$, there exist points $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ as well as points $\check{a}_i, \check{b}_i, \check{c}_i$ with $\check{a}_i < \check{b}_i < \check{c}_i$, such that $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$, as shown in Fig. 1(d).

The following theorems are the main results of this section. We show that for each $1 \le k \le n$, there exist parameters with which (1) admits 3^k equilibria. Let card($\bullet$) denote the cardinality of set $\bullet$.

Theorem 1: If $\mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l \cup \mathcal{B}_3^3 = \mathcal{N}$ and $k = card(\mathcal{B}_3^3) \ge 1$, then there exist 3^k equilibria in (1).

Proof: We consider 3^k disjoint closed regions in $\mathbb{R}^n$

$$\tilde{\Omega}^{\mathbf{w}} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \tilde{\Omega}_i^{w_i}\right\} \qquad (4)$$
$$\mathbf{w} = (w_1, \ldots, w_n)$$
$$w_i = \text{“}l\text{”}, \text{“}m\text{”}, \text{“}r\text{”},\ \text{for } i \in \mathcal{B}_3^3$$
$$w_i = \text{“}s\text{”},\ \text{for } i \in \mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l$$

where $\tilde{\Omega}_i^l = [\check{a}_i, \hat{a}_i]$, $\tilde{\Omega}_i^m = [\hat{b}_i, \check{b}_i]$, $\tilde{\Omega}_i^r = [\check{c}_i, \hat{c}_i]$, and $\tilde{\Omega}_i^s = [\check{m}_i, \hat{m}_i]$ are compact intervals, as shown in Figs. 1 and 2. Let us take $\tilde{\Omega}^{\mathbf{w}}$ as any one of these regions. For any given $(\zeta_1, \ldots, \zeta_n) \in \tilde{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$h_i(x_i) = 0 \qquad (5)$$

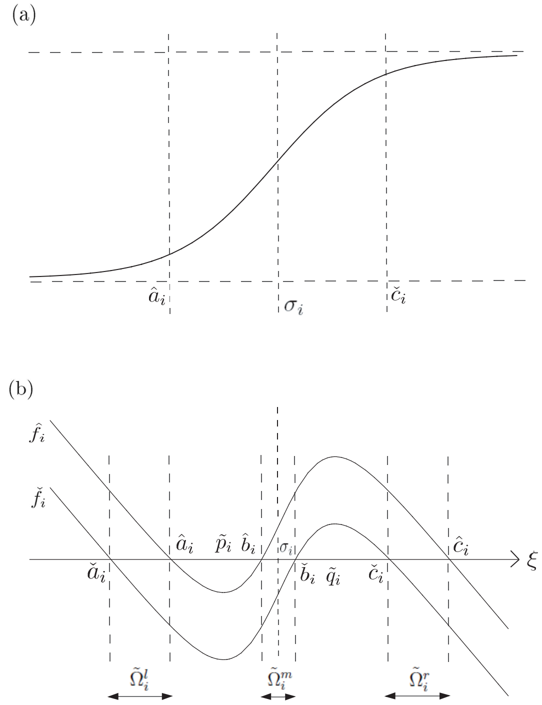

Fig. 2. (a) Graph of activation function g_i in class $\mathcal{A}$. (b) Configurations of functions $\hat{f}_i$ and $\check{f}_i$ for $i \in \mathcal{B}_3^3$.

where $h_i(x_i) := -\mu_i x_i + (\alpha_{ii} + \beta_{ii})g_i(x_i) + \sum_{j=1,j\neq i}^{n} (\alpha_{ij} + \beta_{ij})g_j(\zeta_j) + I_i$, $i \in \mathcal{N}$. Note that each h_i is a vertical shift of f_i in (3) and lies between $\hat{f}_i$ and $\check{f}_i$. Thus, for $i \in \mathcal{B}_3^3$, one can always find three solutions to (5), which lie in regions $\tilde{\Omega}_i^l$, $\tilde{\Omega}_i^m$, and $\tilde{\Omega}_i^r$, respectively. For $i \in \mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l$, there exists one solution to (5), which lies in region $\tilde{\Omega}_i^s$. Next, we consider a mapping $\Phi^{\mathbf{w}} : \tilde{\Omega}^{\mathbf{w}} \to \tilde{\Omega}^{\mathbf{w}}$ defined by

$$\Phi^{\mathbf{w}}(\zeta_1, \ldots, \zeta_n) = (\tilde{x}_1, \ldots, \tilde{x}_n)$$

where $\tilde{x}_i$ is the solution of (5) lying in $\tilde{\Omega}_i^{w_i}$. The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, since each g_i is continuous. It follows from the Brouwer's fixed-point theorem that there exists one fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ of $\Phi^{\mathbf{w}}$ in $\tilde{\Omega}^{\mathbf{w}}$, which is also a zero of the function $\mathbf{F}$, where $\mathbf{F}$ is defined in (2). Consequently, there exist 3^k equilibria for (1) and each of the 3^k disjoint regions $\tilde{\Omega}^{\mathbf{w}}$ contains one of the equilibria. ■

We note that what was obtained in [30] is the existence of 3^n equilibria for the n-neuron system (1). Theorem 1 establishes the existence of 3^k equilibria for (1) via Brouwer's fixed-point theorem. Next, we further apply the contraction mapping theorem to assert the existence of exact 3^k equilibria for (1), under additional conditions. Let $L_i := g_i'(\sigma_i)$, $i \in \mathcal{N}$.

Theorem 2: Assume that $\mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l \cup \mathcal{B}_3^3 = \mathcal{N}$ with $k = card(\mathcal{B}_3^3) \geq 1$. For each $i \in \mathcal{N}$, fix a $\theta_i \in (0, \mu_i)$ and then define

$$\bar{L}_i := \begin{cases} \frac{\mu_i - \theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } i \in \mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l \\ L_i, & \text{if } i \in \mathcal{B}_3^3. \end{cases} \tag{6}$$

If the parameters satisfy

$$\theta_i > \sum_{j=1,j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}| \tag{7}$$

and

$$g_i'(\xi) \begin{cases} < \frac{\mu_i - \theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } \xi \in [\check{m}_i, \hat{m}_i],\ i \in \mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l \\ < \frac{\mu_i - \theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } \xi \in (-\infty, \hat{a}_i]\cup[\check{c}_i, \infty),\ i \in \mathcal{B}_3^3 \\ > \frac{\mu_i + \theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } \xi \in [\hat{b}_i, \check{b}_i],\ i \in \mathcal{B}_3^3 \end{cases} \tag{8}$$

for all $i \in \mathcal{N}$, then there exist exactly 3^k equilibria in (1), and each region $\tilde{\Omega}^{\mathbf{w}}$, defined in (4), contains exactly one of these 3^k equilibria.

Proof: Let $\tilde{\Omega}^{\mathbf{w}}$, $\mathbf{w} = (w_1, \ldots, w_n)$, be any one of the 3^k regions defined in (4). We shall show that $\Phi^{\mathbf{w}}$ defined in Theorem 1 is a contraction map; there, hence, exists exactly one equilibrium lying in $\tilde{\Omega}^{\mathbf{w}}$. Assume that $\Phi^{\mathbf{w}}(\mathbf{y}) = \mathbf{y}^*$, $\Phi^{\mathbf{w}}(\mathbf{x}) = \mathbf{x}^*$, i.e., for each $i = 1, \ldots, n$

$$\begin{cases} -\mu_i y_i^* + (\alpha_{ii} + \beta_{ii})g_i(y_i^*) + \sum_{j=1,j\neq i}^{n}(\alpha_{ij} + \beta_{ij})g_j(y_j) \\ +I_i = 0 \\ -\mu_i x_i^* + (\alpha_{ii} + \beta_{ii})g_i(x_i^*) + \sum_{j=1,j\neq i}^{n}(\alpha_{ij} + \beta_{ij})g_j(x_j) \\ +I_i = 0. \end{cases}$$

Then, subtracting one equation from the other, we obtain

$$(x_i^* - y_i^*)[\mu_i - (\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*)] - \sum_{j=1,j\neq i}^{n} (\alpha_{ij} + \beta_{ij})g_j'(\eta_j^*)[x_j - y_j] = 0$$

where ξ_i^* is some number between x_i^* and y_i^*, and η_j^* is some number between x_j and y_j. Let us divide the discussion into four cases.

1) If $w_i = m$, where $i \in \mathcal{B}_3^3$, then $x_i^*, y_i^*, \xi_i^* \in [\hat{b}_i, \check{b}_i]$, and $g_i'(\xi_i^*) > (\mu_i + \theta_i)/(\alpha_{ii} + \beta_{ii})$ by (8). Hence

$$\begin{aligned} |x_i^* - y_i^*| &= \left| \sum_{j=1,j\neq i}^{n} (\alpha_{ij} + \beta_{ij})g_j'(\eta_j^*)(x_j - y_j) \right| \Big/ |(\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*) - \mu_i| \\ &\leq \left\{ \left[\sum_{j=1,j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}| \right] / \theta_i \right\} \|\mathbf{x} - \mathbf{y}\|_\infty \end{aligned}$$

where $[\sum_{j=1,j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}|]/\theta_i < 1$, owing to (7).

2) If $i \in \mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l$, then $x_i^*, y_i^*, \xi_i^* \in [\check{m}_i, \hat{m}_i]$. Thus, $0 \leq g_i'(\xi_i^*) < (\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii})$, due to (8). It follows that

$$|(\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*) - \mu_i| = \mu_i - (\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*) > \theta_i.$$

Subsequently

$$\begin{aligned} &|x_i^* - y_i^*| \\ &\leq \left\{ \left[\sum_{j=1,j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}| \right] \Big/ |(\alpha_{ii} + \beta_{ii})g_i'(\xi_i^*) - \mu_i| \right\} \\ &\cdot \|\mathbf{x} - \mathbf{y}\|_\infty \\ &< \left\{ \left[\sum_{j=1,j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}| \right] \Big/ \theta_i \right\} \|\mathbf{x} - \mathbf{y}\|_\infty \end{aligned}$$

where $[\sum_{j=1,j\neq i}^{n} \bar{L}_j |\alpha_{ij} + \beta_{ij}|]/\theta_i < 1$, owing to (7).

The situation for $w_i = r$ or l, $i \in \mathcal{B}_3^3$, is similar. Therefore, $\Phi^{\mathbf{w}}$ is a contraction mapping and there exists a unique fixed point $\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)$ of $\Phi^{\mathbf{w}}$, lying in $\tilde{\Omega}^{\mathbf{w}}$. Restated, for each $i = 1, \ldots, n$

$$-\mu_i\overline{x}_i + \alpha_{ii}g_i(\overline{x}_i) + \sum_{j=1, j\neq i}^{n} \alpha_{ij}g_j(\overline{x}_j) + \sum_{j=1}^{n}\beta_{ij}g_j(\overline{x}_j) + J_i = 0. \quad (9)$$

Thus, $\overline{\mathbf{x}}$ is the unique equilibrium point of (1) lying in $\tilde{\Omega}^{\mathbf{w}}$. On the other hand, if $\overline{\mathbf{x}}$ is an equilibrium point of (1), then its components satisfy (2), and thus must lie in some $\tilde{\Omega}^{\mathbf{w}}$. System (1), therefore, admits exactly 3^k equilibria. ■

Notably, (8) describes how the slope of g_i is smaller or larger than $\mu_i/(\alpha_{ii} + \beta_{ii})$ on the indicated regions with the help of θ_i. In fact, (7) prefers smaller $|\alpha_{ij} + \beta_{ij}|$, $i \neq j$. Moreover, (7) favors larger $\theta_i \in (0, \mu_i)$, since each $\bar{L}_j$ in the right-hand side of (7) is lowered if θ_j is larger, as seen from the definition of $\bar{L}_j$.

The existence of equilibrium for (1) in Theorem 1 is the most clear-cut situation, as for each i, the intersection scenario with the horizontal axis for the upper function $\hat{f}_i$ is identical to the one for the lower function $\check{f}_i$. We shall pursue more complicated cases where the upper and lower functions have disparate intersection configurations with the horizontal axis in Section IV.

III. Global Convergence of Dynamics in (1)

In this section, we shall discuss the globally convergent dynamics for (1), i.e., every solution of (1) converges to one of the equilibria. This implicates that almost every orbit of (1) converges to one of the stable equilibria. More precisely, the orbits, which do not lie on the stable manifolds of the unstable equilibria, will converge to one of the stable equilibria. To conclude globally convergent dynamics, we shall apply the newly developed sequential contracting scheme. We describe this scheme and formulation for (1) in Section III-A. We then implement the scheme to the case of 3^k equilibria for the n-neuron system in Section III-B.

