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Oscillation Theorems for Fourth-Order Partial Difference Equations

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Abstract—The behavior of the solutions of nonlinear partial difference equations of fourth order is discussed. We give some sufficient conditions for the oscillation of nontrivial solutions of the given equation by using the weighted techniques.

Keywords—Oscillation, Partial difference equations.

1. INTRODUCTION

In this paper, we are concerned with oscillatory behavior of the solutions of nonlinear difference equations. For self-adjoint second-order linear difference equation

$$\Delta\left(P_n \Delta x_n\right) + Q_n x_n = 0, \qquad \forall\, n \in \mathbf{N} = \{1, 2, \dots\}, \tag{1.1}$$

where

$$P_n > 0, \qquad \forall\, n = 1, 2, \dots. \tag{1.2}$$

The oscillatory behavior of (1.1) has been extensively discussed by several authors (see [1–4]). However, they deal with (1.1) under the assumption

$$\sum_{i=1}^{\infty} Q_i = \infty. \tag{1.3}$$

Therefore, it is interesting to discuss (1.1) without the required assumption (1.3). Some oscillation results for ordinary differential equations are obtained in [5,6]. Following this direction,

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we establish some oscillation results for (1.1) in [7], and as an application, we can discuss the oscillation in the discrete analogue for second-order nonlinear wave equations.

The principal objective of this paper is to establish some oscillation theorems for partial difference equations of fourth order which extend the results of [7].

Let Δ_1, Δ_2, Δ_1^k be partial difference operators defined as $\Delta_1 u_{m,n} = u_{m+1,n} - u_{m,n}$, $\Delta_2 u_{m,n} = u_{m,n+1} - u_{m,n}$, and $\Delta_1^k u_{m,n} = \Delta_1^{k-1}(\Delta_1 u_{m,n})$. We shall consider the following partial difference equation which is the discrete analogue for the nonlinear wave equation of fourth order:

$$\begin{aligned} &\Delta_2(\alpha_{n-1}\Delta_2 u_{m,n-1}) + \beta_{n-1}\Delta_2 u_{m,n-1} + \gamma_n \Delta_1^4 A_0(n;u) u_{m-2,n} \\ &\quad - \delta_n \Delta_1^2(A(m-1,n;u)u_{m-1,n}) + b(m,n,u_{m,n})u_{m,n} = 0, \\ &\quad \forall\, m \in \mathbf{\Omega}, \qquad n \in \mathbf{N}, \\ &u_{0,n} = u_{M+1,n} = 0, \quad \Delta_1^2 u_{-1,n} = \Delta_1^2 u_{M,n} = 0, \qquad \forall\, n \in \mathbf{N}, \end{aligned} \tag{1.4}$$

where $M \in \mathbf{N}$, $\mathbf{\Omega} = \{1,2,\dots,M\}$, $\mathbf{S} = \{0,1,2,\dots\}$, $\alpha_n, \gamma_n, \delta_n > 0$, $\beta_n \in \mathbf{R}$, $\forall\, n \in \mathbf{N}$. Here, b is a function of m, n, and u. A is a function in (m,n) and depends on some quantities related to $u_{m,n}$ itself or its difference. For example,

$$A(m,n;u) = \rho + \theta_m \sum_{t=1}^{M} (\Delta u_{t,n})^2$$

and

$$b(m,n,u) = u^{2p}, \qquad p \in \mathbf{N}.$$

And A_0 is a function of n only and depends on some quantities related to $u_{m,n}$ or its difference.

The contents of this paper are organized as follows. In Section 2, we give some preliminaries including two oscillation results for ordinary difference equations and discrete Green's formula. In Sections 3 and 4, we shall discuss oscillations of the solution of some partial difference equations based on some results for the self-adjoint second-order linear equation given in Section 2. Two different kinds of average techniques are used in later sections, respectively. The main results are given in Theorems 3.3 and 4.4.

