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Economic design of product specifications for a complete inspection plan

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In a complete inspection plan, all outgoing items are subject to inspection, and the items failing to conform to the predetermined specifications are reworked so that their quality characteristics are exactly equal to the expected value (target value). This paper presents an economic model to determine the most profitable specifications for the complete inspection plan with the considerations of (*a*) economic loss caused by quality deviations, (*b*) rework cost, and (*c*) inspection cost. The solution to the cases in which the inspection cost is a constant provides important insights of the model. A detailed solution procedure is also given to a model in which the inspection cost is a linear function of the width of the specifications, the quality characteristic is normally distributed, and other loss and cost functions are assumed of a quadratic or linear form.

1. Introduction

In today's highly competitive business environment, producers receive an increasing demand from consumers to produce high quality products at low cost. Consumer's perception of quality is mainly obtained from the perceived performance as compared to the expected performance (Shaw 1982, Taguchi 1984). It is well-known that the items produced by the same production process vary in performance due to some inevitable random variations in materials, machine operations, and human operations. Many Japanese producers try to eliminate the quality variation by complete inspection and necessary rework so that the quality characteristics of the outgoing items are exactly equal to the expected value (target value). Their approach, although incurring high inspection and rework costs, yields high outgoing quality. Largely due to the advancement of robotics and automatic inspection equipment, the complete inspection scheme becomes feasible and attractive to many production processes. Several reports of successful applications have indicated that this approach is cost effective (Baird *et al.* 1982). As a result, the role of inspection in quality assurance is changing from passive maintenance (Dodge 1950) to aggressive alteration of the outgoing distribution of the quality characteristic.

Recent papers (Bisgaard *et al.* 1984, Carlesson 1984, Hunter and Kartha 1977) and several earlier papers (Bettes 1962, Burr 1967, Springer 1951) discuss the selection of the most profitable process mean for given product specifications. In their approach, all outgoing items are subject to an acceptance inspection; the items which conform to the predetermined specifications are accepted and sold at the regular price, and the rejected items are sold at reduced prices. Further, it is assumed that the quality characteristic is normally distributed with known variance, and the per-item manufacturing cost is a linear function of the quality

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characteristic. Inspection cost is not considered in their formulation; however, it is reasonable to assume that in their problems the cost is constant and does not affect the solutions. The most economic process mean is determined by maximizing the expected profit, which is a function of the expected selling price and the expected manufacturing cost. The models are typically applicable to the products of which the quality characteristics are a measure of weight, volume, number, or concentration, such as packages of sugar and canned food.

For many industrial products, there are often target values, for example, zero running error for watches and a specific output voltage for power circuits, and specification limits need to be set for inspection and control purposes. The payoff of using tight specifications is a high degree of consistency in product performance. However, this will be at the expense of (*a*) high costs associated with the disposition of the rejected items, such as reworking, scrapping, or downgrading, and (*b*) a high inspection cost due to the requirements of using more precise measuring instruments, longer processing time (Pryor 1982, Stover 1984), and skilled labour. Hence, the selection of specifications should be based on the balance of the outgoing quality level and the costs of rework and inspection. Grant and Leavenworth (1972) indicate that in practice product specifications are often set with little or no critical consideration of the various factors involved.

The purpose of this paper is to develop an economic model for selecting the most profitable specification limits for a given target value in a complete inspection plan. The paper is organized as follows. In the next section, the issues regarding the single summary measures of process quality level are raised and discussed, and it is shown that the commonly used percent defective rate is not a good quality measure. We introduce a value-loss function to describe the relationship between the market value of a product and its quality deviation from the target value. Then, a quality measure based on the value-loss function and the distribution of the quality characteristic is proposed. In § 3, we develop an economic model for the determination of the most profitable specification limits for a complete inspection plan. In § 4, the optimal solution is given to the cases in which inspection cost is a constant, and some important insights of the model are discussed. In § 5, we discuss the model in which the inspection cost is a linear function of the width (tightness) of the specification limits, the quality characteristic is normally distributed, the rework cost is a constant, and the value-loss function is quadratic. A detailed solution procedure for this model is given. Some discussion and extensions of the model are given in § 6.

2. Quality measures and value-loss function

Single summary measures of process quality level are often used in planning and evaluating a production process, thus the selection of such measures is critical to production decisions. By far the most common quality measure is the percent of the items failing to conform to the predetermined product specifications (percent defective rate) (Dodge and Romig 1959, Hald 1960). Taguchi (1981, 1984) used the television sets produced by a Japanese company as an example to illustrate that the percent defective rate is not a good quality measure. In Fig. 1, the solid line is the distribution of the colour density, the quality characteristic, of the television sets produced in a Japanese plant, and the dashed line represents that of a subsidiary plant located in the United States. Both plants used the same

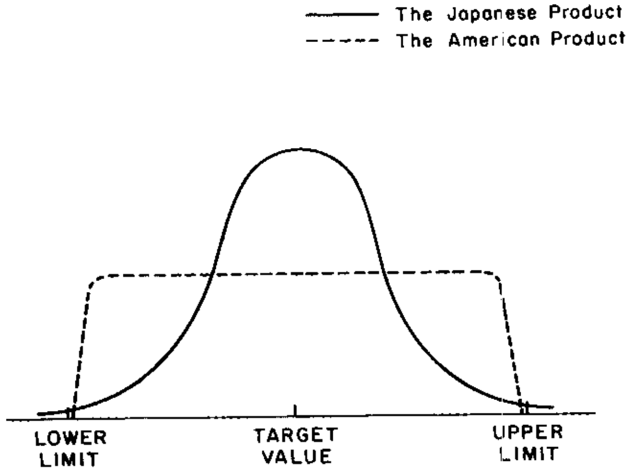

Figure 1

product specification limits as shown in the figure. The color density of the Japanese product was normally distributed with a small proportion falling outside the specification limits and that of the American product distributed evenly within the specification limits. Using the percent defective rate, the quality of the American product is rated higher than that of the Japanese product. However, a market survey (Taguchi 1984) indicated the opposite result. The reason for the failure of using the percent defective rate in this example is simple: the items conforming to the specifications are not necessarily good quality items, and only those items of which the colour densities are very close to the target value are good quality items. The figure shows that the Japanese plant produced a larger proportion of good quality items than the American plant. In other words, a larger proportion of the consumers buying the Japanese TV sets received good quality items than that of those buying the American TV sets.