A. Sequential Contracting

An upper function and a lower function for each component of the equations were formulated to locate the equilibria in Section II. To study the dynamics, such a formulation is insufficient and it is tempting to design a sequence of upper and lower dynamics to capture the asymptotic behaviors of the solutions. This gives rise to the sequential contracting scheme. The idea is actually quite natural. We organize the coupling terms in (1) so that the dynamics corresponding to each component equation can be controlled by some scalar equations. We start by constructing a preliminary upper equation and a lower equation for each component. Such upper and lower dynamics can usually be constructed due to the dissipative property of the coupled systems. The upper and lower equations are designed to have their own dynamics contracted to some compact intervals. Each component for the original dynamics of (1) is then trapped in the interval after certain time. We then construct finer upper and lower dynamics, so that the original dynamics are attracted to even more concentrated regions after a later time. We then formulate a criterion for contraction under, which these nested intervals collapse into points, as the iterative constructions of upper and lower dynamics carry on continuously.

Notably, (1) is dissipative, as observed from the equations that the summation terms in (1) are bounded. Such a property was formally justified in [18]. Therefore, a solution $\mathbf{x}(t) = \mathbf{x}(t; t_0, \phi)$ of (1), starting from arbitrary $\phi \in C([-\tau_M, 0], \mathbb{R}^n)$ at $t = t_0$, exists on $[t_0, \infty)$. In the following discussion, we fix an initial condition ϕ. Its evolution $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ is then a fixed function defined on $[t_0, \infty)$ and the ith component $x_i(t)$ satisfies

$$\dot{x}_i(t) = -\mu_i x_i(t) + \alpha_{ii}g_i(x_i(t)) + \beta_{ii}g_i(x_i(t - \tau_{ii})) + w_i \quad (10)$$

for all $t \geq t_0$, where $w_i := \sum_{j\neq i}\{\alpha_{ij}g_j(x_j(t)) + \beta_{ij}g_j(x_j(t-\tau_{ij}))\} + I_i$. Herein, each w_i, $i \in \mathcal{N}$, is a function of t, which is defined from the solution $\mathbf{x}(t)$; i.e., w_i varies with respect to solutions. For a fixed solution $\mathbf{x}(t)$, w_i is indeed a function of t, i.e., $w_i = w_i(t)$, and (10) is like a scalar equation for each i.

For later use, we define for each $i \in \mathcal{N}$

$$w_i^{\max}(\infty) := \lim_{T\to\infty} w_i^{\max}(T), w_i^{\min}(\infty) := \lim_{T\to\infty} w_i^{\min}(T)$$

where $w_i^{\max}(T) := \sup\{w_i(t) \mid t \geq T\}$ is nonincreasing in T and $w_i^{\min}(T) := \inf\{w_i(t) \mid t \geq T\}$ is nondecreasing in T. We define the following preliminary upper and lower bounds for (10), respectively:

$$\hat{h}_i(\xi) := -\mu_i\xi + 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) \quad (11)$$

$$\check{h}_i(\xi) := -\mu_i\xi - 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\min}(t_0) \quad (12)$$

where $\hat{h}_i$ and $\check{h}_i$ are linear decreasing functions with unique zeros: $\hat{A}_i^h$ and $\check{A}_i^h$, where $\hat{A}_i^h := [2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0)]/\mu_i$ and $\check{A}_i^h := [-2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\min}(t_0)]/\mu_i$, respectively. Notably, $\hat{h}_i(\check{A}_i^h) = -\check{h}_i(\hat{A}_i^h) = 4(|\alpha_{ii}|+|\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0) \geq 0$. Applying the arguments similar to those for [33, Lemma 2.1] reveals that for each $i \in \mathcal{N}$

$$\check{h}_i(x_i(t)) + (|\alpha_{ii}| + |\beta_{ii}|)\rho_i \leq \dot{x}_i(t) \leq \hat{h}_i(x_i(t)) - (|\alpha_{ii}| + |\beta_{ii}|)\rho_i$$

for all $t \geq t_0$. Consequently, there exists a t_ϕ (depending on ϕ) such that $x_i(t)$ enters and remains in interval $[\check{A}_i^h, \hat{A}_i^h]$ for all i and $t \geq t_\phi$. Accordingly, we can construct the second preliminary upper and lower bounds for (10)

$$\hat{f}_i^{(0)}(\xi, T) := \begin{cases} \hat{\gamma}_i(\xi, T) - \beta_{ii}L_i\tau_{ii}\check{h}_i\left(\hat{A}_i^h\right) & \text{if } \beta_{ii} \geq 0 \\ \hat{\gamma}_i(\xi, T) - \beta_{ii}L_i\tau_{ii}\hat{h}_i\left(\check{A}_i^h\right) & \text{if } \beta_{ii} < 0 \end{cases} \quad (13)$$

$$\check{f}_i^{(0)}(\xi, T) := \begin{cases} \check{\gamma}_i(\xi, T) - \beta_{ii}L_i\tau_{ii}\hat{h}_i\left(\check{A}_i^h\right) & \text{if } \beta_{ii} \geq 0 \\ \check{\gamma}_i(\xi, T) - \beta_{ii}L_i\tau_{ii}\check{h}_i\left(\hat{A}_i^h\right) & \text{if } \beta_{ii} < 0 \end{cases} \quad (14)$$

where

$$\hat{\gamma}_i(\xi, T) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + w_i^{\max}(T)$$
$$\check{\gamma}_i(\xi, T) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + w_i^{\min}(T)$$

for some $T \geq t_0$. It is not difficult to see that condition (M1) (in the Appendix) implies $|\alpha_{ii}| + |\beta_{ii}| > 0$, and thus

$$\begin{aligned}\check{h}_i(\xi) &< \check{f}_i^{(0)}(\xi, t_0) \leq \check{f}_i^{(0)}(\xi, T)\\ &\leq \hat{f}_i^{(0)}(\xi, T) \leq \hat{f}_i^{(0)}(\xi, t_0) < \hat{h}_i(\xi)\end{aligned} \tag{15}$$

for all $T \geq t_0$ and $\xi \in \mathbb{R}$. From (15), conditions (M1) and (M2) (in the Appendix) imply that there exists a unique zero $\check{m}_i^{(0)}(T)$ [resp., $\hat{m}_i^{(0)}(T)$] of $\check{f}_i^{(0)}(\cdot, T) = 0$ [resp., $\hat{f}_i^{(0)}(\cdot, T) = 0$] for each $T \geq T_0$, where T_0 is defined in condition (M2). Moreover

$$\check{A}_i^h < \check{m}_i^{(0)}(T_0) \leq \check{m}_i^{(0)}(T) \leq \hat{m}_i^{(0)}(T) \leq \hat{m}_i^{(0)}(T_0) < \hat{A}_i^h.$$

By arguments similar to those for [33, Lemma 2.2], we can show that for any $T \geq \max\{t_\phi + \tau_{ii}, T_0\}$

$$\begin{aligned}&\check{f}_i^{(0)}(x_i(t), T) + |\beta_{ii}|(|\alpha_{ii}| + |\beta_{ii}|)\rho_i L_i \tau_{ii}\\ &\quad\leq \dot{x}_i(t) \leq \hat{f}_i^{(0)}(x_i(t), T) - |\beta_{ii}|(|\alpha_{ii}| + |\beta_{ii}|)\rho_i L_i \tau_{ii}\end{aligned}$$

for $t \geq T$. Consequently, $x_i(t)$ enters and remains in interval $[\check{m}_i^{(0)}(T), \hat{m}_i^{(0)}(T)]$ contained in $[\check{A}_i^h, \hat{A}_i^h]$ after certain time. Then, iteratively applying arguments based on constructing finer upper $\hat{f}_i^{(k)}$ and lower bounds $\check{f}_i^{(k)}$ for (10) allows us to establish the convergence of $x_i(t)$ to some compact interval, say J_i, as $t \to \infty$. The formulation of such $\hat{f}_i^{(k)}$ and $\check{f}_i^{(k)}$ and the precise statement of this convergence (Proposition 2) are arranged in the Appendix. Herein, we say that a real-valued function $y(t)$ converges to a compact interval J if $d(y(t), J) := \inf\{|y(t) - \zeta| : \zeta \in J\} \to 0$, as $t \to \infty$. On the other hand, another possible configuration of upper and lower functions leads to the convergence of $x_i(t)$ to one of three compact intervals, which is summarized in Proposition 3 (in the Appendix). As performed component by component successively, this sequential contracting scheme converts the convergence of solution $\mathbf{x}(t)$ to an equilibrium of (1) into solving a corresponding linear system of algebraic equations via the following proposition.

Proposition 1: Let $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ be a fixed solution of (1). Assume that for every $i \in \mathcal{N}$, there exists a compact interval J_i of length d_i, such that $x_i(t)$ converges to J_i and d_i satisfies

$$d_i \leq \left[w_i^{\max}(\infty) - w_i^{\min}(\infty)\right]\big/\eta_i$$

for some $\eta_i > 0$, and there exist a compact interval $\tilde{J}_i$ and a $\tilde{L}_i \geq 0$, such that $J_i \subseteq \tilde{J}_i$ and

$$g_i'(\xi) \leq \tilde{L}_i \text{ for all } \xi \in \tilde{J}_i.$$

Let $\mathbf{M} := [m_{ij}]_{1\leq i,j\leq n}$ with $m_{ii} := \eta_i$, $m_{ij} := -(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j$ for $i \neq j$. If the Gauss–Seidel iteration for solving the linear system

$$\mathbf{Mv} = \mathbf{0} \tag{16}$$

converges to zero, the unique solution of (16), or equivalently, $\tilde{\lambda}_M < 1$, then every d_i degenerates into zero, and the solution $\mathbf{x}(t)$ of (1) converges to a singleton, where

$$\tilde{\lambda}_M := \max_{1\leq i\leq n}\{|\lambda_i| : \lambda_i : \text{ eigenvalue of } (D_\mathbf{M} - L_\mathbf{M})^{-1}U_\mathbf{M}\}$$

and $\mathbf{M} = D_\mathbf{M} - L_\mathbf{M} - U_\mathbf{M}$ with $D_\mathbf{M}$, $-L_\mathbf{M}$, and $-U_\mathbf{M}$ the diagonal, strictly lower triangular and strictly upper triangular parts of $\mathbf{M}$, respectively.

Proof: We first claim that for each $i \in \mathcal{N}$, there exists a sequence of compact intervals $\{J_i^{(k)}\}_{k=0}^{\infty}$ with $J_i^{(k)} \supseteq J_i$, and the length $d_i^{(k)}$ of $J_i^{(k)}$ satisfies

$$d_i^{(k)} = \left\{\sum_{j=1}^{i-1}(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j d_j^{(k)} + \sum_{j=i+1}^{n}(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j d_j^{(k-1)}\right\}\Bigg/\eta_i \tag{17}$$

for every $k \in \mathbb{N}$, where $J_i^{(0)} = \tilde{J}_i$. Let us sketch the arguments for the claim through induction. Assume that the claim holds for $k = \tilde{k} - 1 \in \mathbb{N}$ and all $i \in \mathcal{N}$ for some $\tilde{k} \geq 2$, and it holds for $k = \tilde{k}$ and every $i \in \{1, \ldots, \ell - 1\}$ with $1 \leq \ell - 1 < n$. That is, $x_i(t)$ converges to the compact interval $J_i^{(\tilde{k})}$ (resp., $J_i^{(\tilde{k}-1)}$) if $i = 1, \ldots, \ell-1$ (resp., $i = \ell+1, \ldots, n$). Set

$$\begin{aligned}\check{W}_\ell^{(\tilde{k})}(\infty) &:= I_\ell + \sum_{j=1}^{\ell-1}\min_{\xi,\eta\in J_j^{(\tilde{k})}\cap\tilde{J}_j}\{\alpha_{\ell j}g_j(\xi) + \beta_{\ell j}g_j(\eta)\}\\ &\quad+ \sum_{j=\ell+1}^{n}\min_{\xi,\eta\in J_j^{(\tilde{k}-1)}\cap\tilde{J}_j}\{\alpha_{\ell j}g_j(\xi) + \beta_{\ell j}g_j(\eta)\}\\ \hat{W}_\ell^{(\tilde{k})}(\infty) &:= I_\ell + \sum_{j=1}^{\ell-1}\max_{\xi,\eta\in J_j^{(\tilde{k})}\cap\tilde{J}_j}\{\alpha_{\ell j}g_j(\xi) + \beta_{\ell j}g_j(\eta)\}\\ &\quad+ \sum_{j=\ell+1}^{n}\max_{\xi,\eta\in J_j^{(\tilde{k}-1)}\cap\tilde{J}_j}\{\alpha_{\ell j}g_j(\xi) + \beta_{\ell j}g_j(\eta)\}.\end{aligned}$$