2. PRELIMINARIES

In this section, we shall give two oscillation theorems for self-adjoint linear equation (1.1) which will be used later. A nontrivial solution $\{x_n\}_{n=1}^{\infty}$ of the difference equation (1.1) is oscillatory if for every $n \in \mathbf{N}$, there exists $n_1, n_2 \geq n$ such that $x_{n_1} x_{n_2} \leq 0$. Otherwise, it is nonoscillatory. The difference equation (1.1) is said to be oscillatory if it has no nonoscillatory nontrivial solution.

THEOREM A. *(See [3].) If*

$$\sum_{i=1}^{\infty} \frac{1}{P_i} = \infty$$

and

$$\sum_{i=1}^{\infty} Q_i = \infty$$

hold, then (1.1) is oscillatory.

THEOREM B. *(See [7].) Assume that the following assumptions hold.*

(A1) $P_n > 0$, $\forall\, n = 1,2,\dots$.

(A2) *There exists a nonnegative integer N_0 such that*

$$P_n \leq 1 \quad \textit{and} \quad Q_n > 0, \qquad \forall\, n \geq N_0.$$

(A3) *There exists a nonnegative integer k such that the series*

$$H_n^{(0)} = \sum_{i=n}^{\infty} Q_i \quad \text{and} \quad H_n^{(m)} \equiv \sum_{i=n}^{\infty} iQ_iH_i^{(m-1)}, \qquad m \geq 1,$$

converges for $m = 0, 1, 2, \ldots, k$, and

$$\sum_{i=1}^{\infty} iQ_iH_i^{(k)} = \infty.$$

Then (1.1) is oscillatory.

LEMMA C. *(See [1,8].) Consider the following Sturm-Liouville system:*

$$\begin{aligned} \Delta^2\phi_{n-1} + \lambda\phi_n &= 0, \qquad \forall\, n \in \mathbf{\Omega}, \\ \phi_0 = \phi_{M+1} &= 0. \end{aligned} \tag{2.1}$$

Let λ_1 be the smallest eigenvalue of system (2.1) and $\phi_n^{(1)}$ be an eigenfunction corresponding to λ_1. Then $\lambda_1 > 0$ and $\phi_n^{(1)} > 0$, $\forall\, n \in \mathbf{\Omega}$.

Let $Ly_n = \Delta(a_{n-1}\Delta y_{n-1}) + b_ny_n$. The discrete version of Green's formula is given as follows.

LEMMA D. *(See [1,3].) Let $\{y_n\}_{n=0}^{M+1}$ and $\{z_n\}_{n=0}^{M+1}$ be two sequences. Then we have*

$$\sum_{n=1}^{M} z_nLy_n - y_nLz_n = \{a_nW[z_n, y_n]\}_{n=0}^{M},$$

where $W[z_n, y_n]$ is called the Casoratian of z_n and y_n, and is denoted by the determinant

$$W[z_n, y_n] = \begin{vmatrix} z_n & y_n \\ \Delta z_n & \Delta y_n \end{vmatrix}.$$

LEMMA E. *(See [7].) Let $N \in \mathbf{N}$ and $\{U_n\}_{n\in S}$ be a sequence such that*

$$U_n \geq 0, \qquad \forall\, n \geq N,$$

and

$$\Delta(a_{n-1}\Delta U_{n-1}) + b_{n-1}\Delta U_{n-1} + c_n^*U_n \leq 0, \qquad \forall\, n \geq N,$$

where

$$a_n, c_n^* > 0, \qquad \forall\, n \in \mathbf{S}.$$

If we assume

$$b_n < a_n, \qquad \forall\, n \in \mathbf{S},$$

then we have

(a)
$$U_n > 0, \qquad \forall\, n \geq N;$$

(b) *there exists $\{V_n\}_{n=N}^{\infty}$ such that*

$$V_n \geq U_n > 0, \qquad \forall\, n \geq N,$$

and

$$\Delta(a_{n-1}\Delta V_{n-1}) + b_{n-1}\Delta V_{n-1} + c_n^*V_n = 0, \qquad \forall\, n \geq N + 1.$$

3. OSCILLATION RESULT (I)

In this section, we shall follow [9] to discuss oscillation of solutions for the fourth-order partial differential equation (1.4). A nontrivial solution $u_{m,n}$ of (1.4) is said to be oscillatory if for every $N \in \mathbf{N}$, there exists $m_1, m_2 \in \boldsymbol{\Omega}$ and $n_1, n_2 \geq N$ such that $u_{m_1,n_1} u_{m_2,n_2} \leq 0$. Otherwise, it is nonoscillatory.