The above discussion suggests that a quality measure should be based on the distribution of the quality deviation from the target value and the market reaction to the deviation as well. Furthermore, a quality measure is also expected to reflect changes, when occurring, in either or both the market conditions and the manufacturing technology. This concept is gradually accepted by practitioners, and may prevail in the near future. In the remaining of this section, we discuss the value-loss function and suggest a quality measure.

Let x be the quality characteristic of interest, and M be its target value. Assume that the item with x being exactly equal to M has a market value of P dollars. If the measured value of x associated with an item is not equal to M, the market value of the item suffers a loss due to the quality deviation. Let the loss be described by $L_1(v)$, the value-loss function, where v is the quality deviation $x - M$. Several $L_1(v)$'s have been used in literature. The most often seen $L_1(v)$'s are of a piecewise linear form. Examples are the cost of acceptance used in the bayesian attribute acceptance sampling (Hald 1960) and the producer's give-away cost function used in (Bisgaard *et al.* 1984, Carlesson 1984, Hunter and Kartha 1977). These two functions are illustrated in Fig. 2. The first function has been used in attribute inspection: an item is defective if its v is outside the interval $[a, b]$ and is non-defective otherwise. The cost (loss) caused by passing a defective

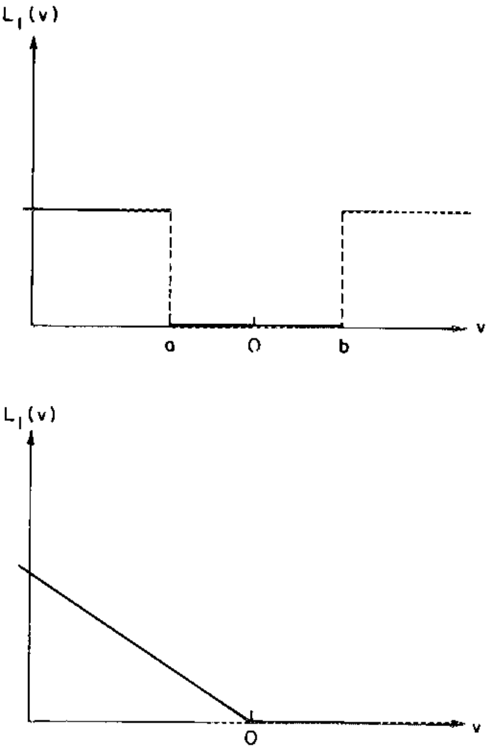

Figure 2

item is assumed to be a constant and the cost of passing a non-defective item is zero. The target value of this function is the interval $[a, b]$. The second function describes the situation where an item is sold at the regular price if its x exceeds a lower limit, and is sold at reduced prices otherwise. The price reduction is a linear function of the quality deviation between x and the lower limit. This function is often used when x is a measure of quantity, such as weight, volume, number, or concentration (Bisgaard *et al.* 1984).

Taguchi (1984) suggests the following quadratic loss function:

$$L_1(v) = kv^2 \tag{1}$$

where k is a positive constant. He obtained the quadratic function by expanding $L_1(v)$ into a Taylor's series, skipping the terms of an order higher than two, and applying the condition that $L_1'(0) = 0$. His argument of skipping the high-order terms does not seem reasonable; however, the loss function has been proved to be appropriate in many applications. Although Taguchi's quadratic loss function will be used in the later sections, at present, we do not specify $L_1(v)$ in order to obtain more general results. Nevertheless, throughout this paper $L_1(v)$ is assumed to be positive and non-decreasing in $|v|$.

Now, we propose the following quality measure:

$$Q_p = 1 - \int_v L_1(v) f(v) \, dv / P \tag{2}$$

where $f(v)$ is the probability density function of v. Essentially, Q_p takes into account the marketing conditions as well as the stochastic nature of the production process. As a result, this measure can reflect the changes in the manufacturing technology and the market conditions, such as consumer's quality consciousness, foreign competition, and government regulations. Thus, continuously updating the measure provides management with current economic information about the process quality level. Notice that the percent defective rate is simply a special case of Q_p when $L_1(v)$ takes a special form.

As an example, if $L_1(v)$ is Taguchi's quadratic loss function and the expected value of v is zero,

$$Q_p = 1 - k\sigma^2/P \tag{3}$$

where σ^2 is the variance of v. According to this quality measure, a process with a high variance is graded low. Particularly, in the TV example the American product had a higher variance in colour density than the Japanese product. Consequently, Q_p can adequately indicate the quality difference between the two products.

3. Improving outgoing quality by inspection

Complete inspection of important parts and outgoing items has been used in many production systems. There have been reports of successful implementations of complete variable inspection plans in Japanese manufacturing systems (Baird *et al.* 1982). Their approach measures the quality characteristic of every outgoing item, and reworks the item, if necessary, so that the quality characteristic of the item is exactly equal to the target value. The method is intended to achieve a high degree of consistency of product performance, and becomes increasingly attractive, particularly as the inspection cost and the rework cost are lowered by the advancement of robotics and automatic inspection instruments. A similar approach is complete acceptance inspection with predetermined product specifications, in which only the items failing to meet the specifications are reworked. The main difference between these two approaches is in the extent of alteration of the process distribution. As a matter of fact, the complete variable inspection approach can be viewed as a special case of the complete acceptance inspection approach when extremely tight specifications are used. Therefore, our discussion is based on the latter approach. As already discussed, the selection of the most profitable product specifications is based on the tradeoff between the outgoing quality and the costs of rework and inspection. The remainder of this section is devoted to the development of an economic model describing these relationships.