In respecting the term w_ℓ associated with the solution $(x_1(t), \ldots, x_n(t))$ defined in (10), and the definition of $w_\ell^{\min}(\infty)$ and $w_\ell^{\max}(\infty)$, we obtain

$$\check{W}_\ell^{(\tilde{k})}(\infty) \leq w_\ell^{\min}(\infty) \leq w_\ell^{\max}(\infty) \leq \hat{W}_\ell^{(\tilde{k})}(\infty)$$

noting that $x_i(t)$ converges to $J_i^{(\tilde{k})} \cap \tilde{J}_i$ (resp., $J_i^{(\tilde{k}-1)} \cap \tilde{J}_i$) if $i = 1, \ldots, \ell - 1$ (resp., $i = \ell + 1, \ldots, n$). Thus, $x_\ell(t)$ converges to J_ℓ with its length d_ℓ satisfying

$$\begin{aligned}d_\ell &\leq \left[w_\ell^{\max}(\infty) - w_\ell^{\min}(\infty)\right]\big/\eta_\ell\\ &\leq \left[\hat{W}_\ell^{(\tilde{k})}(\infty) - \check{W}_\ell^{(\tilde{k})}(\infty)\right]\big/\eta_\ell\\ &\leq \left\{\sum_{j=1}^{\ell-1}(|\alpha_{\ell j}| + |\beta_{\ell j}|)\tilde{L}_j d_j^{(\tilde{k})} + \sum_{j=\ell+1}^{n}(|\alpha_{\ell j}| + |\beta_{\ell j}|)\tilde{L}_j d_j^{(\tilde{k}-1)}\right\}\Bigg/\eta_\ell = d_\ell^{(\tilde{k})}\end{aligned}$$

where the last inequality follows from applying the mean valve theorem to $\hat{W}_\ell^{(\tilde{k})}(\infty) - \check{W}_\ell^{(\tilde{k})}(\infty)$, recalling $g_i'(\xi) \le \tilde{L}_i$ if $\xi \in J_i^{(k)} \cap \tilde{J}_i$ and $J_i^{(k)} \cap \tilde{J}_i \subseteq \tilde{J}_i$, for all $i \in \mathcal{N}$ and $k \in \mathbb{N}$. These computations also explain the formulation of $d_i^{(k)}$ in (17). Subsequently, $x_\ell(t)$ converges to a compact interval $J_\ell^{(\tilde{k})}$ that contains J_ℓ and whose length is exactly $d_\ell^{(\tilde{k})}$. That is, the assertion holds for $k = \tilde{k}$ and $i = \ell$ as well. We thus justify the claim.

We observe that $\{(d_1^{(k)}, \ldots, d_n^{(k)})\}_{k\in\mathbb{N}}$ is exactly the Gauss–Seidel iteration for solving the linear system (16). From the previous arguments, it follows that $(x_1(t), \ldots, x_n(t))$ converges to a singleton if the iteration $\{(d_1^{(k)}, \ldots, d_n^{(k)})\}_{k\in\mathbb{N}}$ satisfies $d_i^{(k)} \to 0$ as $k \to \infty$, for all $i \in \mathcal{N}$. In addition, $\tilde{\lambda}_M < 1$ is a sufficient and necessary condition for such a convergence of the Gauss–Seidel iteration for (16) [15]. ■

The convergence to zero for Gauss–Seidel iteration in Proposition 1 can be assured if $\mathbf{M}$ is strictly diagonal dominant [37].

Lemma 2: Under the strictly diagonal dominance of $\mathbf{M}$: $\eta_i > \sum_{j\neq i}\{(|\alpha_{ij}| + |\beta_{ij}|)\tilde{L}_j\}$ for all $i \in \mathcal{N}$, the Gauss–Seidel iteration converges to zero, the unique solution of (16).

B. Global Convergence to 3^k Equilibria for (1)

Next, we investigate the convergence of dynamics in (1) with 3^k equilibria. Notice that the formulation of $\check{h}_i$, $\hat{h}_i$ in (11) and (12), and $\hat{f}_i^{(0)}(\cdot, T)$, $\check{f}_i^{(0)}(\cdot, T)$ in (13) and (14), depends on a given solution $\mathbf{x}(t)$. To formulate the convergence condition for all solutions of (1), we need to employ the following solution-independent upper and lower functions:

$$\hat{F}_i(\xi) = \hat{\Gamma}_i(\xi) + \tilde{k}_i^+, \;\; \check{F}_i(\xi) = \check{\Gamma}_i(\xi) + \tilde{k}_i^- \tag{18}$$

where $\tilde{k}_i^+ := \sum_{j=1, j\neq i}^n (|\alpha_{ij}| + |\beta_{ij}|)\rho_j + I_i$, $\tilde{k}_i^- := -\sum_{j=1, j\neq i}^n (|\alpha_{ij}| + |\beta_{ij}|)\rho_j + I_i$, and

$$\hat{\Gamma}_i(\xi) := \begin{cases} -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\check{H}_i(\hat{A}_i^H) & \text{if } \beta_{ii} \ge 0 \\ -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\hat{H}_i(\check{A}_i^H) & \text{if } \beta_{ii} < 0 \end{cases}$$

$$\check{\Gamma}_i(\xi) := \begin{cases} -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\hat{H}_i(\check{A}_i^H) & \text{if } \beta_{ii} \ge 0 \\ -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) - \beta_{ii}L_i\tau_{ii}\check{H}_i(\hat{A}_i^H) & \text{if } \beta_{ii} < 0 \end{cases}$$

and $\hat{A}_i^H$ and $\check{A}_i^H$ are, respectively, the unique zeros of the linear decreasing functions $\hat{H}_i$ and $\check{H}_i$ defined as

$$\hat{H}_i(\xi) := -\mu_i\xi + 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + \tilde{k}_i^+$$
$$\check{H}_i(\xi) := -\mu_i\xi - 2(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + \tilde{k}_i^-.$$

$\hat{F}_i$ and $\check{F}_i$ are also vertical shifts of $\hat{f}_i$ and $\check{f}_i$ and $\hat{f}_i^{(0)}(\cdot, T)$, $\check{f}_i^{(0)}(\cdot, T)$ defined in Sections II and III-A, respectively. It is not difficult to verify

$$\check{H}_i(\xi) \le \check{h}_i(\xi) \le \hat{h}_i(\xi) \le \hat{H}_i(\xi) \tag{19}$$

$$\check{F}_i(\xi) \le \check{f}_i^{(0)}(\xi, T) \le \hat{f}_i^{(0)}(\xi, T) \le \hat{F}_i(\xi) \tag{20}$$

for all $T \ge t_0$ and $\xi \in \mathbb{R}$. Moreover

$$\check{F}_i(\xi) \le \check{f}_i(\xi) \le \hat{f}_i(\xi) \le \hat{F}_i(\xi) \tag{21}$$

for all $\xi \in \mathbb{R}$, as $\hat{H}_i(\check{A}_i^H) \ge 0$ and $\check{H}_i(\hat{A}_i^H) \le 0$.

When $\mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l \cup \mathcal{B}_3^3 = \mathcal{N}$ holds, we further consider the condition

$$\begin{cases} \check{F}_i(\tilde{p}_i) > 0 & \text{if } i \in \mathcal{B}_r^r \\ \hat{F}_i(\tilde{q}_i) < 0 & \text{if } i \in \mathcal{B}_l^l \\ \hat{F}_i(\tilde{p}_i) < 0, \;\; \check{F}_i(\tilde{q}_i) > 0 & \text{if } i \in \mathcal{B}_3^3 \end{cases} \tag{22}$$

where $\tilde{p}_i, \tilde{q}_i$ are defined in Lemma 1. Under (22), there exists a unique zero $\check{m}_i^F$ (resp., $\hat{m}_i^F$) to $\check{F}_i$ (resp., $\hat{F}_i$), if $i \in \mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l$, and there exist exactly three zeros $\hat{a}_i^F$, $\hat{b}_i^F$, $\hat{c}_i^F$ (resp., $\check{a}_i^F$, $\check{b}_i^F$, $\check{c}_i^F$) to $\hat{F}_i$ (resp., $\check{F}_i$), if $i \in \mathcal{B}_3^3$, and $\check{a}_i^F \le \hat{a}_i^F < \tilde{p}_i < \hat{b}_i^F \le \check{b}_i^F < \tilde{q}_i < \check{c}_i^F \le \hat{c}_i^F$. Let

$$\tau_{ii}^c := \frac{(|\alpha_{ii}| + |\beta_{ii}|)\rho_i}{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + 2\sum_{j=1, j\neq i}^n (|\alpha_{ij}| + |\beta_{ij}|)\rho_j]}.$$

To conclude the stability of 2^k out of these 3^k equilibria, we further need the following functions:

$$\hat{I}_i(\xi) := -\mu_i\xi + \alpha_{ii}g_i(\xi) + \sum_{j=1, j\neq i}^n |\alpha_{ij}|\rho_j + \sum_{j=1}^n |\beta_{ij}|\rho_j + I_i$$

$$\check{I}_i(\xi) := -\mu_i\xi + \alpha_{ii}g_i(\xi) - \sum_{j=1, \neq i}^n |\alpha_{ij}|\rho_j - \sum_{j=1}^n |\beta_{ij}|\rho_j + I_i$$

for $i \in \mathcal{N}$. Notably

$$\check{I}_i(\xi) \le \check{f}_i(\xi) < \hat{f}_i(\xi) \le \hat{I}_i(\xi) \tag{23}$$

for all $\xi \in \mathbb{R}$. Moreover, if $L_i > \mu_i/\alpha_{ii} > 0$, there exist exactly two points $\bar{p}_i$ and $\bar{q}_i$ with $\bar{p}_i < \sigma_i < \bar{q}_i$ such that $g_i'(\bar{p}_i) = g_i'(\bar{q}_i) = \mu_i/\alpha_{ii}$. If in addition that $\check{I}_i(\bar{q}_i) > 0$ and $\hat{I}_i(\bar{p}_i) < 0$, then there exist exactly three zeros $\hat{a}_i^I$, $\hat{b}_i^I$, and $\hat{c}_i^I$ (resp., $\check{a}_i^I$, $\check{b}_i^I$, and $\check{c}_i^I$) for $\hat{I}_i$ (resp., $\check{I}_i$), where $\check{a}_i^I \le \hat{a}_i^I < \bar{p}_i < \hat{b}_i^I \le \check{b}_i^I < \bar{q}_i < \check{c}_i^I \le \hat{c}_i^I$.

We assume $\mathcal{B}_3^3 \neq \emptyset$ and consider the following condition.

Condition (I): $L_i > \mu_i/\alpha_{ii} > 0$, $\check{I}_i(\bar{q}_i) > 0$, and $\hat{I}_i(\bar{p}_i) < 0$, for $i \in \mathcal{B}_3^3$.

Assuming $\mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l \cup \mathcal{B}_3^3 = \mathcal{N}$ with $\text{card}(\mathcal{B}_3^3) = k \ge 1$, we define the following 2^k regions:

$$\begin{aligned} \bar{\mathbf{\Omega}}^{\mathbf{w}} &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \bar{\mathbf{\Omega}}_i^{w_i}\} \\ \mathbf{w} &= (w_1, \ldots, w_n) \\ w_i &= \text{“}l\text{”}, \text{“}r\text{”}, \;\text{for } i \in \mathcal{B}_3^3 \\ w_i &= \text{“}s\text{”}, \;\text{for } i \in \mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l \end{aligned} \tag{24}$$

where $\bar{\mathbf{\Omega}}_i^l = (-\infty, \hat{b}_i^I)$, $\bar{\mathbf{\Omega}}_i^r = (\check{b}_i^I, \infty)$, and $\bar{\mathbf{\Omega}}_i^s = \mathbb{R}$, under condition (I). Notably, $\tilde{\mathbf{\Omega}}^{\mathbf{w}} \subseteq \bar{\mathbf{\Omega}}^{\mathbf{w}}$ and $\tilde{\mathbf{\Omega}}^{\mathbf{w}} \cap \bar{\mathbf{\Omega}}^{\mathbf{w}'} = \emptyset$ if $\mathbf{w} \neq \mathbf{w}'$ due to (23). Denote by $\bar{\mathbf{x}}_{\mathbf{w}}$ the equilibrium lying in $\bar{\mathbf{\Omega}}^{\mathbf{w}}$. We can then establish the global convergence of dynamics for (1) and show that each equilibrium $\bar{\mathbf{x}}_{\mathbf{w}}$ is attracting in the sense that every solution evolved from initial value in $\bar{\mathbf{\Omega}}^{\mathbf{w}}$ converges to $\bar{\mathbf{x}}_{\mathbf{w}}$.