We shall obtain sufficient conditions for the oscillation of a nontrivial solution of (1.4) under the following conditions.

(B1) b is nonnegative in $\boldsymbol{\Omega} \times \mathbf{N} \times \mathbf{R}$.
(B2) There exists a constant $a > 0$ such that $A(m, n; u) \geq a$ in $\boldsymbol{\Omega} \times \mathbf{N} \times \mathbf{R}$.
(B3) There exists a constant $a_0 \geq 0$ such that $A_0(n; u) > a_0 \geq 0$ in $\mathbf{N} \times \mathbf{R}$.

THEOREM 3.1. *Consider the difference equation*

$$\Delta(\alpha_{n-1}\Delta V_{n-1}) + \beta_{n-1}\Delta V_{n-1} + \left(\lambda_1^2 a_0 \gamma_n + \lambda_1 a \delta_n\right) V_n = 0, \qquad \forall n \in \mathbf{N}, \tag{3.1}$$

where α_n, β_n, γ_n, δ_n are defined as in (1.4) and λ_1 is the smallest eigenvalue of system (2.1). Assume that (B1)–(B3) hold, and

$$\beta_n < \alpha_n, \qquad \forall n \in \mathbf{S}. \tag{3.2}$$

Then every nontrivial solution of system (1.4) is oscillatory provided that every solution of equation (3.1) is oscillatory.

PROOF. Suppose (1.4) has an eventually positive nontrivial solution $u_{m,n}$, say $u_{m,n} > 0, \forall m \in \boldsymbol{\Omega}$, $n \geq N$, for some $N \in \mathbf{N}$. We set

$$U_n = \sum_{m=1}^{M} u_{m,n}\phi_m^{(1)}, \qquad \forall n \in \mathbf{S}, \tag{3.3}$$

where $\phi_m^{(1)}$ is an eigenfunction of system (2.1) corresponding to its smallest eigenvalue λ_1. Then U_n is an eventually positive solution. If we multiply equation (1.4) by $\phi_m^{(1)}$ and sum from $m = 1$ to M, then the following expression is obtained:

$$\begin{gathered}
\sum_{m=1}^{M} \Delta_2(\alpha_{n-1}\Delta_2 u_{m,n-1})\phi_m^{(1)} + \sum_{m=1}^{M} \beta_{n-1}\Delta_2 u_{m,n-1}\phi_m^{(1)} \\
+ \sum_{m=1}^{M} \gamma_n A_0(n;u)\Delta_1^4 u_{m-2,n}\phi_m^{(1)} - \sum_{m=1}^{M} \delta_n \Delta_1^2(A(m-1,n;u)u_{m-1,n})\phi_m^{(1)} \\
+ \sum_{m=1}^{M} b(m,n,u_{m,n})u_{m,n}\phi_m^{(1)} = 0, \qquad \forall m \in \boldsymbol{\Omega}, \quad n \in \mathbf{N}.
\end{gathered} \tag{3.4}$$

By Lemma D, the boundary conditions

$$u_{0,n} = u_{M+1,n} = 0, \quad \Delta_1^2 u_{-1,n} = \Delta_1^2 u_{M,n} = 0, \qquad \forall n \in \mathbf{N}$$