Consider in a continuous production process, each outgoing item is inspected to determine if it satisfies predetermined specifications. Let $\delta = (\delta_1, \delta_2)$ denote the specification limits of v, where δ_1 and δ_2 are the lower limit and the upper limit, respectively. Therefore, if the measured value of v associated with an item is smaller than δ_1 or larger than δ_2, the item is rejected and subject to rework, so that its v value is exactly equal to zero. In this section $f(v)$ is not given a specific form for the same reason as that for $L_1(v)$ given in the last section, that is, to keep the generality of the results. However, in § 5, the normal density function is used as $f(v)$.

The outgoing density function of v after inspection and rework is a mixture of a truncated $f(v)$ and a point distribution at 0, that is

$$\theta(v) = \begin{cases} f(v) & \delta_1 \leqslant v \leqslant \delta_2 \\ 1 - \int_{\delta_1}^{\delta_2} f(v)\, dv & v = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Since when $v = 0$ $L_1(v) = 0$, the outgoing quality level, a function of δ,

$$\begin{aligned} \theta Q_p(\delta) &= 1 - \int_v L_1(v)\theta(v)\, dv/P \\ &= 1 - \int_{\delta_1}^{\delta_2} L_1(v) f(v)\, dv/P \\ &= 1 - \int_{-\infty}^{\infty} L_1(v) f(v)\, dv/P \\ &\quad + \left[\int_{-\infty}^{\delta_1} L_1(v) f(v)\, dv + \int_{\delta_2}^{\infty} L_1(v) f(v)\, dv\right]\Big/P \\ &= Q_p + \left[\int_{-\infty}^{\delta_1} L_1(v) f(v)\, dv + \int_{\delta_2}^{\infty} L_1(v) f(v)\, dv\right]\Big/P \end{aligned} \tag{5}$$

It is clear, as indicated in the last expression, that the quality level is improved by eliminating the value-loss associated with the rejected items. The economic payoff of such improvement is $P \cdot (\theta Q_p(\delta) - Q_p)$. As expected, the economic payoff is larger as tighter specifications are used.

The cost of reworking an item may depend on its v value; let $L_2(v)$ denote the cost function which is also assumed to be positive and non-decreasing in $|v|$. Since only the rejected items are reworked, the per-item expected cost of rework is given by

$$\mathrm{CR}(\delta) = \int_{-\infty}^{\delta_1} L_2(v) f(v)\, dv + \int_{\delta_2}^{\infty} L_2(v) f(v)\, dv \tag{6}$$

Finally, the last component of the model is the cost of inspection. Since the cost may be a function of the tightness of the specifications (the magnitude of $\delta_2 - \delta_1$), we use $\mathrm{CI}(\delta)$ to denote the function. Consequently, the per-item expected net profit is

$$\mathrm{PR}(\delta) = \int_{-\infty}^{\delta_1} [L_1(v) - L_2(v)] f(v)\, dv + \int_{\delta_2}^{\infty} [L_1(v) - L_2(v)] f(v)\, dv - \mathrm{CI}(\delta) \tag{7}$$

The most profitable $\delta^* = (\delta_1^*, \delta_2^*)$ is obtained by maximizing the function $\mathrm{PR}(\delta)$. Note that $\mathrm{PR}(\delta^*)$ may be negative, indicating that complete inspection is not profitable, which may be caused by relatively high inspection and rework costs as compared to economic gain in quality level. Therefore, it is necessary to evaluate $\mathrm{PR}(\delta^*)$ to justify the use of the complete inspection scheme.

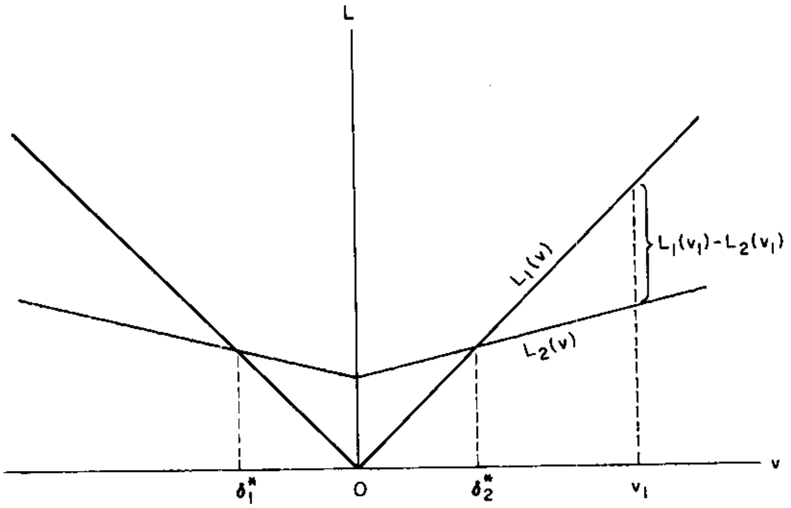

Figure 3

4. Constant inspection cost

The discussion in this section is based on the assumption that the inspection cost is the constant C_{I}. The model (7) becomes

$$\mathrm{PR}(\delta) = \int_{\infty}^{\delta_1} [L_1(v) - L_2(v)] f(v)\, dv + \int_{\delta_2}^{\infty} [L_1(v) - L_2(v)] f(v)\, dv - C_{\mathrm{I}} \qquad (8)$$