Theorem 3: Assume that $\mathcal{M} \cup \mathcal{B}_r^r \cup \mathcal{B}_l^l \cup \mathcal{B}_3^3 = \mathcal{N}$, (7) and (22) hold, and for each $i \in \mathcal{N}$

$$|\beta_{ii}|\tau_{ii} < \tau_{ii}^c \tag{25}$$

and

$$g_i'(\xi)\begin{cases} < \frac{\mu_i-\theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } \xi\in\left[\check{m}_i^F,\hat{m}_i^F\right],\ i\in\mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l \\ < \frac{\mu_i-\theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } \xi\in\left(-\infty,\hat{a}_i^F\right]\cup\left[\check{c}_i^F,\infty\right),\ i\in\mathcal{B}_3^3 \\ > \frac{\mu_i+\theta_i}{\alpha_{ii}+\beta_{ii}}, & \text{if } \xi\in\left[\hat{b}_i^F,\check{b}_i^F\right],\ i\in\mathcal{B}_3^3 \end{cases} \tag{26}$$

for some $\theta_i\in(0,\mu_i)$. Then, (1) achieves global convergence to the 3^k equilibria provided that the Gauss–Seidel iteration for the linear algebraic system (16) converges to zero, the unique solution, where $m_{ii}=(1-2|\beta_{ii}|L_i\tau_{ii})\theta_i$ for $i\in\mathcal{N}$, $m_{ij}=-(|\alpha_{ij}|+|\beta_{ij}|)\bar{L}_j$ for $i,j\in\mathcal{N}$, $i\neq j$, and $\bar{L}_j$ is defined in (6). If condition (I) holds in addition, then the 2^k out of these 3^k equilibria are attracting; more precisely, every solution $\mathbf{x}(t)$ evolved from an initial value in $\bar{\Omega}^{\mathbf{w}}$ converges to $\bar{\mathbf{x}}_{\mathbf{w}}$, as $t\to\infty$, where $\bar{\Omega}^{\mathbf{w}}$ is defined in (24).

Proof: Based on (21), it can be verified that (26) implies (8). Consequently, (1) admits exactly 3^k equilibria under the conditions imposed by Theorem 2. Next, let us prove the convergence of dynamics. Let $(x_1(t),\ldots,x_n(t))$ be an arbitrary solution of (1). Then, each component $\dot{x}_i(t)$ can be written in the form (10). If $i\in\mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l$, then there exists a unique zero $\check{m}_i^f$ (resp., $\hat{m}_i^f$) to $\check{f}_i^{(0)}(\cdot,t_0)$ [resp., $\hat{f}_i^{(0)}(\cdot,t_0)$] where $[\check{m}_i^f,\hat{m}_i^f]\subseteq[\check{m}_i^F,\hat{m}_i^F]$, by (20) and the first two inequalities in (22). If $i\in\mathcal{B}_3^3$, then there exist exactly three zeros $\hat{a}_i^f$, $\hat{b}_i^f$, and $\hat{c}_i^f$ (resp., $\check{a}_i^f$, $\check{b}_i^f$, and $\check{c}_i^f$) to $\hat{f}_i^{(0)}(\cdot,t_0)$ [resp., $\check{f}_i^{(0)}(\cdot,t_0)$] by the last inequality in (22); in addition, $[\check{a}_i^f,\hat{a}_i^f]\subseteq[\check{a}_i^F,\hat{a}_i^F]$, $[\hat{b}_i^f,\check{b}_i^f]\subseteq[\hat{b}_i^F,\check{b}_i^F]$, and $[\check{c}_i^f,\hat{c}_i^f]\subseteq[\check{c}_i^F,\hat{c}_i^F]$. Then, it is not difficult to verify that component $x_i(t)$ satisfies (M1)–(M3) if $i\in\mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l$, and satisfies (B1)–(B3) for $i\in\mathcal{B}_3^3$, under (22), (25), and (26) (see the Appendix). By Propositions 2 and 3, we obtain the convergence of $x_i(t)$ to an interval J_i of length d_i for every $i\in\mathcal{N}$, where d_i satisfies

$$d_i\leq\left[w_i^{\max}(\infty)-w_i^{\min}(\infty)\right]/[(1-2|\beta_{ii}|L_i\tau_{ii})\theta_i]. \tag{27}$$

Notably, $J_i\subseteq[\check{m}_i^F,\hat{m}_i^F]$ if $i\in\mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l$, and $J_i\subseteq[\check{a}_i^F,\hat{a}_i^F]\cup[\hat{b}_i^F,\check{b}_i^F]\cup[\check{c}_i^F,\hat{c}_i^F]\subseteq[\check{a}_j^F,\hat{c}_j^F]$ if $i\in\mathcal{B}_3^3$. If $i\in\mathcal{B}_3^3$, which of the intervals $[\check{a}_i^F,\hat{a}_i^F]$, $[\hat{b}_i^F,\check{b}_i^F]$, $[\check{c}_i^F,\hat{c}_i^F]$ J_i is actually contained in depends on the initial condition; the detailed arguments are similar to [33, Proposition 4]. Moreover, $g_j'(\xi)\leq\bar{L}_j$, where

$$\bar{L}_j=\begin{cases}\frac{\mu_j-\theta_j}{\alpha_{jj}+\beta_{jj}} & \text{if } j\in\mathcal{M}\cup\mathcal{B}_r^r\cup\mathcal{B}_l^l,\ \xi\in\left[\check{m}_j^F,\hat{m}_j^F\right]\\ L_j & \text{if } j\in\mathcal{B}_3^3,\ \xi\in\left[\check{a}_j^F,\hat{c}_j^F\right].\end{cases}$$

By Proposition 1, it follows that $(x_1(t),\ldots,x_n(t))$ converges to a singleton if the Gauss–Seidel iteration for (16) converges to zero, the unique solution of (16), where $m_{ii}=(1-2|\beta_{ii}|L_i\tau_{ii})\theta_i$ if $i\in\mathcal{N}$ and $m_{ij}=-(|\alpha_{ij}|+|\beta_{ij}|)\bar{L}_j$ if $i,j\in\mathcal{N}$ and $i\neq j$.

Below, let us show that $\bar{\Omega}^{\mathbf{w}}$ is positively invariant under the solution flow of (1). Let $(x_1(t),\ldots,x_n(t))$ be a solution evolved from an initial condition lying $\bar{\Omega}^{\mathbf{w}}$. From (1), we obtain

$$\check{I}_i(x_i(t))\leq\dot{x}_i(t)\leq\hat{I}_i(x_i(t)). \tag{28}$$

From (28) and the configurations of $\check{I}_i$ and $\hat{I}_i$, for $i\in\mathcal{B}_3^3$, we see that $\dot{x}_i(t)<0$, should $x_i(t)$ stay in $(\hat{a}_i^I,\hat{b}_i^I)\cup(\hat{c}_i^I,\infty)$, and $\dot{x}_i(t)>0$, should $x_i(t)$ stay in $(-\infty,\check{a}_i^I)\cup(\check{b}_i^I,\check{c}_i^I)$ under condition (I). Consequently, for $i\in\mathcal{B}_3^3$, $x_i(t)$ remains in $(-\infty,\hat{b}_i^I)$ [resp., $(\check{b}_i^I,\infty)$] and actually converges to $[\check{a}_i^I,\hat{a}_i^I]$ (resp., $[\check{c}_i^I,\hat{c}_i^I]$) if $w_i=l$ (resp., r). We hence justify the positive invariance of $\bar{\Omega}^{\mathbf{w}}$. Subsequently, every solution evolved from an initial value in $\bar{\Omega}^{\mathbf{w}}$ converges to $\bar{\mathbf{x}}_{\mathbf{w}}$, due to the global convergence of (1). ■

Corollary 1: If $\mathbf{M}$ is strictly diagonal-dominant, i.e., $(1-2|\beta_{ii}|L_i\tau_{ii})\theta_i>\sum_{j\neq i}\{(|\alpha_{ij}|+|\beta_{ij}|)\bar{L}_j\}$, for all $i\in\mathcal{N}$, then the Gauss–Seidel iteration converges to zero, and the assertions of Theorem 3 hold under the same assumptions.

Remark 1: 1) In the proof of Theorem 3, it was shown that every solution converges to a single point as $t\to\infty$. This point is certainly an equilibrium of (1). 2) Theorem 3 concludes that the 2^k equilibria out of these 3^k equilibria are attracting in (1). By applying the arguments in [30, Th. 3.4], we can further prove that these 2^k equilibria are asymptotically stable and the other (3^k-2^k) equilibria are unstable without additional assumptions.

We stress that to establish analytic theory to conclude the global dynamics for (1), several additional conditions have been imposed in Theorem 3. These are certainly sufficient conditions that are formulated according to the mathematical methodologies. It is likely that the convergent dynamics already holds under the assumption of Theorem 1.

IV. Other Cases of Multistability

In this section, we elaborate on the other cases of multiple equilibria in (1). These cases are more complicated than the one in Section II. We illustrate the idea for $n=2$.

System (1) with $n=2$ reads as

$$\frac{dx_1(t)}{dt}=-\mu_1x_1(t)+\alpha_{11}g_1(x_1(t))+\alpha_{12}g_2(x_2(t))+\beta_{11}g_1(x_1(t-\tau_{11}))+\beta_{12}g_2(x_2(t-\tau_{12}))+I_1 \tag{29}$$

$$\frac{dx_2(t)}{dt}=-\mu_2x_2(t)+\alpha_{21}g_1(x_1(t))+\alpha_{22}g_2(x_2(t))+\beta_{21}g_1(x_1(t-\tau_{21}))+\beta_{22}g_2(x_2(t-\tau_{22}))+I_2. \tag{30}$$

The upper and lower functions are now

$$\hat{f}_i(\xi)=-\mu_i\xi+(\alpha_{ii}+\beta_{ii})g_i(\xi)+|\alpha_{ij}+\beta_{ij}|\rho_j+I_i$$
$$\check{f}_i(\xi)=-\mu_i\xi+(\alpha_{ii}+\beta_{ii})g_i(\xi)-|\alpha_{ij}+\beta_{ij}|\rho_j+I_i$$

where $i,j\in\{1,2\}$ and $j\neq i$. For this two-neuron system, there are four basic types: 1) $(\mathcal{M},\mathcal{M})$; 2) $(\mathcal{M},\mathcal{B})$; 3) $(\mathcal{B},\mathcal{M})$; and 4) $(\mathcal{B},\mathcal{B})$, which correspond to $1,2\in\mathcal{M}$, $1\in\mathcal{M}$ and $2\in\mathcal{B}$, $1\in\mathcal{B}$ and $2\in\mathcal{M}$, and $1,2\in\mathcal{B}$, respectively, according to the notation in Section II. We further denote the following subtypes of these four types.

Notation 1: Denote subtype $\binom{m}{m}\binom{m}{m}$ if $1,2\in\mathcal{M}$, $\binom{m}{m}\binom{\star}{\bullet}$ if $1\in\mathcal{M}$ and $2\in\mathcal{B}_\bullet^\star$, $\binom{\star}{\bullet}\binom{m}{m}$ if $1\in\mathcal{B}_\bullet^\star$ and $2\in\mathcal{M}$, and $\binom{\star}{\bullet}\binom{\star'}{\bullet'}$ if $1\in\mathcal{B}_\bullet^\star$ and $2\in\mathcal{B}_{\bullet'}^{\star'}$.