and

$$\phi_0^{(1)} = \phi_{M+1}^{(1)} = 0,$$

we then have

$$\begin{aligned}
\sum_{m=1}^{M} \Delta_1^4 A_0(n;u) u_{m-2,n}\phi_m^{(1)} &= \sum_{m=1}^{M} \Delta_1^2 (A_0(n;u)\Delta_1^2 u_{m-2,n})\phi_m^{(1)} \\
&= \sum_{m=1}^{M} \Delta_1^2 A_0(n;u) u_{m-1,n}\Delta^2\phi_{m-1}^{(1)} + W\left[\phi_m^{(1)}, A_0(n;u)\Delta_1^2 u_{m-1,n}\right]_{m=0}^{M} \\
&= \sum_{m=1}^{M} \Delta_1^2 A_0(n;u) u_{m-1,n}\Delta^2\phi_{m-1}^{(1)} \\
&= -\lambda_1 \sum_{m=1}^{M} \Delta_1^2 A_0(n;u) u_{m-1,n}\phi_m^{(1)} \\
&= -\lambda_1 \left\{ \sum_{m=1}^{M} A_0(n;u) u_{m,n}\Delta^2\phi_{m-1}^{(1)} + W\left[\phi_m^{(1)}, A_0(n;u) u_{m,n}\right]_{m=0}^{M} \right\} \\
&= -\lambda_1 \sum_{m=1}^{M} A_0(n;u) u_{m,n}\Delta^2\phi_{m-1}^{(1)} \\
&= \lambda_1^2 \sum_{m=1}^{M} A_0(n;u) u_{m,n}\phi_m^{(1)}.
\end{aligned}$$

Hence, by Assumption (B3), we get

$$\sum_{m=1}^{M} \gamma_n \Delta_1^4 A_0(n;u) u_{m-2,n}\phi_m^{(1)} \geq \lambda_1^2 a_0 \gamma_n U_n. \tag{3.5}$$

Similarly, we see that

$$\begin{aligned}
&\sum_{m=1}^{M} \Delta_1^2 (A(m-1,n;u) u_{m-1,n})\phi_m^{(1)} \\
&\qquad = \sum_{m=1}^{M} A(m,n;u) u_{m,n}\Delta^2\phi_{m-1}^{(1)} + W\left[\phi_m^{(1)}, A(m,n;u) u_{m,n}\right]_{m=0}^{M} \\
&\qquad = \sum_{m=1}^{M} A(m,n;u) u_{m,n}\Delta^2\phi_{m-1}^{(1)} \\
&\qquad = -\lambda_1 \sum_{m=1}^{M} A(m,n;u) u_{m,n}\phi_m^{(1)}.
\end{aligned}$$

By Assumption (B2), we then obtain

$$\sum_{m=1}^{M} \delta_n \Delta_1^2 (A(m-1,n;u) u_{m-1,n})\phi_m^{(1)} \leq -\lambda_1 a \delta_n U_n, \qquad \forall\, n \in \mathbf{N}. \tag{3.6}$$

From (3.4)–(3.6), we get

$$\Delta(\alpha_{n-1}\Delta U_{n-1}) + \beta_{n-1}\Delta U_{n-1} + \left(\lambda_1^2 a_0 \gamma_n + \lambda_1 a \delta_n\right) U_n \leq 0.$$

By Lemma E, equation (3.1) has an eventually positive solution. Finally, by replacing $u_{m,n}$ with $-u_{m,n}$ if system (1.4) has an eventually negative solution, we obtain that equation (3.1) has an eventually negative solution.

Note that equation (3.1) can be written in the self-adjoint form

$$\Delta\left(P_{n-1}\Delta V_{n-1}\right) + Q_n V_n = 0, \qquad \forall n \in \mathbf{N},$$

where

$$P_n = \alpha_n R_n$$

and

$$Q_n = \left(\lambda_1^2 a_0 \gamma_n + \lambda_1 a \delta_n\right) R_n, \tag{3.7}$$

where

$$R_n = \prod_{i=0}^{n-1} \frac{\alpha_i}{\alpha_i - \beta_i}. \tag{3.8}$$

Therefore, by Theorems A and B, we get the following sufficient conditions for the oscillation of equation (1.4).

THEOREM 3.2. *Assume that (B1)–(B3) and (3.2) hold. If*

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n R_n} = \infty$$

and

$$\sum_{n=1}^{\infty} \left(\lambda_1^2 a_0 \gamma_n + \lambda_1 a \delta_n\right) R_n = \infty,$$

then every solution of (1.4) is oscillatory.