Setting the first derivatives of (8) with respect to δ_1 and δ_2 to be zeros, we derive the necessary conditions for the optimum δ^* as follows:

$$\left.\begin{array}{l} L_1(\delta_1^*) = L_2(\delta_1^*) \\ L_1(\delta_2^*) = L_2(\delta_2^*) \end{array}\right\} \qquad (9)$$

The result that δ^* is independent of $f(v)$ is interesting. Let us explain it graphically. In Fig. 3, straight lines are drawn to represent $L_1(v)$ and $L_2(v)$ for easy visualization. In this case, δ_1^* and δ_2^* are uniquely determined. Consider a given v, say v_1, the vertical distance between $L_1(v_1)$ and $L_2(v_1)$ is the per-item net profit when v is equal to v_1. It is clear that rework is not profitable when v is between δ_1^* and δ_2^*, since the rework cost is higher than the economic gain in quality. On the other hand, rework is profitable when v is smaller than δ_1^* or larger than δ_2^*. Consequently, we derive a general conclusion that the specification limits should be set in the way that rework is profitable to every rejected item, which is, in fact, exactly implied by eqn. (9). Note that $\delta_1^* = -\delta_2^*$ if both $L_1(v)$ and $L_2(v)$ are symmetric with respect to 0. The above discussions are based on the assumption that $L_1(v)$ and $L_2(v)$ intersect only at two points. When there exist multiple intersections, it may be economical to use multiple specification limits. That is to apply the conclusion drawn from the above discussion to determine the disposition of each region bounded by two adjacent intersection points. Incidentally, it should be mentioned that by the same reasoning, the complete variable inspection plan, represented by $\delta_1^* = \delta_2^* = 0$, should be used when $L_1(v)$ is larger than $L_2(v)$ for all v except at 0.

In some cases, the complete inspection approach may not be a true economic decision. For instance, in the TV example, if δ^* is looser than the specification limits used, complete inspection will not at all improve the outgoing quality level of the American plant. Therefore, it is necessary to consider $f(v)$ to justify the use of the complete inspection scheme.

5. Linear inspection cost function

The reasoning given in the last section does not directly apply to the cases in which the inspection cost is a function of the specification limits. In this section, we develop a solution procedure for a model in which the per-item cost of inspection is a linear function of the width of the specifications $\delta_2 - \delta_1$. Since the cost functions and $f(v)$ are symmetric with respect to zero, the optimal specifications are symmetric with respect to zero, that is $\delta_1^* = -\delta_2^*$. Hence the inspection cost can be written as:

$$\mathrm{CI}(\delta) = \begin{cases} S - s\delta_1 & \delta_1 \leqslant S/s \\ 0 & \delta_1 > S/s \end{cases} \tag{10}$$

where S and s are positive constants. For practical meanings, let us assume that $\mathrm{CI}(\delta)$ is positive in the range of interest. Additional assumptions of the model are: (*a*) $L_1(v)$ is represented by Taguchi's quadratic loss function, (*b*) the per-item rework cost is a constant r, and (*c*) v is normally distributed with mean 0 and variance σ^2.

Without loss of generality, we assume that $f(v)$ is the standardized normal density function $\phi(\cdot)$. When $\delta_1 \leqslant S/s$, the model (7) becomes

$$\mathrm{PR}(\delta) = 2k\int_{\delta_1}^{\infty} v^2\phi(v)\,dv - 2r\int_{\delta_1}^{\infty}\phi(v)\,dv - (S - s\delta_1) \tag{11}$$

The incomplete second moment $\int_{\delta_1}^{\infty} v^2\phi(v)\,dv$ needed for evaluating $\mathrm{PR}(\delta)$ is tabulated as Table 1. Note that, if s is very small, applying the result given in the last section, δ_1^* can be obtained by solving the equation $k\delta_1^{*2} = r$ which gives $\delta_1^* = \sqrt{(r/k)}$.

Differentiating eqn. (11) with respect to δ_1, we obtain

$$\mathrm{PR}'(\delta) = 2(r - k\delta_1^2)\phi(\delta_1) + s \tag{12}$$

which implies that the local optimum, if it exists, is greater than $\sqrt{(r/k)}$. The result makes intuitive sense, since rework may be no longer economical when v is equal to or slightly larger than $\sqrt{(r/k)}$ due to the inclusion of the decreasing inspection cost function.

The second derivative

$$\mathrm{PR}''(\delta) = 2\delta_1(k\delta_1^2 - (2k + r))\phi(\delta_1) \tag{13}$$

which indicates that $(2 + (r/k))^{1/2}$ is a point of inflection, and the function $\mathrm{PR}(\delta)$ is concave downward when δ_1 is smaller than $(2 + (r/k))^{1/2}$ and concave upward when δ_1 is larger than $(2 + (r/k))^{1/2}$. It suggests that the local optimum, if it exists, must be smaller than $(2 + (r/k))^{1/2}$.