TABLE I

SUBTYPES IN $(\mathcal{M},\mathcal{M})$, $(\mathcal{M},\mathcal{B})$, $(\mathcal{B},\mathcal{M})$, AND $(\mathcal{B},\mathcal{B})$

Type	Subtype	Cases
$(\mathcal{M},\mathcal{M})$	T_1	$\binom{m}{m}\binom{m}{m}$
$(\mathcal{M},\mathcal{B})$	T_2	$\binom{m}{m}\binom{r}{r}, \binom{m}{m}\binom{l}{l}$
	T_3	$\binom{m}{m}\binom{3}{3}$
	T_4	$\binom{m}{m}\binom{r}{3}, \binom{m}{m}\binom{3}{l}$
	T_5	$\binom{m}{m}\binom{r}{l}$
$(\mathcal{B},\mathcal{M})$	T_6	$\binom{r}{r}\binom{m}{m}, \binom{l}{l}\binom{m}{m}$
	T_7	$\binom{3}{3}\binom{m}{m}$
	T_8	$\binom{r}{3}\binom{m}{m}, \binom{3}{l}\binom{m}{m}$
	T_9	$\binom{r}{l}\binom{m}{m}$
$(\mathcal{B},\mathcal{B})$	T_{10}	$\binom{r}{r}\binom{r}{r}, \binom{l}{l}\binom{l}{l}, \binom{r}{r}\binom{l}{l}, \binom{l}{l}\binom{r}{r}$
	T_{11}	$\binom{r}{r}\binom{3}{3}, \binom{l}{l}\binom{3}{3}, \binom{3}{3}\binom{r}{r}, \binom{3}{3}\binom{l}{l}$
	T_{12}	$\binom{r}{r}\binom{r}{3}, \binom{r}{r}\binom{3}{l}, \binom{l}{l}\binom{r}{3}, \binom{l}{l}\binom{3}{l}$ $\binom{r}{3}\binom{r}{r}, \binom{r}{3}\binom{l}{l}, \binom{3}{l}\binom{r}{r}, \binom{3}{l}\binom{l}{l}$
	T_{13}	$\binom{r}{r}\binom{r}{l}, \binom{l}{l}\binom{r}{l}, \binom{r}{l}\binom{r}{r}, \binom{r}{l}\binom{l}{l}$
	T_{14}	$\binom{3}{3}\binom{3}{3}$
	T_{15}	$\binom{3}{3}\binom{r}{3}, \binom{3}{3}\binom{3}{l}, \binom{r}{3}\binom{3}{3}, \binom{3}{l}\binom{3}{3}$
	T_{16}	$\binom{3}{3}\binom{r}{l}, \binom{r}{l}\binom{3}{3}$
	T_{17}	$\binom{r}{3}\binom{r}{3}, \binom{3}{l}\binom{3}{l}$
	T_{18}	$\binom{r}{3}\binom{3}{l}, \binom{3}{l}\binom{r}{3}$
	T_{19}	$\binom{r}{3}\binom{r}{l}, \binom{3}{l}\binom{r}{l}, \binom{r}{l}\binom{r}{3}, \binom{r}{l}\binom{3}{l}$
	T_{20}	$\binom{r}{l}\binom{r}{l}$

In these notations, the first (second) column corresponds to the configurations of upper and lower functions for the first neuron $i = 1$ (second neuron $i = 2$). The notation $\mathcal{B}^{\star}_{\bullet}$ is as defined in Section II. We list these 20 subtypes T_k, $k = 1, \ldots, 20$, in Table I. The cases of $T_1, T_2, T_3, T_6, T_7, T_{10}, T_{11}$, and T_{14} have been discussed in Section II.

We need the following notation to further describe the geometric configurations of the upper and lower functions.

Notation 2:

1) For an interval $\mathcal{I} = [\varrho, \varsigma]$, we say that $\mathcal{I} > 0$ (< 0) if $\varrho > 0$ ($\varsigma < 0$). Denote $c\mathcal{I} := [c\varrho, c\varsigma]$, if $c \geq 0$, $c\mathcal{I} := [c\varsigma, c\varrho]$, if $c < 0$; $\beta + \mathcal{I} := [\beta + \varrho, \beta + \varsigma]$, for a real number β; $g_i(\mathcal{I}) := [g_i(\varrho), g_i(\varsigma)]$ for increasing function g_i.
2) For intervals $\mathcal{I}_k = [\varrho_k, \varsigma_k]$, $k = 1, 2$, $\mathcal{I}_1 + \mathcal{I}_2 := [\varrho_1 + \varrho_2, \varsigma_1 + \varsigma_2]$, an interval.
3) For a given $\eta \in \mathbb{R}$, define functions $f_i^{\eta}(\xi) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + (\alpha_{ij} + \beta_{ij})g_j(\eta) + I_i$, where $i, j \in \{1, 2\}$ and $j \neq i$.
4) For each ξ, define intervals $K_i(\xi; \mathcal{I}) := -\mu_i\xi + (\alpha_{ii} + \beta_{ii})g_i(\xi) + (\alpha_{ij} + \beta_{ij})g_j(\mathcal{I}) + I_i = \{f_i^{\eta}(\xi) : \eta \in \mathcal{I}\}$, where $i, j \in \{1, 2\}$, $j \neq i$. Regarding ξ as a variable, denote by $K_i(\xi; [\varrho, \varsigma])$ an interval-valued function of ξ.

Note that the graph of $K_i(\xi; [\varrho, \varsigma])$ is a strip that lies between the graphs of functions $f_i^{\varrho}(\xi)$ and $f_i^{\varsigma}(\xi)$.

Herein, we demonstrate the analysis for the existence of equilibria in three cases: 1) $\binom{r}{r}\binom{r}{3} \in T_{12}$; 2) $\binom{3}{3}\binom{r}{3} \in T_{15}$; and 3) $\binom{3}{3}\binom{r}{l} \in T_{16}$. First, let us take the case $\binom{r}{r}\binom{r}{3}$ to introduce further notation. In this case, we have $\check{f}_1(\check{m}_1) = \hat{f}_1(\hat{m}_1) = 0$ with $\check{m}_1 < \hat{m}_1$. Denote $S_1 := [\check{m}_1, \hat{m}_1]$, an interval. If $K_2(\tilde{p}_2; S_1) < 0$, the strip between the graphs of functions $f_2^{\check{m}_1}$ and $f_2^{\hat{m}_1}$ intersects the horizontal axis by three intervals. Here, we take $\alpha_{21} + \beta_{21} > 0$ to introduce the following notation and the case of opposite sign can be treated similarly.

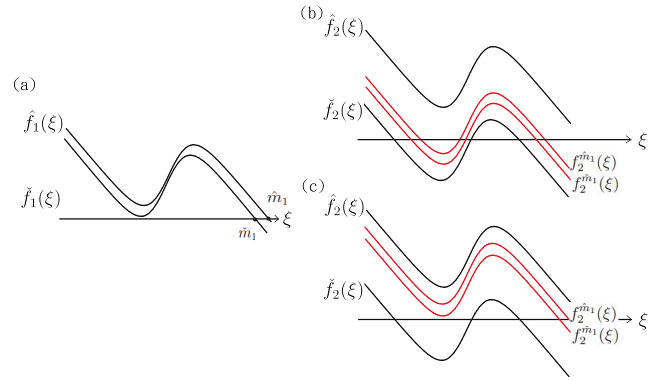

Fig. 3. Configuration under $\alpha_{21} + \beta_{21} > 0$. (a) Graphs of $\hat{f}_1(\xi)$ and $\check{f}_1(\xi)$. (b) $K_2(\tilde{p}_2; S_1) < 0$: strip $K_2(\xi; S_1)$ bounded by the graphs of $f_2^{\hat{m}_1}$ and $f_2^{\check{m}_1}$ intersects ξ-axis by three intervals. (c) $K_2(\tilde{p}_2; S_1) > 0$: strip $K_2(\xi; S_1)$ intersects ξ-axis by one interval.

Each of functions $f_2^{\check{m}_1}$ and $f_2^{\hat{m}_1}$ has three zeros, say $f_2^{\check{m}_1}(\check{a}_2^{(1)}) = f_2^{\check{m}_1}(\check{b}_2^{(1)}) = f_2^{\check{m}_1}(\check{c}_2^{(1)}) = 0$, and $f_2^{\hat{m}_1}(\hat{a}_2^{(1)}) = f_2^{\hat{m}_1}(\hat{b}_2^{(1)}) = f_2^{\hat{m}_1}(\hat{c}_2^{(1)}) = 0$ with $\check{a}_2^{(1)} < \hat{a}_2^{(1)} < \hat{b}_2^{(1)} < \check{b}_2^{(1)} < \check{c}_2^{(1)} < \hat{c}_2^{(1)}$ [Fig. 3(a) and (b)]. Denote the intervals $A_2^{S_1} := [\check{a}_2^{(1)}, \hat{a}_2^{(1)}]$, $B_2^{S_1} := [\hat{b}_2^{(1)}, \check{b}_2^{(1)}]$, and $C_2^{S_1} := [\check{c}_2^{(1)}, \hat{c}_2^{(1)}]$. Let us explain the notation: $A_2^{S_1}$ is obtained by the intersections of the left arms of $f_2^{\check{m}_1}$ and $f_2^{\hat{m}_1}$ with the horizontal axis; in addition, the values of $F_2(x_1, x_2)$ lie between $f_2^{\check{m}_1}(x_2)$ and $f_2^{\hat{m}_1}(x_2)$, when x_1 is restricted to S_1. Similar interpretation applies to $B_2^{S_1}$ and $C_2^{S_1}$.

If $K_2(\tilde{p}_2; S_1) > 0$, then each of functions $f_2^{\check{m}_1}(\xi)$ and $f_2^{\hat{m}_1}(\xi)$ has one zero, say $f_2^{\check{m}_1}(\check{m}_2^{(1)}) = 0$ and $f_2^{\hat{m}_1}(\hat{m}_2^{(1)}) = 0$ with $\check{m}_2^{(1)} < \hat{m}_2^{(1)}$ [Fig. 3(c)]. Denote the interval $S_2^{S_1} = [\check{m}_2^{(1)}, \hat{m}_2^{(1)}]$. In this case, the strip between the graphs of $f_2^{\check{m}_1}$ and $f_2^{\hat{m}_1}$ intersects the horizontal axis by one interval.

Theorem 4: Consider (29) and (30) with the case $\binom{r}{r}\binom{r}{3}$. There exists one equilibrium if $K_2(\tilde{p}_2; S_1) > 0$, and three equilibria if $K_2(\tilde{p}_2; S_1) < 0$.

Proof: For the case $K_2(\tilde{p}_2; S_1) < 0$, we consider the following three regions:

$$\Omega^{\mathbf{w}} := \left\{(x_1, x_2) \in \mathbb{R}^2 | x_i \in \Omega_i^{w_i}\right\}$$
$$\mathbf{w} = (w_1, w_2), w_1 = s, w_2 = l,\ m,\ r$$

where $\Omega_1^s = S_1$, $\Omega_2^l = A_2^{S_1}$, $\Omega_2^m = B_2^{S_1}$, and $\Omega_2^r = C_2^{S_1}$. Let $\Omega^{\mathbf{w}}$ be any one of these regions. For any given $(\zeta_1, \zeta_2) \in \Omega^{\mathbf{w}}$, we solve for x_i in

$$-\mu_i x_i + (\alpha_{ii} + \beta_{ii})g_i(x_i) + (\alpha_{ij} + \beta_{ij})g_j(\zeta_j) + I_i = 0 \quad (31)$$

for $i, j \in \{1, 2\}$ and $j \neq i$. Note that $\check{f}_1(\xi) \leq f_1^{\eta}(\xi) \leq \hat{f}_1(\xi)$ for all $\eta \in \Omega_2^{w_2}$ and $f_2^{\check{m}_1}(\xi) \leq f_2^{\eta}(\xi) \leq f_2^{\hat{m}_1}(\xi)$ [resp., $f_2^{\hat{m}_1}(\xi) \leq f_2^{\eta}(\xi) \leq f_2^{\check{m}_1}(\xi)$] for all $\eta \in S_1$ if $\alpha_{21} + \beta_{21} > 0$ (resp., $\alpha_{21} + \beta_{21} < 0$). Accordingly, one can always find three solutions to (31), which lie in regions Ω_i^l, Ω_i^m, and Ω_i^r, respectively. We define a mapping $\Phi^{\mathbf{w}} : \Omega^{\mathbf{w}} \to \Omega^{\mathbf{w}}$ by $\Phi^{\mathbf{w}}(\zeta_1, \zeta_2) = (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_i$ is the solution of (31). The mapping $\Phi^{\mathbf{w}}$ as defined is continuous, as in the proof of Theorem 1. It follows from the Brouwer's fixed-point theorem