THEOREM 3.3. *Suppose that (B1)–(B3) and (3.2) hold. Assume that*

(C1) *there exists a nonnegative integer N such that*

$$\alpha_n R_n \leq 1, \qquad \forall n \geq N;$$

(C2)

$$\sum_{n=1}^{\infty} \left(\lambda_1^2 a_0 \gamma_n + \lambda_1 a \delta_n\right) R_n < \infty.$$

Let

$$H_n^{(0)} = \sum_{j=n}^{\infty} \left(\lambda_1^2 a_0 \gamma_j + \lambda_1 a \delta_j\right) R_j, \qquad \forall n \in \mathbf{N},$$

and

$$H_n^{(m)} = \sum_{j=n}^{\infty} j \left(\lambda_1^2 a_0 \gamma_j + \lambda_1 a \delta_j\right) R_j H_j^{(m-1)}, \qquad \forall m \geq 1.$$

Assume further that

(C3) *there exists a nonnegative integer k such that the series*

$$H_n^{(m)} < \infty$$

converges for $m = 1, 2, \ldots, k$, and

$$\sum_{j=1}^{\infty} j \left(\lambda_1^2 a_0 \gamma_j + \lambda_1 a \delta_j\right) R_j H_j^{(k)} = \infty.$$

Then every solution of (1.4) is oscillatory.

As a consequence of Theorems 3.2 and 3.3, we have the following result.

EXAMPLE 3.4. Consider the system

$$\Delta_2^2 u_{m,n-1} + \frac{1}{n^p}\Delta_1^4 u_{m-2,n} - \frac{1}{n^q}\Delta_1^2 u_{m-1,n} + u_{m,n}^3 = 0,$$
$$\forall m \in \{1,2,\dots,M\}, \qquad n \in \mathbf{N},$$
$$u_{0,n} = u_{M+1,n} = 0, \quad \Delta_1^2 u_{-1,n} = \Delta_1^2 u_{M,n} = 0, \qquad \forall n \in \mathbf{N}.$$

If $p \le 1$ or $q \le 1$, by Theorem 3.2, the system is oscillatory. Furthermore, since all assumptions of Theorem 3.3 are valid under the condition $1 < p < 2$ or $1 < q < 2$, hence, the system is oscillatory if $p < 2$ or $q < 2$.

EXAMPLE 3.5. The discrete analogue of the Woinowsky-Krieger nonlinear elastic model [10]

$$u_{tt} + \alpha u_{xxxx} - \left(\beta + k\int_0^L u_x^2\,dx\right) u_{xx} + u^{2p+1} = 0$$

is given by

$$\Delta_2(\alpha_{n-1}\Delta_2 u_{m,n-1}) + \gamma_n \Delta_1^4 u_{m-2,n} - \delta_n A(n;u)\Delta_1^2 u_{m-1,n} + (u_{m,n})^{2p+1} = 0,$$
$$\forall m \in \{1,2,\dots,M\}, \qquad n, p \in \mathbf{N},$$
$$u_{0,n} = u_{M+1,n} = 0, \quad \Delta_1^2 u_{-1,n} = \Delta_1^2 u_{M,n} = 0, \qquad \forall n \in \mathbf{N},$$

here

$$A(n;u) = \rho + \theta_n \sum_{m=1}^{M} (\Delta_1 u_{m,n})^2,$$

for $\rho > 0$ and $\theta_n \ge 0$. The oscillatory criteria can be obtained by Theorems 3.2 and 3.3.

4. OSCILLATION RESULT (II)

In this section, we shall use another kind of averaging method given in [11] to find some oscillation criteria for the following partial difference equation with different boundary conditions:

$$\begin{gathered}\Delta_2(\alpha_{n-1}\Delta_2 u_{m,n-1}) + \beta_{n-1}\Delta_2 u_{m,n-1} + \gamma_n \Delta_1^4 u_{m-2,n} \\ - \delta_n \Delta_1^2 \left(A(m-1,n;u) u_{m-1,n}\right) + \eta_n u_{m,n} + b(m,n,u_{m,n}) u_{m,n} = 0, \\ \forall m \in \mathbf{\Omega}, \qquad n \in \mathbf{N}, \\ u_{0,n} = u_{M+1,n} = 0, \quad \Delta_1 u_{0,n} = \Delta_1 u_{M,n} = 0, \qquad \forall n \in \mathbf{N},\end{gathered} \tag{4.1}$$

where $M \in \mathbf{N}$, $\mathbf{\Omega} = \{1,2,\dots,M\}$, $\mathbf{S} = \{0,1,2,\dots\}$, $\alpha_n, \gamma_n, \delta_n, \eta_n > 0$, $\beta_n \in \mathbf{R}$, $\forall n \in \mathbf{N}$.