Summing up the results, the local optimum δ_1^* is between $\sqrt{(r/k)}$ and $(2 + (r/k))^{1/2}$ provided that the optimum exists. To check the existence of δ_1^*, we need only to evaluate $\mathrm{PR}'(\delta)$ at $\delta_1 = (2 + (r/k)^{1/2})$. Since $\mathrm{PR}'(\delta)$ at $\delta_1 = 0$ is positive,

δ	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
0·0	0·5000	0·5000	0·5000	0·5000	0·5000	0·5000	0·5000	0·5000	0·4999	0·4999
0·1	0·4999	0·4998	0·4998	0·4997	0·4996	0·4996	0·4995	0·4994	0·4992	0·4991
0·2	0·4989	0·4988	0·4986	0·4984	0·4982	0·4980	0·4977	0·4974	0·4971	0·4968
0·3	0·4965	0·4961	0·4958	0·4954	0·4949	0·4945	0·4940	0·4935	0·4930	0·4925
0·4	0·4919	0·4913	0·4907	0·4900	0·4893	0·4886	0·4878	0·4871	0·4863	0·4854
0·5	0·4846	0·4837	0·4827	0·4818	0·4808	0·4798	0·4787	0·4776	0·4765	0·4754
0·6	0·4742	0·4730	0·4717	0·4704	0·4691	0·4678	0·4664	0·4650	0·4635	0·4621
0·7	0·4605	0·4590	0·4574	0·4558	0·4542	0·4525	0·4508	0·4490	0·4473	0·4454
0·8	0·4436	0·4417	0·4398	0·4379	0·4359	0·4339	0·4319	0·4299	0·4278	0·4257
0·9	0·4235	0·4214	0·4192	0·4169	0·4147	0·4124	0·4101	0·4078	0·4054	0·4030
1·0	0·4006	0·3982	0·3947	0·3933	0·3908	0·3882	0·3857	0·3831	0·3805	0·3779
1·1	0·3753	0·3727	0·3700	0·3673	0·3646	0·3619	0·3592	0·3564	0·3537	0·3509
1·2	0·3481	0·3453	0·3425	0·3396	0·3368	0·3340	0·3311	0·3282	0·3254	0·3225
1·3	0·3196	0·3167	0·3138	0·3109	0·3079	0·3050	0·3021	0·2992	0·2962	0·2933
1·4	0·2904	0·2874	0·2845	0·2816	0·2786	0·2757	0·2728	0·2698	0·2669	0·2640
1·5	0·2611	0·2582	0·2553	0·2524	0·2495	0·2466	0·2437	0·2408	0·2380	0·2351
1·6	0·2323	0·2294	0·2266	0·2238	0·2210	0·2182	0·2154	0·2127	0·2099	0·2072
1·7	0·2044	0·2017	0·1990	0·1964	0·1937	0·1910	0·1884	0·1858	0·1832	0·1806
1·8	0·1780	0·1755	0·1730	0·1704	0·1680	0·1655	0·1630	0·1606	0·1582	0·1558
1·9	0·1534	0·1510	0·1487	0·1464	0·1441	0·1418	0·1395	01373	0·1351	0·1329
2·0	0·1307	0·1286	0·1265	0·1244	0·1223	0·1202	0·1182	0·1161	0·1142	0·1122
2·1	0·1102	0·1083	0·1064	0·1045	0·1026	0·1008	0·0990	0·0972	0·0954	0·0937
2·2	0·0919	0·0902	0·0886	0·0869	0·0853	0·0836	0·0820	0·0805	0·0789	0·0774
2·3	0·0759	0·0744	0·0729	0·0715	0·0701	0·0686	0·0673	0·0659	0·0646	0·0632
2·4	0·0619	0·0607	0·0594	0·0582	0·0569	0·0557	0·0546	0·0534	0·0523	0·0511
2·5	0·0500	0·0489	0·0479	0·0468	0·0458	0·0448	0·0438	0·0428	0·0418	0·0409
2·6	0·0400	0·0391	0·0382	0·0373	0·0364	0·0356	0·0348	0·0340	0·0332	0·0324
2·7	0·0316	0·0309	0·0301	0·0294	0·0287	0·0280	0·0273	0·0266	0·0260	0·0253
2·8	0·0247	0·0241	0·0235	0·0229	0·0223	0·0218	0·0212	0·0207	0·0202	0·0196
2·9	0·0191	0·0186	0·0181	0·0177	0·0172	0·0168	0·0163	0·0159	0·0153	0·0150
3·0	0·0146	0·0143	0·0139	0·0135	0·0131	0·0128	0·0124	00121	0·0117	0·0114
3·1	0·0111	0·0108	0·0105	0·0102	0·0099	0·0096	0·0093	0·0091	0·0088	0·0086
3·2	0·0083	0·0081	0·0078	0·0076	0·0074	0·0072	0·0070	0·0068	0·0066	0·0064
3·3	0·0062	0·0060	0·0058	0·0056	0·0055	0·0053	0·0051	0·0050	0·0048	0·0047
3·4	0·0045	0·0044	0·0042	0·0041	0·0040	0·0039	0·0037	0·0036	0·0035	0·0034
3·5	0·0033	0·0032	0·0031	0·0030	0·0029	0·0028	0·0027	0·0026	0·0025	0·0024
3·6	0·0024	0·0023	0·0022	0·0021	0·0021	0·0020	0·0019	0·0019	0·0018	0·0017
3·7	0·0017	0·0016	0·0016	0·0015	0·0015	0·0014	0·0014	0·0013	0·0013	0·0012
3·8	0·0012	0·0011	0·0011	0·0011	0·0010	0·0010	0·0010	0·0009	0·0009	0·0009
3·9	0·0008	0·0008	0·0008	0·0007	0·0007	0·0007	0·0007	0·0006	0·0006	0·0006
4·0	0·0006	0·0005	0·0005	0·0005	0·0005	0·0005	0·0005	0·0004	0·0004	0·0004
4·1	0·0004	0·0004	0·0004	0·0003	0·0003	0·0003	0·0003	0·0003	0·0003	0·0003
4·2	0·0003	0·0003	0·0002	0·0002	0·0002	0·0002	0·0002	0·0002	0·0002	0·0002
4·3	0·0002	0·0002	0·0002	0·0002	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001
4·4	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001
4·5	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001	0·0001
4·6	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

Table 1. Incomplete second moment.