TABLE II

CRITERIA FOR VARIOUS NUMBERS OF EQUILIBRIUM POINTS FOR THE CASE $\binom{3}{3}\binom{r}{3}$

Criteria	♯ Equi.
$\alpha_{21}+\beta_{21}>0$	
$K_2(\tilde{p}_2;A_1)>0$	3
$K_2(\tilde{p}_2;A_1)<0<K_2(\tilde{p}_2;B_1)$	5
$K_2(\tilde{p}_2;B_1)<0<K_2(\tilde{p}_2;C_1)$	7
$K_2(\tilde{p}_2;C_1)<0$	9
$\alpha_{21}+\beta_{21}<0$	
$K_2(\tilde{p}_2;C_1)>0$	3
$K_2(\tilde{p}_2;C_1)<0<K_2(\tilde{p}_2;B_1)$	5
$K_2(\tilde{p}_2;B_1)<0<K_2(\tilde{p}_2;A_1)$	7
$K_2(\tilde{p}_2;A_1)<0$	9

TABLE III

CRITERIA FOR VARIOUS NUMBERS OF EQUILIBRIUM POINTS FOR THE CASE $\binom{3}{3}\binom{r}{l}$

Criteria	♯ Equi.
$\alpha_{21}+\beta_{21}>0$	
$K_2(\tilde{p}_2;A_1)>0$	3
$K_2(\tilde{p}_2;\bullet)<0$ and $K_2(\tilde{q}_2;\bullet)>0$ for $\bullet=A_1,B_1$ or C_1	5
$K_2(\tilde{p}_2;B_1)<0$, $K_2(\tilde{p}_2;C_1)>0$, $K_2(\tilde{q}_2;A_1)>0$	7
$K_2(\tilde{p}_2;C_1)<0$, $K_2(\tilde{q}_2;B_1)>0$, $K_2(\tilde{q}_2;A_1)<0$	7
$K_2(\tilde{p}_2;C_1)<0$, $K_2(\tilde{q}_2;A_1)>0$	9
$\alpha_{21}+\beta_{21}<0$	
$K_2(\tilde{q}_2;C_1)<0$	3
$K_2(\tilde{p}_2;\bullet)<0$ and $K_2(\tilde{q}_2;\bullet)>0$ for $\bullet=A_1,B_1$ or C_1	5
$K_2(\tilde{p}_2;B_1)<0$, $K_2(\tilde{p}_2;A_1)>0$, $K_2(\tilde{q}_2;C_1)>0$	7
$K_2(\tilde{p}_2;A_1)<0$, $K_2(\tilde{q}_2;B_1)>0$, $K_2(\tilde{q}_2;C_1)<0$	7
$K_2(\tilde{p}_2;A_1)<0$, $K_2(\tilde{q}_2;C_1)>0$	9

that there exists a fixed point $\bar{\mathbf{x}}=(\bar{x}_1,\bar{x}_2)$ of $\Phi^{\mathbf{w}}$ in $\mathbf{\Omega}^{\mathbf{w}}$, which is also a zero of $\mathbf{F}$ in (2). Consequently, there exist three equilibria for (29) and each region $\mathbf{\Omega}^{\mathbf{w}}$ contains one equilibrium.

If $K_2(\tilde{p}_2;S_1)>0$, then by constructing a continuous mapping on the region $\mathbf{\Omega}^{\mathbf{w}} := S_1\times S_2^{S_1}$, where $S_2^{S_1} := [\check{m}_2^{(1)},\hat{m}_2^{(1)}]$ and $\check{f}_2^{\check{m}_1}(\check{m}_2^{(1)}) = \hat{f}_2^{\hat{m}_1}(\hat{m}_2^{(1)}) = 0$ [resp., $\check{f}_2^{\check{m}_1}(\check{m}_2^{(1)}) = \hat{f}_2^{\hat{m}_2}(\hat{m}_2^{(1)}) = 0$] if $\alpha_{21}+\beta_{21}>0$ (resp., <0), and by similar arguments, we conclude that there exists an equilibrium for (29), which lies in the region $\mathbf{\Omega}^{\mathbf{w}}$. ■

Denote $A_1 := [\check{a}_1,\hat{a}_1]$, $B_1 := [\hat{b}_1,\check{b}_1]$, and $C_1 := [\check{c}_1,\hat{c}_1]$. By similar arguments, we can obtain the following existence of multiple equilibria.

Theorem 5: Consider the case $\binom{3}{3}\binom{r}{3}$ or $\binom{3}{3}\binom{r}{l}$. Then, there exist parameters such that (29) and (30) have three, five, seven, or nine equilibria. The criterion for each of these existence is listed in Tables II and III, respectively.

Proof: We only show the case $\binom{3}{3}\binom{r}{3}$. The other cases are similar.

If $\alpha_{21}+\beta_{21}>0$ and $K_2(\tilde{p}_2;A_1)>0$ or if $\alpha_{21}+\beta_{21}<0$ and $K_2(\tilde{p}_2;C_1)>0$, then $K_2(\xi;\bullet)$ intersects the ξ-axis by one interval, say $S_2^{\bullet}$, for $\bullet = A_1$, B_1, and C_1. Thus, there exist three regions: $A_1\times S_2^{A_1}$, $B_1\times S_2^{B_1}$, and $C_1\times S_2^{C_1}$, and each can be regarded as region $\mathbf{\Omega}^{\mathbf{w}}$ in the proof of Theorem 4; hence, there are three equilibria.

If $\alpha_{21}+\beta_{21}>0$, $K_2(\tilde{p}_2;A_1)<0$, and $K_2(\tilde{p}_2;B_1)>0$, then $K_2(\xi;A_1)$ intersects the ξ-axis by three intervals, say $A_2^{A_1}$, $B_2^{A_1}$, $C_2^{A_1}$, and $K_2(\xi;\bullet)$ intersects the ξ-axis by one interval, say $S_2^{\bullet}$, for $\bullet = B_1, C_1$. Then, there exist five regions $A_1\times A_2^{A_1}$, $A_1\times B_2^{A_1}$, $A_1\times C_2^{A_1}$, $B_1\times S_2^{B_1}$, and $C_1\times S_2^{C_1}$ and hence five equilibria. Same assertion holds if $\alpha_{21}+\beta_{21}<0$, $K_2(\tilde{p}_2;C_1)<0$, and $K_2(\tilde{p}_2;B_1)>0$.

If $\alpha_{21}+\beta_{21}>0$, $K_2(\tilde{p}_2;B_1)<0$, and $K_2(\tilde{p}_2;C_1)>0$ or $\alpha_{21}+\beta_{21}<0$, $K_2(\tilde{p}_2;B_1)<0$, and $K_2(\tilde{p}_2;A_1)>0$, there are seven regions to be considered as previous cases and hence seven equilibria.

If $\alpha_{21}+\beta_{21}>0$, $K_2(\tilde{p}_2;C_1)<0$ or $\alpha_{21}+\beta_{21}<0$, and $K_2(\tilde{p}_2;A_1)<0$, there are nine regions to be considered and hence nine equilibria. ■

We can apply sequential contracting formulation similar to Theorem 3 to obtain global convergence of dynamics for the cases of multiple equilibria in Theorems 4 and 5. As the above analysis is performed componentwise, it certainly can be extended to the general n-neuron system (1).

V. PIECEWISE-LINEAR ACTIVATION FUNCTIONS

It is straightforward to extend the discussions in Sections II–IV to (1) with piecewise-linear activation functions

$$g_i(\xi)=\begin{cases} u_i & \text{if } -\infty<\xi<\bar{p}_i \\ u_i+\frac{v_i-u_i}{\bar{q}_i-\bar{p}_i}(\xi-\bar{p}_i) & \text{if } \bar{p}_i\le\xi\le\bar{q}_i \\ v_i & \text{if } \bar{q}_i<\xi<\infty. \end{cases} \quad (32)$$

This class of functions includes the standard activation function in cellular neural networks

$$g_i(\xi)=\frac{1}{2}(|\xi+1|-|\xi-1|),\quad i\in\mathcal{N}.$$

In this case, we adapt the setting $\mathcal{M}$ and $\mathcal{B}$ to

$$\mathcal{M} := \left\{i\in\mathcal{N}|\ \frac{v_i-u_i}{\bar{q}_i-\bar{p}_i}\le\frac{\mu_i}{\alpha_{ii}+\beta_{ii}}\right\}$$
$$\mathcal{B} := \left\{i\in\mathcal{N}|\ 0<\frac{\mu_i}{\alpha_{ii}+\beta_{ii}}<\frac{v_i-u_i}{\bar{q}_i-\bar{p}_i}\right\}.$$

Further classification according to the intersections of $\hat{f}_i$ and $\check{f}_i$ with the horizontal axis, as those shown in Fig. 1, can be formulated with $\tilde{p}_i$ and $\tilde{q}_i$ replaced by $\bar{p}_i$ and $\bar{q}_i$, respectively.

All the results in Sections II–IV can thus be extended to (1) with such piecewise-linear activation functions. The extension also holds if the middle part of g_i for $\bar{p}_i\le\xi\le\bar{q}_i$ in (32) is replaced by an increasing function connecting the two saturation segments.

VI. NUMERICAL ILLUSTRATIONS

In this section, we present two examples for (1) with $n=2$ or $n=3$ to illustrate the theorems in Sections II–IV. We take the activation function as $g_i(\xi)=\tanh(\xi)$, and thus $\sigma_i=0$, $\rho_i=1$, $L_i=1$ for $i=1,2,3$.

Example 1: This example illustrates Theorem 5 for the case $n=2$ and $\binom{3}{3}\binom{r}{3}$. We consider the parameters

$$\mu_1=1,\quad \alpha_{11}=1.5,\quad \alpha_{12}=0.07,\quad \beta_{11}=0.1,\quad \beta_{12}=0.08$$
$$\mu_2=1,\quad \alpha_{21}=0.1,\quad \alpha_{22}=1.4,\quad \beta_{21}=0.1,\quad \beta_{22}=0.1$$
$$I_1=-0.05,\quad I_2=0.32$$
$$\tau_{11}=\tau_{22}=0.05,\quad \tau_{12}=\tau_{21}=10.$$

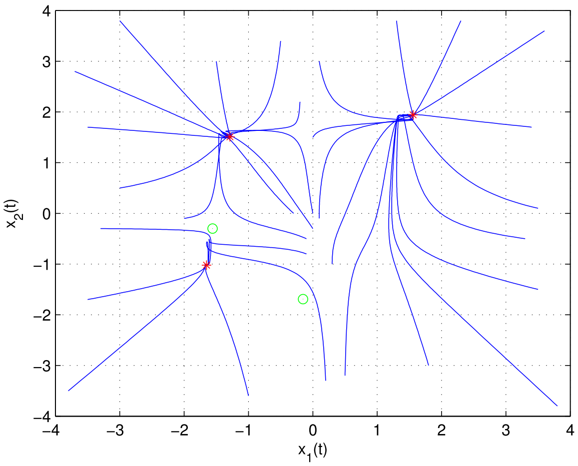

Fig. 4. Numerical simulation for Example 1.