LEMMA 4.1. *Let* $\{w_n\}_{n=0}^{M+1}$ *be a given sequence and let*

$$g_{i,j} = \begin{cases} \dfrac{(M+1-i)\,j}{M+1}, & 0 \le j \le i, \\ \dfrac{(M+1-j)\,i}{M+1}, & i \le j \le M+1. \end{cases} \tag{4.2}$$

Then we have

(i)
$$g_{i,j} \ge 0, \qquad \text{for } 0 \le i, j \le M+1,$$

(ii)
$$\sum_{j=1}^{M} g_{i,j}\Delta^2 w_{j-1} = -w_i + g_{i,M} w_{M+1} + g_{i,1} w_0, \tag{4.3}$$

(iii)

$$\sum_{j=1}^{M} g_{i,j}\Delta^4 w_{j-2} = -w_{i-1} + 2w_i - w_{i+1} + (-2g_{i,M} + g_{i,M-1})\, w_{M+1} \\ - g_{i,M}w_M - g_{i,1}w_1 + (g_{i,2} - 2g_{i,1})\, w_0. \tag{4.4}$$

PROOF. By Lemma D.

Hereafter, we shall find some sufficient conditions for the oscillation of a nontrivial solution of (4.1) under the following conditions:

(C1) b is a nonnegative function in $\mathbf{\Omega} \times \mathbf{N} \times \mathbf{R}$;
(C2) $\eta_n > 0$, $\forall\, n \in \mathbf{N}$;
(C3) $\beta_n < \alpha_n$, $\forall\, n \in \mathbf{N}$;
(C4) there exists a constant $a > 0$ such that $A(m,n\,;u) \geq a$ in $\mathbf{\Omega} \times \mathbf{N} \times \mathbf{R}$.

THEOREM 4.2. *Consider the difference equation*

$$\Delta(\alpha_{n-1}\Delta V_{n-1}) + \beta_{n-1}\Delta V_{n-1} + \eta_n V_n = 0, \qquad \forall\, n \in \mathbf{N} \tag{4.5}$$

Assume that (C1)–(C4) hold. Then every nontrivial solution of system (4.1) is oscillatory, if every solution of equation (4.5) is oscillatory.

PROOF. Suppose (4.1) has an eventually positive nontrivial solution $u_{m,n}$, say $u_{m,n} > 0$, $\forall\, m \in \mathbf{\Omega}$, $n \geq N$, for some $N \in \mathbf{N}$. Let

$$U_n = \sum_{t,m=1}^{M} g_{t,m}u_{m,n}, \qquad \forall\, n \in \mathbf{S}. \tag{4.6}$$

Then U_n is an eventually positive function. If we multiply equation (4.1) by $g_{t,m}$ and sum on m and t from 1 to M, then the following expression is obtained:

$$\begin{aligned} &\sum_{t,m=1}^{M} \Delta_2(\alpha_{n-1}\Delta_2 u_{m,n-1})g_{t,m} + \sum_{t,m=1}^{M} \beta_{n-1}\Delta_2 u_{m,n-1}g_{t,m} \\ &+ \sum_{t,m=1}^{M} \gamma_n \Delta_1^4 u_{m-2,n}g_{t,m} - \sum_{t,m=1}^{M} \delta_n \Delta_1^2(A(m-1,n;u)u_{m-1,n})g_{t,m} \\ &+ \sum_{t,m=1}^{M} \eta_n u_{m,n}g_{t,m} + \sum_{t,m=1}^{M} b(m,n,u_{m,n})u_{m,n}g_{t,m} = 0, \qquad \forall\, m \in \mathbf{\Omega}, \quad n \in \mathbf{N}. \end{aligned} \tag{4.7}$$