and if $PR'(\delta)$ is positive at $\delta_1 = (2 + (r/k))^{1/2}$ then $PR(\delta)$ is, in fact, a strictly increasing function. Due to the fact that $\text{limit}_{\delta_1 \to \infty} PR(\delta) = 0$, the complete inspection plan is not economical. On the other hand, if $PR'(\delta)$ is negative at $\delta_1 = (2 + (r/k))^{1/2}$ then there exists δ_1^* at which $PR'(\delta) = 0$. According to eqn. (11), δ_1^* is determined by k/s and r/s. Although a simple form solution is not

k/s	r/s 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
3	1·02	1·26	1·54	*	*	*	*	*	*	*	*	*	*	*	*
4	0·83	1·01	1·17	1·32	1·48	1·65	*	*	*	*	*	*	*	*	*
5	0·73	0·88	1·01	1·13	1·24	1·35	1·46	1·57	1·68	1·82	*	*	*	*	*
6	0·65	0·79	0·90	1·01	1·10	1·19	1·28	1·37	1·45	1·53	1·61	1·70	1·79	1·90	2·06
7	0·60	0·72	0·82	0·92	1·01	1·09	1·16	1·24	1·31	1·37	1·44	1·51	1·57	1·64	1·71
8	0·55	0·67	0·76	0·85	0·93	1·00	1·07	1·14	1·20	1·26	1·32	1·38	1·44	1·49	1·55
9	0·52	0·63	0·72	0·80	0·87	0·94	1·00	1·07	1·12	1·18	1·23	1·29	1·34	1·39	1·43
10	0·49	0·59	0·68	0·75	0·82	0·89	0·95	1·00	1·06	1·11	1·16	1·21	1·26	1·30	1·35
11	0·47	0·56	0·64	0·71	0·78	0·84	0·90	0·95	1·00	1·05	1·10	1·14	1·19	1·23	1·27
12	0·45	0·54	0·61	0·68	0·74	0·80	0·86	0·91	0·96	1·00	1·05	1·09	1·13	1·17	1·21
13	0·43	0·51	0·59	0·65	0·71	0·77	0·82	0·87	0·92	0·96	1·00	1·04	1·08	1·12	1·16
14	0·41	0·49	0·57	0·63	0·69	0·74	0·79	0·84	0·88	0·92	0·96	1·00	1·04	1·08	1·11
15	0·40	0·48	0·54	0·61	0·66	0·71	0·76	0·81	0·85	0·89	0·93	0·97	1·00	1·04	1·07
16	0·38	0·46	0·53	0·59	0·64	0·69	0·73	0·78	0·82	0·86	0·90	0·93	0·97	1·00	1·04
17	0·37	0·45	0·51	0·57	0·62	0·67	0·71	0·75	0·79	0·83	0·87	0·90	0·94	0·97	1·00
18	0·36	0·43	0·50	0·55	0·60	0·65	0·69	0·73	0·77	0·81	0·84	0·88	0·91	0·94	0·97
19	0·35	0·42	0·48	0·54	0·58	0·63	0·67	0·71	0·75	0·78	0·82	0·85	0·88	0·91	0·94
20	0·34	0·41	0·47	0·52	0·57	0·61	0·65	0·69	0·73	0·76	0·80	0·83	0·86	0·89	0·92
21	0·33	0·40	0·46	0·51	0·55	0·60	0·64	0·68	0·71	0·74	0·78	0·81	0·84	0·87	0·90
22	0·32	0·39	0·45	0·50	0·54	0·58	0·62	0·66	0·69	0·73	0·76	0·79	0·82	0·85	0·87
23	0·32	0·38	0·44	0·48	0·53	0·57	0·61	0·64	0·68	0·71	0·74	0·77	0·80	0·83	0·85
24	0·31	0·37	0·43	0·47	0·52	0·56	0·59	0·63	0·66	0·70	0·73	0·75	0·78	0·81	0·84
25	0·30	0·37	0·42	0·46	0·51	0·55	0·58	0·62	0·65	0·68	0·71	0·74	0·77	0·79	0·82
26	0·30	0·36	0·41	0·46	0·50	0·54	0·57	0·60	0·64	0·67	0·70	0·72	0·75	0·78	0·80
27	0·29	0·35	0·40	0·45	0·49	0·52	0·56	0·59	0·62	0·65	0·68	0·71	0·74	0·76	0·79
28	0·29	0·34	0·39	0·44	0·48	0·52	0·55	0·58	0·61	0·64	0·67	0·70	0·72	0·75	0·77
29	0·28	0·34	0·39	0·43	0·47	0·51	0·54	0·57	0·60	0·63	0·66	0·68	0·71	0·73	0·76
30	0·28	0·33	0·38	0·42	0·46	0·50	0·53	0·56	0·59	0·62	0·65	0·67	0·70	0·72	0·75