Herein, $\alpha_{12} + \beta_{12} = 0.15 > 0$, $\alpha_{21} + \beta_{21} = 0.2 > 0$. Moreover, $\tilde{p}_1 = -0.7127085$, $\tilde{q}_1 = 0.7127085$, $\tilde{p}_2 = -0.6584789$, $\tilde{q}_2 = 0.6584789$, $k_1^+ = 0.1$, $k_1^- = -0.2$, $k_2^+ = 0.52$, and $k_2^- = 0.12$. Let us examine the condition in Theorem 5. We compute to find intervals $A_1 := [\check{a}_1, \hat{a}_1] = [-1.69581395, -1.26304012]$ and $B_1 := [\hat{b}_1, \check{b}_1] = [-0.17106488, 0.37914872]$. Condition $K_2(\tilde{p}_2; A_1) < 0 < K_2(\tilde{p}_2; B_1)$ in Table II can be directly justified

$$K_2(\tilde{p}_2; A_1) = [-0.0745232, -0.0579265] < 0$$
$$K_2(\tilde{p}_2; B_1) = [0.0785704, 0.1848471] > 0.$$

The parameters chosen here also satisfy the convergence theorem that is not included in this paper. It can be observed in Fig. 4 that almost all trajectories converge to three equilibria marked in red (the other two equilibria are marked in green). These five equilibria can be computed numerically as

$$(1.5582, 1.9426),\ (-0.15341, 1.691),\ (-1.2876, 1.5082),$$
$$(-1.6522, -1.0216),\ (-1.5578, -0.30005).$$

Example 2: This example illustrates Theorems 2 and 3 for (1) with $n = 3$. We consider the parameters

$$(\mu_i) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (\alpha_{ij}) = \begin{pmatrix} 1.8 & 0.05 & 0 \\ 0.05 & 1.9 & 0 \\ 0 & 0.05 & 0.6 \end{pmatrix}$$
$$(I_i) = \begin{pmatrix} 0.05 \\ 0 \\ 0.15 \end{pmatrix}, \quad (\beta_{ij}) = \begin{pmatrix} 0.2 & 0 & 0.05 \\ 0 & 0.1 & 0.05 \\ 0.05 & 0 & 0.1 \end{pmatrix}.$$

It is straightforward to see $i = 1, 2 \in \mathcal{B}_3^3$ and $i = 3 \in \mathcal{M}$, and thus $\mathrm{card}(\mathcal{B}_3^3) = 2$. In addition, we set $\tau_{ii} = 0.1$, $\tau_{ij} = 12$ for $i, j = 1, 2, 3$, $i \neq j$. We compute to find $\tilde{p}_1 = \tilde{p}_2 = -0.8813736$, $\tilde{q}_1 = \tilde{q}_2 = 0.8813736$, $k_1^+ = 0.15$, $k_1^- = -0.05$, $k_2^+ = 0.1$, $k_2^- = -0.1$, $k_3^+ = 0.25$, and $k_3^- = 0.05$. First, let us examine the conditions in Theorem 2. For (7), for $i = 1 \in \mathcal{B}_3^3$, we take $\theta_1 = 0.3 \in (0, 1)$; then

$$\theta_1 = 0.3 > \bar{L}_2|\alpha_{12} + \beta_{12}| + \bar{L}_3|\alpha_{13} + \beta_{13}| = 0.107.$$

For $i = 2 \in \mathcal{B}_3^3$, we take $\theta_2 = 0.4 \in (0, 1)$; then

$$\theta_2 = 0.4 > \bar{L}_1|\alpha_{21} + \beta_{21}| + \bar{L}_3|\alpha_{23} + \beta_{23}| = 0.107.$$

For $i = 3 \in \mathcal{M}$, we take $\theta_3 = 0.2 \in (0, 1)$; then

$$\theta_3 = 0.2 > \bar{L}_1|\alpha_{31} + \beta_{31}| + \bar{L}_2|\alpha_{32} + \beta_{32}| = 0.1.$$

For (8), for $i = 1 \in \mathcal{B}_3^3$, we take $\theta_1 = 0.3 \in (0, 1)$; then for $\xi \in (-\infty, \hat{a}_1] \cup [\check{c}_1, \infty)$

$$\max g_1'(\xi) = 0.25098 < 0.35 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}$$

and for $\xi \in [\hat{b}_1, \check{b}_1]$

$$\min g_1'(\xi) = 0.97723 > 0.65 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}.$$

For $i = 2 \in \mathcal{B}_3^3$, we take $\theta_2 = 0.4 \in (0, 1)$; then

$$\max g_2'(\xi) = 0.23747 < 0.3 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}$$

for $\xi \in (-\infty, \hat{a}_2] \cup [\check{c}_2, \infty)$, and

$$\min g_2'(\xi) = 0.98996 > 0.7 = \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}$$

for $\xi \in [\hat{b}_2, \check{b}_2]$. For $i = 3 \in \mathcal{M}$, we take $\theta_3 = 0.2 \in (0, 1)$; then for $\xi \in [\check{m}_3, \hat{m}_3]$

$$\max g_3'(\xi) = 0.97402 < 1.14286 = \frac{\mu_3 - \theta_3}{\alpha_{33} + \beta_{33}}.$$

Next, let us examine the conditions in Theorem 3. For (22), for $i = 1 \in \mathcal{B}_3^3$, we have

$$\hat{F}_1(\tilde{p}_1) = -0.2188399 < 0, \quad \check{F}_1(\tilde{q}_1) = 0.3188399 > 0.$$

For $i = 2 \in \mathcal{B}_3^3$, we have

$$\hat{F}_2(\tilde{p}_2) = -0.3508400 < 0, \quad \check{F}_2(\tilde{q}_2) = 0.3508400 > 0.$$

For (25)

$$\beta_{11}\tau_{11} = 0.02 < \tau_{11}^c = 0.2439024$$
$$\beta_{22}\tau_{22} = 0.01 < \tau_{11}^c = 0.2439024$$
$$\beta_{33}\tau_{33} = 0.01 < \tau_{11}^c = 0.2333333.$$

We need the following quantities to examine (26):

$$\hat{a}_1^F = -1.4941831, \quad \hat{b}_1^F = -0.3387856, \quad \hat{c}_1^F = 2.2719172$$
$$\check{a}_1^F = -2.1616717, \quad \check{b}_1^F = 0.2210641, \quad \check{c}_1^F = 1.6413113$$
$$\hat{a}_2^F = -1.6850199, \quad \hat{b}_2^F = -0.1862482, \quad \hat{c}_2^F = 2.1258384$$
$$\check{a}_2^F = -2.1258384, \quad \check{b}_2^F = 0.1862482, \quad \check{c}_2^F = 1.6850199$$
$$\hat{m}_3^F = 0.7054875, \quad \check{m}_3^F = 0.0664389.$$

For (26), for $i = 1 \in \mathcal{B}_3^3$, we take $\theta_1 = 0.3 \in (0, 1)$; then for $\xi \in (-\infty, \hat{a}_1^F] \cup [\check{c}_1^F, \infty)$

$$\max g_1'(\xi) = 0.30935 < 0.35 = \frac{\mu_1 - \theta_1}{\alpha_{11} + \beta_{11}}$$

and for $\xi \in [\hat{b}_1^F, \check{b}_1^F]$

$$\min g_1'(\xi) = 0.89704 > 0.65 = \frac{\mu_1 + \theta_1}{\alpha_{11} + \beta_{11}}.$$

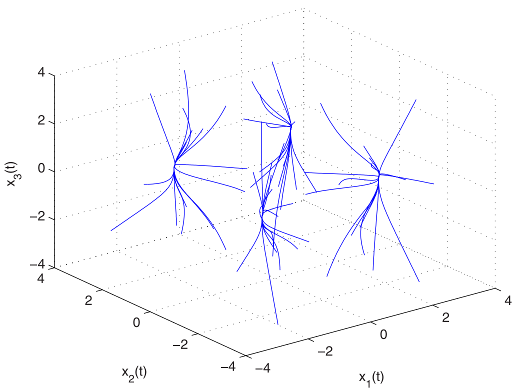

Fig. 5. Numerical simulation for Example 2.

For $i = 2 \in \mathcal{B}_3^3$, we take $\theta_2 = 0.4 \in (0, 1)$; then

$$\max g_2'(\xi) = 0.26046 < 0.3 = \frac{\mu_2 - \theta_2}{\alpha_{22} + \beta_{22}}$$

for $\xi \in (-\infty, \hat{a}_2^F] \cup [\check{c}_2^F, \infty)$, and for $\xi \in [\hat{b}_2^F, \check{b}_2^F]$

$$\min g_2'(\xi) = 0.96648 > 0.7 = \frac{\mu_2 + \theta_2}{\alpha_{22} + \beta_{22}}.$$

For $i = 3 \in \mathcal{M}$, we take $\theta_3 = 0.2 \in (0, 1)$; then

$$\max g_3'(\xi) = 0.99860 < 1.14286 = \frac{\mu_3 - \theta_3}{\alpha_{33} + \beta_{33}}$$

for $\xi \in [\check{m}_3^F, \hat{m}_3^F]$.

A direct computation yields

$$\mathbf{M} = \begin{pmatrix} 0.288 & -0.05 & -0.057 \\ -0.05 & 0.392 & -0.057 \\ -0.05 & -0.05 & 0.19544 \end{pmatrix}$$

which is strictly diagonal-dominant. Fig. 5 shows that almost all trajectories converge to four of the nine equilibria, as concluded by Corollary 1. Numerical computation shows that these nine equilibria are

(2.0630, 2.0056, 0.6439), (1.9989, −0.0731, 0.5392)
(−0.1236, 1.9368, 0.5334), (−1.8331, 0.0320, 0.3220)
(−1.7684, 1.8830, 0.4430), (1.9427, −1.8312, 0.4401)
(−0.0696, −0.0167, 0.4286), (−0.0174, −1.8977, 0.3145)
(−1.9022, −1.9615, 0.1762).

VII. Conclusion

We have developed an approach that combines the geometric structure embedded in the equations with Brouwer's fixed-point theorem or contracting mapping principle to conclude the existence of multiple equilibria in the general delay neural network (1). We further employed a new methodology, named sequential contracting, to investigate the global dynamics and convergence of solutions to one of the equilibrium points. The approach unfolds from constructing suitable lower and upper equations iteratively for (1). Effective designs of lower and upper dynamics can then capture the asymptotic behaviors of the coupled systems. Such a formulation has the advantage that the nonlinear terms in the equations are not overmanipulated by linearization or other treatments, and hence exploits the intrinsic nonlinear nature of the system. This methodology and analysis are completely different from the commonly or previously adopted approaches, such as Lyapunov function technique and matrix inequality, for concluding global dynamics. With the present new methodology, we are able to establish new multistability scenarios for neural networks with delays and with smooth sigmoidal or piecewise-linear activation functions. Our approach can be applied to investigate the asymptotic behaviors in other neural networks and nonlinear systems. In addition to multistability for neural networks, the idea of sequential contracting has been successfully applied to study the asymptotic phases in an integro-differential equation modeling T-cell differentiation [10], synchronous behaviors in a gene regulatory model [17], and synchronization for coupled cells [32].

Appendix

Sequences of upper and lower functions used in Section III

$$\hat{f}_i^{(k)}(\xi, T) := \begin{cases} \hat{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \check{f}_i^{(k-1)}\big(\hat{m}_i^{(k-1)}(T), T\big) & \text{if } \beta_{ii} \geq 0 \\ \hat{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \hat{f}_i^{(k-1)}\big(\check{m}_i^{(k-1)}(T), T\big) & \text{if } \beta_{ii} < 0 \end{cases}$$

$$\check{f}_i^{(k)}(\xi, T) := \begin{cases} \check{\gamma}_i(\xi, T) - \beta_{ii} L_i \tau_{ii} \hat{f}_i^{(k-1)}\big(\check{m}_i^{(k-1)}(T), T\big) & \text{if } \beta_{ii} \geq 0 \\ \check{\gamma}(\xi, T) - \beta_{ii} L_i \tau_{ii} \check{f}_i^{(k-1)}\big(\hat{m}_i^{(k-1)}(T), T\big) & \text{if } \beta_{ii} < 0 \end{cases}$$

for $T \geq T_0$, $k \in \mathbb{N}$, where $\hat{m}_i^{(k)}(T)$ [resp., $\check{m}_i^{(k)}(T)$] is the unique solution of $\hat{f}_i^{(k)}(\xi, T) = 0$ [resp., $\check{f}_i(\xi, T) = 0$]. They are vertical shifts of $\check{f}_i^{(0)}(\xi, T)$ and $\hat{f}_i^{(0)}(\xi, T)$.

Three conditions for monostable scenario of upper and lower functions are as follows.

Condition (M1): $|\beta_{ii}|\tau_{ii} < (|\alpha_{ii}| + |\beta_{ii}|)\rho_i/\{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0)]\}$.

Condition (M2): There exists a $T_0 \geq t_0$ such that $\hat{f}_i^{(0)}(\cdot, T_0)$ and $\check{f}_i^{(0)}(\cdot, T_0)$ have unique zeros, $\hat{m}_i^{(0)}(T_0)$ and $\check{m}_i^{(0)}(T_0)$, respectively.

Condition (M3): $g_i'(\xi) < (\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii})$ for all $\xi \in [\check{m}_i^{(0)}(T_0), \hat{m}_i^{(0)}(T_0)]$ for some $\theta_i \in (0, \mu_i)$.