By Lemma 4.1 and the boundary condition

$$u_{0,n} = u_{M+1,n} = 0,$$

we have

$$\sum_{m=1}^{M} g_{t,m}\Delta_1^2\left(A(m-1,n;u)u_{m-1,n}\right) = -A(t,n;u)u_{t,n} \leq 0. \tag{4.8}$$

And from the boundary condition

$$\Delta_1 u_{0,n} = \Delta_1 u_{M,n} = 0, \qquad \forall\, n \in \mathbf{N},$$

we also have

$$\sum_{t,m=1}^{M} g_{t,m}\Delta_1^4 u_{m-2,n} = \sum_{t=1}^{M}(-u_{t-1,n} + 2u_{t,n} - u_{t+1,n}) \\ = \Delta_1 u_{0,n} - \Delta_1 u_{M,n} = 0. \tag{4.9}$$

Note that

$$\sum_{t,m=1}^{M} \eta_n u_{m,n} g_{t,m} = \eta_n U_n \tag{4.10}$$

and

$$\sum_{t,m=1}^{M} b(m,n,u_{m,n}) u_{m,n} g_{t,m} \geq 0. \tag{4.11}$$

In view of (4.7)–(4.11), we get

$$\Delta(\alpha_{n-1}\Delta U_{n-1}) + \beta_{n-1}\Delta U_{n-1} + \eta_n U_n \leq 0. \tag{4.12}$$

By Lemma E, equation (4.5) has an eventually positive solution. Finally, if system (4.1) has an eventually negative solution, replacing $u_{m,n}$ by $-u_{m,n}$, we obtain that equation (4.5) has an eventually negative solution.

Equation (4.5) can be written in the self-adjoint form

$$\Delta\left(P_{n-1}\Delta V_{n-1}\right) + Q_n V_n = 0, \qquad \forall n \in \mathbf{N},$$

where

$$P_n = \alpha_n R_n$$

and

$$Q_n = \eta_n R_n, \tag{4.13}$$

where R_n is given in (3.8).

Since Theorems A and B are applicable, hence, we get the following sufficient conditions for the oscillation of equation (4.1).

THEOREM 4.3. *Assume that (C1)–(C4) hold. If*

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n R_n} = \infty$$

and

$$\sum_{n=1}^{\infty} \eta_n R_n = \infty,$$

then every solution of (4.1) is oscillatory.

THEOREM 4.4. *Assume that (C1)–(C4) hold. Suppose that*

(D1) *there exists a nonnegative integer N such that*

$$\alpha_n R_n \leq 1, \qquad \forall n \geq N,$$

and

(D2)

$$\sum_{n=1}^{\infty} \eta_n R_n < \infty.$$

Let

$$H_n^{(0)} = \sum_{j=n}^{\infty} \eta_j R_j, \qquad \forall\, n \in \mathbf{N},$$

and

$$H_n^{(m)} = \sum_{j=n}^{\infty} j\eta_j R_j H_j^{(m-1)}, \qquad \forall\, m \geq 1.$$

Assume further that

(D3) *there exists a nonnegative integer k such that the series*

$$H_n^{(m)} < \infty$$

converges for $m = 0, 1, 2, \dots, k$, and

$$\sum_{j=1}^{\infty} j\eta_j R_j H_j^{(k)} = \infty.$$

Then every solution of (4.1) is oscillatory.

As a consequence of Theorems 4.3 and 4.4, we have the following result.

EXAMPLE 4.5. The system

$$\Delta_2^2 u_{m,n-1} + \delta_n \Delta_1^4 u_{m-2,n} - \gamma_n \Delta_1^2 u_{m-1,n} + \frac{1}{n^p} u_{m,n} + u_{m,n}^3 = 0,$$
$$\forall\, m \in \{1, 2, \dots, M\}, \qquad n \in \mathbf{N},$$
$$u_{0,n} = u_{M+1,n} = 0, \quad \Delta_1 u_{0,n} = \Delta_1 u_{M,n} = 0, \qquad \forall\, n \in \mathbf{N},$$

is oscillatory if $p < 2$ for any δ_n, $\gamma_n > 0$, for $n \in \mathbf{N}$.

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