k/s	r/s 16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
3	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
4	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
5	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
6	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
7	1·78	1·85	1·93	2·03	*	*	*	*	*	*	*	*	*	*	*
8	1·60	1·66	1·71	1·77	1·82	1·88	1·95	2·02	2·11	*	*	*	*	*	*
9	1·48	1·53	1·58	1·62	1·67	1·71	1·76	1·81	1·86	1·91	1·96	2·02	2·08	2·15	*
10	1·39	1·43	1·47	1·51	1·56	1·60	1·64	1·68	1·72	1·76	1·80	1·84	1·88	1·92	1·97
11	1·31	1·35	1·39	1·43	1·47	1·50	1·54	1·58	1·61	1·65	1·68	1·72	1·75	1·79	1·82
12	1·25	1·29	1·32	1·36	1·39	1·43	1·46	1·50	1·53	1·56	1·59	1·63	1·66	1·69	1·72
13	1·19	1·23	1·26	1·30	1·33	1·36	1·40	1·43	1·46	1·49	1·52	1·55	1·58	1·61	1·64
14	1·15	1·18	1·21	1·25	1·28	1·31	1·34	1·37	1·40	1·43	1·45	1·48	1·51	1·54	1·56
15	1·10	1·14	1·17	1·20	1·23	1·26	1·29	1·32	1·34	1·37	1·40	1·43	1·45	1·48	1·50
16	1·07	1·10	1·13	1·16	1·19	1·22	1·24	1·27	1·30	1·32	1·35	1·37	1·40	1·42	1·45
17	1·03	1·06	1·09	1·12	1·15	1·18	1·20	1·23	1·25	1·28	1·30	1·33	1·35	1·38	1·40
18	1·00	1·03	1·06	1·09	1·11	1·14	1·17	1·19	1·22	1·24	1·26	1·29	1·31	1·33	1·36
19	0·97	1·00	1·03	1·06	1·08	1·11	1·13	1·16	1·18	1·21	1·23	1·25	1·27	1·30	1·32
20	0·95	0·98	1·00	1·03	1·05	1·08	1·10	1·13	1·15	1·17	1·19	1·22	1·24	1·26	1·28
21	0·92	0·95	0·98	1·00	1·03	1·05	1·07	1·10	1·12	1·14	1·16	1·19	1·21	1·23	1·25
22	0·90	0·93	0·95	0·98	1·00	1·03	1·05	1·07	1·09	1·11	1·14	1·16	1·18	1·20	1·22
23	0·88	0·91	0·93	0·96	0·98	1·00	1·02	1·05	1·07	1·09	1·11	1·13	1·15	1·17	1·19
24	0·86	0·89	0·91	0·93	0·96	0·98	1·00	1·02	1·04	1·06	1·09	1·11	1·12	1·14	1·16
25	0·84	0·87	0·89	0·91	0·94	0·96	0·98	1·00	1·02	1·04	1·06	1·08	1·10	1·12	1·14
26	0·83	0·85	0·87	0·90	0·92	0·94	0·96	0·98	1·00	1·02	1·04	1·06	1·08	1·10	1·12
27	0·81	0·83	0·86	0·88	0·90	0·92	0·94	0·96	0·98	1·00	1·02	1·04	1·06	1·08	1·09
28	0·80	0·82	0·84	0·86	0·88	0·90	0·92	0·94	0·96	0·98	1·00	1·02	1·04	1·06	1·07
29	0·78	0·80	0·83	0·85	0·87	0·89	0·91	0·93	0·95	0·96	0·98	1·00	1·02	1·04	1·05
30	0·77	0·79	0·81	0·83	0·85	0·87	0·89	0·91	0·93	0·95	0·97	0·98	1·00	1·02	1·04

* Complête inspection is not economical.

Table 2. Optimal specification limit δ_1^*.

available, an approximate δ_1^* can be found easily by numerical methods. A direct search algorithm, known as the Bisection Method (Conte and Boor 1980) is used in this study. Essentially, the algorithm locates the root of $\mathrm{PR}'(\delta) = 0$ in a sequence of intervals of decreasing size. The initial interval used in the search is $(\sqrt{(r/k)},\ (2 + (r/k)^{1/2})$. The time for the search is just a fraction of a second. Table 2 gives the results of a wide range of combinations of the cost parameters. The program will be kept available for those who are interested in the problems with the cost parameters outside the range of the table. Since at the end point $\text{limit}_{\delta_1 \to \infty}\ \mathrm{PR}(\delta) = 0$, δ_1^* is the only candidate for the optimal specification limit. Therefore, if $\mathrm{PR}(\delta^*)$ is positive, δ^* represents the optimal specifications, and if $\mathrm{PR}(\delta^*)$ is negative, the complete inspection scheme is not economical. Note that when $\delta_1 > S/s$, $\mathrm{CI}(\delta)$ is zero. Therefore, if δ_1^* is found to be greater than S/s, δ_1^* is not the optimum. In this case, if $\sqrt{(r/k)} > S/s$, the true optimum $\delta_1^* = \sqrt{(r/k)}$, otherwise δ_1^* is at the boundary S/s.

Now, we use an example to illustrate the use of Tables 1 and 2 in making the most economic inspection decision.

Example 1. In a complete inspection plan, an automatic test equipment (ATE) is used to determine the position of a pointer relative to a marking on a gauge. The deviation between the pointer and the marking follows the standard normal distribution with a unit of 0·01 mm. To determine whether or not an item conforms to the specifications, if tighter inspection specifications are used, it requires more complex computer softwares and longer processing time to establish the ATE's knowledge base as well as to produce more clear machine vision (image). Thus the cost associated with the inspection process is higher when tighter specifications are used. Suppose that the inspection cost can be approximated by the linear function (10) with $S = \$10$ and $s = \$1/0{\cdot}01$ mm. For an item with a deviation, the position of the pointer can be adjusted to the target value with a cost of \$32. Further, the quality loss can be adequately described by the quadratic function (1) with $k = \$16/(0{\cdot}01\ \text{mm})^2$. We compute $k/s = 8$ and $r/s = 16$ and find that the corresponding δ_1^* from Table 2 is $1{\cdot}60 \times 0{\cdot}01$ mm. According to the inspection plan, approximately 11% of the outgoing items need to be reworked. The values of incomplete second moment and the cumulative distribution function found in Table 1 are 0·2323 and 0·0548, respectively. Hence, the expected per-item gain in quality is \$7·4336 ($2 \times 16 \times 0{\cdot}2323$), and the expected per-item rework cost is \$3·5072 ($2 \times 32 \times 0{\cdot}0548$). The per-item inspection cost is \$6·8. The per-item expected net profit is \$7·4336-\$3·5072-\$6·8 = \$$-2{\cdot}8736$, which indicates that the complete inspection approach is not economical.