Under conditions (M1)–(M3), the configuration of the upper and lower functions forces $x_i(t)$ to converge to an interval whose length can be estimated.

Proposition 2: Assume that conditions (M1)–(M3) hold for some $i \in \mathcal{N}$. Then, $x_i(t)$ satisfying (10) converges to $[\underline{m}_i, \overline{m}_i]$, where

$$\overline{m}_i - \underline{m}_i \leq \big[w_i^{\max}(\infty) - w_i^{\min}(\infty)\big]/[(1 - 2|\beta_{ii}|L_i\tau_{ii})\theta_i].$$

Three conditions for bistable scenario of upper and lower functions are as follows.

Condition (B1): $L_i > \mu_i/(\alpha_{ii} + \beta_{ii}) > 0$, $|\beta_{ii}|\tau_{ii} < (|\alpha_{ii}| + |\beta_{ii}|)\rho_i/\{L_i[4(|\alpha_{ii}| + |\beta_{ii}|)\rho_i + w_i^{\max}(t_0) - w_i^{\min}(t_0)]\}$.

Notably, condition (B1) implies $L_i > \mu/(\alpha_{ii} + \beta_{ii})$. There hence exist exactly two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i < \sigma_i < \tilde{q}_i$, satisfying

$$g_i'(\tilde{p}_i) = g'(\tilde{q}_i) = \mu_i/(\alpha_{ii} + \beta_{ii}).$$

Condition (B2): There exists a $T_0 \geq t_0$ such that $\check{f}_i^{(0)}(\tilde{q}_i, T_0) > 0$ and $\hat{f}_i^{(0)}(\tilde{p}_i, T_0) < 0$.

Under condition (B2), there exist exactly three zeros $\hat{a}_i$, $\hat{b}_i$ and $\hat{c}_i$ (resp., $\check{a}_i$, $\check{b}_i$, and $\check{c}_i$) of $\hat{f}_i^{(0)}(\cdot, T_0) = 0$ [resp., $\check{f}_i^{(0)}(\cdot, T_0) = 0$], where $\check{a}_i \leq \hat{a}_i < \tilde{p}_i < \hat{b}_i \leq \check{b}_i < \tilde{q}_i < \check{c}_i \leq \hat{c}_i$. Let $\theta_i \in (0, \mu_i)$ be a fixed number. The third condition is concerned with the slope of function g_i.

Condition (B3):

$$g_i'(\xi) \begin{cases} > (\mu_i + \theta_i)/(\alpha_{ii} + \beta_{ii}) & \text{if } \xi \in [\hat{b}_i, \check{b}_i], \\ < (\mu_i - \theta_i)/(\alpha_{ii} + \beta_{ii}) & \text{if } \xi \in (-\infty, \hat{a}_i] \cup [\check{c}_i, \infty). \end{cases}$$

Under conditions (B1)–(B3), the configuration of the upper and lower functions forces $x_i(t)$ to converge to one of the three intervals whose lengths can be estimated.

Proposition 3: Assume that conditions (B1)–(B3) hold for some $i \in \mathcal{N}$ and some $\theta_i \in (0, \mu_i)$. Then $x_i(t)$ satisfying (10) converges to one of the three disjoint intervals: $[\underline{a}_i, \overline{a}_i]$, $[\underline{b}_i, \overline{b}_i]$, and $[\underline{c}_i, \overline{c}_i]$, where

$$\begin{aligned} 0 &\leq \overline{a}_i - \underline{a}_i, \overline{b}_i - \underline{b}_i, \overline{c}_i - \underline{c}_i \\ &\leq \left[w_i^{\max}(\infty) - w_i^{\min}(\infty)\right] / [(1 - 2|\beta_{ii}| L_i \tau_{ii})\theta_i]. \end{aligned}$$

Propositions 2 and 3 can be verified by applying the arguments parallel to those for [33, Proposition 4]. More precisely, the arguments for Proposition 2 is actually parallel to the subcase $\mathcal{R}$ (or $\mathcal{L}$) considered therein.

REFERENCES

[1] N. Burić and D. Todorović, "Dynamics of FitzHugh–Nagumo excitable systems with delayed coupling," *Phys. Rev. E*, vol. 67, no. 6, p. 066222, Jun. 2003.

[2] S. A. Campbell, R. Edwards, and P. van den Driessche, "Delayed coupling between two neural network loops," *SIAM J. Appl. Math.*, vol. 65, no. 1, pp. 316–335, 2004.

[3] J. Cao, G. Feng, and Y. Wang, "Multistability and multiperiodicity of delayed Cohen Grossberg neural networks with a general class of activation functions," *Phys. D, Nonlinear Phenomena*, vol. 237, no. 13, pp. 1734–1749, 2008.

[4] C.-Y. Cheng, K.-H. Lin, and C.-W. Shih, "Multistability in recurrent neural networks," *SIAM J. Appl. Math.*, vol. 66, no. 4, pp. 1301–1320, 2006.

[5] C.-Y. Cheng, K.-H. Lin, and C.-W. Shih, "Multistability and convergence in delayed neural networks," *Phys. D, Nonlinear Phenomena*, vol. 225, no. 1, pp. 61–74, Jan. 2007.

[6] L. O. Chua, *CNN: A Paradigm for Complexity*. Singapore: World Scientific, 1998.

[7] L. O. Chua and T. Roska, "Stability of a class of nonreciprocal cellular neural networks," *IEEE Trans. Circuits Syst.*, vol. 37, no. 12, pp. 1520–1527, Dec. 1990.

[8] M. A. Cohen and S. Grossberg, "Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," *IEEE Trans. Syst., Man Cybern.*, vol. SMC-13, no. 5, pp. 815–826, Sep./Oct. 1983.

[9] J. Foss, A. Longtin, B. Mensour, and J. Milton, "Multistability and delayed recurrent loops," *Phys. Rev. Lett.*, vol. 76, no. 4, pp. 708–711, Jan. 1996.

[10] A. Friedman, C.-Y. Kao, and C.-W. Shih, "Asymptotic limit in a cell differentiation model with consideration of transcription," *J. Differ. Equ.*, vol. 252, no. 10, pp. 5679–5711, May 2012.

[11] R. H. R. Hahnloser, R. Sarpeshkar, M. A. Mahowald, R. J. Douglas, and H. S. Seung, "Digital selection and analogue amplification coexist in a cortex-inspired silicon circuit," *Nature*, vol. 405, no. 6789, pp. 947–951, Jun. 2000.

[12] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proc. Nat. Acad. Sci. United States Amer.*, vol. 81, no. 10, pp. 3088–3092, 1984.

[13] L. Huang and J. Wu, "Nonlinear waves in networks of neurons with delayed feedback: Pattern formation and continuation," *SIAM J. Math. Anal.*, vol. 34, no. 4, pp. 836–860, 2003.

[14] G. Huang and J. Cao, "Delay-dependent multistability in recurrent neural networks," *Neural Netw.*, vol. 23, no. 2, pp. 201–209, Mar. 2010.

[15] R. E. Kalaba and K. Spingarn, "A criterion for the convergence of the Gauss–Seidel method," *Appl. Math. Comput.*, vol. 4, no. 4, pp. 359–367, Oct. 1978.

[16] J.-H. Li, A. N. Michel, and W. Porod, "Analysis and synthesis of a class of neural networks: Linear systems operating on a closed hypercube," *IEEE Trans. Circuits Syst.*, vol. 36, no. 11, pp. 1405–1422, Nov. 1989.

[17] K.-L. Liao, C.-W. Shih, and J.-P. Tseng, "Synchronized oscillations in a mathematical model of segmentation in zebrafish," *Nonlinearity*, vol. 25, no. 4, pp. 869–904, 2012.

[18] X. Liao and J. Wang, "Global dissipativity of continuous-time recurrent neural networks with time delay," *Phys. Rev. E*, vol. 68, no. 1, p. 016118, 2003.

[19] E. Liz and A. Ruiz-Herrera, "Attractivity, multistability, and bifurcation in delayed Hopfield's model with non-monotonic feedback," *J. Differ. Equ.*, vol. 225, no. 11, pp. 4244–4266, 2013.

[20] Q. Lu, Q. Wang, and X. Shi, "Effects of time delay on synchronization and firing patterns in coupled neuronal systems," in *Delay Differential Equations, Recent Advances and New Directions*. Berlin, Germany: Springer-Verlag, 2009, pp. 1–31.

[21] W. Lu, L. Wang, and T. Chen, "On attracting basins of multiple equilibria of a class of cellular neural networks," *IEEE Trans. Neural Netw.*, vol. 22, no. 3, pp. 381–394, Mar. 2011.

[22] M. Di Marco, M. Forti, M. Grazzini, and L. Pancioni, "Limit set dichotomy and convergence of cooperative piecewise linear neural networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 5, pp. 1052–1062, May 2011.

[23] M. Di Marco, M. Forti, M. Grazzini, and L. Pancioni, "Limit set dichotomy and multistability for a class of cooperative neural networks with delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 9, pp. 1473–1485, Sep. 2012.

[24] M. Di Marco, M. Forti, M. Grazzini, and M. Pancioni, "Necessary and sufficient condition for multistability of neural networks evolving on a closed hypercube," *Neural Netw.*, vol. 54, no. 6, pp. 38–48, Jun. 2014.

[25] C. M. Marcus and R. M. Westervelt, "Stability of analog neural networks with delay," *Phys. Rev. A*, vol. 39, no. 1, pp. 347–359, Jan. 1989.

[26] X. Nie and J. Cao, "Multistability of second-order competitive neural networks with nondecreasing saturated activation functions," *IEEE Trans. Neural Netw.*, vol. 22, no. 11, pp. 1694–1708, Nov. 2011.

[27] X. Nie, J. Cao, and S. Fei, "Multistability and instability of delayed competitive neural networks with nondecreasing piecewise linear activation functions," *Neurocomputing*, vol. 119, pp. 281–291, Nov. 2013.

[28] X. Nie, J. Cao, and S. Fei, "Multistability and instability of competitive neural networks with Mexican-hat-type activation functions," *Abstract Appl. Anal.*, vol. 2014, pp. 1–20, May 2014, Art. ID 901519.

[29] T. Roska and L. O. Chua, "Cellular neural networks with non-linear and delay-type template elements and non-uniform grids," *Int. J. Circuit Theory Appl.*, vol. 20, no. 5, pp. 469–481, Sep./Oct. 1992.

[30] C.-W. Shih and J.-P. Tseng, "Convergent dynamics for multistable delayed neural networks," *Nonlinearity*, vol. 21, no. 10, pp. 2361–2389, 2008.

[31] C.-W. Shih and J.-P. Tseng, "Global synchronization and asymptotic phases for a ring of identical cells with delayed coupling," *SIAM J. Math. Anal.*, vol. 43, no. 4, pp. 1667–1697, 2011.

[32] C.-W. Shih and J.-P. Tseng, "A general approach to synchronization of coupled cells," *SIAM J. Appl. Dyn. Syst.*, vol. 12, no. 3, pp. 1354–1393, 2013.

[33] J.-P. Tseng, "Global asymptotic dynamics of a class of nonlinearly coupled neural networks with delays," *Discrete Continuous Dyn. Syst. A*, vol. 33, no. 10, pp. 4693–4729, 2013.

[34] L. Wang, W. Lu, and T. Chen, "Coexistence and local stability of multiple equilibria in neural networks with piecewise linear nondecreasing activation functions," *Neural Netw.*, vol. 23, no. 2, pp. 189–200, Mar. 2010.

[35] L. Wang and T. Chen, "Complete stability of cellular neural networks with unbounded time-varying delays," *Neural Netw.*, vol. 36, pp. 11–17, Dec. 2012.

[36] J. Wu, *Introduction to Neural Dynamics and Signal Transmission Delay*. Berlin, Germany: Walter de Gruyter, 2001.

[37] D. M. Young, *Iterative Solution of Large Linear Systems*. New York, NY, USA: Academic, 1971.

[38] Z. Zeng, D.-S. Huang, and Z. Wang, "Memory pattern analysis of cellular neural networks," *Phys. Lett. A*, vol. 342, nos. 1–2, pp. 114–128, Jul. 2005.

[39] Z. Zeng and J. Wang, "Complete stability of cellular neural networks with time-varying delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 53, no. 4, pp. 944–955, Apr. 2006.

[40] Z. Zeng and W. X. Zheng, "Multistability of neural networks with time-varying delays and concave-convex characteristics," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 2, pp. 293–305, Feb. 2012.

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