Now, we generalize the above results by relaxing the assumption about the variance of v being 1. It is not difficult to show the following two properties:

$$(a)\quad \int_{\delta_1}^{\infty} kv^2 f(v)\, dv = \int_{\delta_1/\sigma}^{\infty} k' t^2 \phi(t)\, dt \qquad \text{where } k' = k\sigma^2 \tag{14}$$

and

$$(b)\quad \int_{\delta_1}^{\infty} r\,.\,f(v)\, dv = \int_{\delta_1/\sigma}^{\infty} r' \phi(t)\, dt \qquad \text{where } r' = r\sigma \tag{15}$$

The two properties enable Table 2 to be used in the cases that the standard deviation of v is not one. What is needed is to convert k, r, and s into the unit of

k/s	r/s 1–30			31–60			61–90		
	Maximum error	Minimum error	Average error	Maximum error	Minimum error	Average error	Maximum error	Minimum error	Average error
1–30	43	3	10	19	3	5	16	3	5
31–60	34	3	7	4	2	2	3	1	2
61–90	34	2	7	3	1	2	2	1	1

Note: Errors are in %.

Table 3. Summary statistics for approximation error.

standard deviation. Then, the δ_1^* found in Table 2 is also in terms of standard deviations, and $\delta_1^* \cdot \sigma$ gives the specification limit in the original unit. Let us illustrate the procedure in Example 2.

Example 2. Suppose in Example 1 that $k = \$5/(0{\cdot}01 \text{ mm})^2$, $r = 7/0{\cdot}01$ mm, $S = \$5$, $s = \$1/0{\cdot}0$ mm and the mean and the standard deviation of r are 0 and 2, respectively. We obtain $k' = \$5/(0{\cdot}01 \text{ mm})^2 \cdot (2 \times 0{\cdot}01 \text{ mm}/\sigma)^2 = \$20/\sigma^2$, and $r' = \$7/0{\cdot}01 \text{ mm} \cdot (2 \times 0{\cdot}01 \text{ mm}/\sigma) = \$14/\sigma$. The σ_1^* found in Table 2 is $0{\cdot}89\sigma$ (or $1{\cdot}78 \times 0{\cdot}01$ mm). Similar computations as used in Example 1 can be used to evaluate the plan characteristics. According to the inspection plan, 37·34% of items need to be reworked. The expected per-item rework cost is $5·2276, the expected per-item gain in quality is $17·28, and the per-item inspection cost is $3·22. The per-item expected net profit is $8·5804. Therefore inspection with $\delta_1^* = 1{\cdot}98 \times 0{\cdot}01$ mm is the most economical policy.

It may not be so evident, however it is not unexpected, that δ_1^* in Table 2 approximates $\sqrt{(r/k)}$ when k/s and r/s are large. It is of practical convenience to use $\sqrt{(r/k)}$ as an approximate solution. Table 3, giving some summary statistics of the relative approximation error $(\delta_1^* - \sqrt{(r/k)})/\delta_1^*$, indicates that $\sqrt{(r/k)}$ provides a very good approximation when both ratios k/s are r/s are larger than 30.

6. Discussion

We have developed an economic model to determine the most profitable specifications for a complete inspection plan. Solution procedures are derived for the case where the inspection cost is constant as well as the case where the inspection cost is a linear function of the width of the specifications. Although in the latter case several assumptions have been made with respect to the detailed forms of the loss function and cost functions used, the model can be easily modified to deal with more general cases. The model can be also applied to purchasing decisions where options are available to inspect and rework incoming parts or products of different prices and quality levels. An interesting and useful extension of this model would be the inclusion of additional alternatives with respect to the disposition of the rejected items such as scrapping.

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Dans un plan d'inspection complet, tous les articles sortants font l'objet d'une inspection et ceux qui ne sont pas conformes aux specifications prédéterminées sont refaçonnés de sorte à ce que leurs caractéristiques sur le plan de la qualité correspondent parfaitement à la valeur attendue (valeur visée). Cet article présente un modèle économique afin de déterminer les spécifications les plus avantageuses pour un plan d'inspection complet avec les considérations de (*a*) pertes financières dues à des écarts qualitatifs (*b*) coût de refaçon (*c*) coût d'inspection. La solution aux cas où le coût d'inspection est une constante fournit des aperçus importants du modèle. Une procédure de solution détaillée est fournie à un modèle dans lequel le coût d'inspection est une fonction linéaire de la largeur des spécifications, la caractéristique de qualité est normalement distribuée et les autres fonctions de pertes et de coût sont posées comme étant de forme quadratique ou linéaire.

Bei einem Vollprüfplan werden alle fertigen Artikel einer Gütekontrolle unterworfen. Alle Artikel, die den vorgegebenen technischen Daten nicht entsprechen, werden nachbearbeitet, bis ihre Qualitätsmerkmale dem erwarteten Wert (Zielwert) entsprechen. In dieser Abhandlung wird ein ökonomisches Modell vorgestellt, mit dem sich optimale Prüfvorschriften für einen Vollprüfplan aufstellen lassen, der die folgenden Aspekte berücksichtigt: (*a*) Den wirtschaftlichen Verlust bei Qualitätsabweichungen; (*b*) die Nachbearbeitungskosten und (*c*) die Prüfkosten. Die Lösung für die Fälle, bei denen die Prüfkosten konstant sind, trägt viel zum Verständnis des Modells bei. Ferner wird ein detailliertes Lösungsverfahren für ein Modell gegeben, bei dem die Prüfkosten eine lineare Funktion des Umgangs der Gütevorschriften sind, das Qualitätsmerkmal normal verteilt ist und vorausgesetzt wird, daß die Form aller anderen Verlust- und Kostenfunktionen quadratisch oder linear ist.