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MULTIATTRIBUTE BAYESIAN ACCEPTANCE SAMPLING PLANS UNDER NONDESTRUCTIVE INSPECTION*

KWEI TANG, ROBERT PLANTE AND HERBERT MOSKOWITZ
College of Business Administration, Louisiana State University, Baton Rouge, Louisiana 70803
Krannert Graduate School of Management, Purdue University, West Lafayette, Indiana 47907

A methodology for determining optimal sampling plans for Bayesian multiattribute acceptance sampling models is developed. Inspections are assumed to be nondestructive and attributes are classified as scrappable or screenable according to the corrective action required when a lot is rejected on a given attribute. The effects of interactions among attributes on the resulting optimal sampling plan are examined and show that: (1) sampling plans for screenable attributes can be obtained by solving a set of independent single attribute models, (2) interactions of scrappable attributes on screenable attributes and conversely result in smaller sample sizes for screenable attributes than in single attribute plans, and (3) interactions among scrappable attributes result in either smaller sample sizes, lower acceptance probabilities or both, relative to single attribute plans. An iterative subproblem algorithm is developed, which is effective in finding near optimal multiattribute sampling plans having a large number of attributes.
(ACCEPTANCE SAMPLING—MULTIATTRIBUTE; DECISION ANALYSIS; STATISTICS—SAMPLING)

1. Introduction

Dodge and Romig's pioneering work (1929) on sampling inspection established a strong foundation for model development in the scientific selection of acceptance sampling plans for single attribute inspection. Due to the simplicity of their model and its ease in practical implementation, the method has been extended and widely used. Recently, however, economically based (Bayesian) sampling plans have drawn considerable attention (e.g., Guenther 1971, Hald 1960). Unlike traditional approaches which emphasized classical statistical concepts such as maintenance of an acceptable quality level (AQL), the Bayesian approach explicitly considers the costs associated with decisions to accept or reject inspection lots. An optimal sampling plan is selected by minimizing an expected loss function usually based on three cost components; (1) cost of inspection, (2) cost associated with a rejected lot, and (3) costs incurred for each defective item encountered in an accepted lot.

It is more often the case that a lot requires inspection on more than one attribute. In Dodge and Romig (1929) this is accomplished by considering an item as defective if it possesses a defect on one or more attributes, regardless of the number of defects. This in effect treats a multiattribute problem as if it were a single attribute problem. Consequently, the relative importance among attributes is ignored by this model. However, economic models require explicit assessments of the economic consequences associated with *each* attribute. Economic multiattribute acceptance sampling plans have been formulated in Ailor et al. (1975), Schmidt and Bennett (1972), and most recently in Moskowitz et al. (1984). These studies proposed multiattribute models which associate separate inspection, rejection and acceptance costs with each attribute.

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Due to the discrete nature of the optimization problem a discrete search algorithm, based on an extended pattern search, was developed in Moskowitz et al. (1984) and was shown to be efficient for up to three attributes. Further, it was shown that inspection plans could be obtained independently for each attribute provided that a rejected lot is screened on each of the attributes which precipitated rejection of the lot. However, if rejection on one or more attributes causes the lot to be scrapped, then an optimal multiattribute inspection plan must be obtained by considering the cost consequences of all attributes simultaneously.

We extend the work in Moskowitz et al. (1984) through an investigation of the interactions among attributes and the effect of these interactions on an optimal inspection plan. An optimization algorithm is then developed which utilizes subproblem solutions in conjunction with knowledge of these interaction effects. The algorithm is shown to be effective for obtaining multiattribute inspection plans for a larger number of attributes than was previously possible.

2. Assumptions and Classification of Attributes

Here we present the assumptions used in our study, and give a classification scheme for attributes which is based on the disposition of rejected lots (Schmidt and Bennett 1972). Suppose there is an incoming lot consisting of N items for multiattribute inspection. A random sample of n_i items is drawn from the lot for inspection on the ith attribute. If the number of defective items in the sample x_i exceeds the acceptance number c_i, the lot is rejected on attribute i, otherwise the lot is accepted on that attribute. The nature of inspections on all attributes is assumed to be nondestructive, and inspections on each attribute are carried to completion regardless of the inspection outcome on the other attributes (Ailor et al. 1975, Moskowitz et al. 1984, Schmidt and Bennett 1972). Furthermore, the occurrences of attributes on an item are assumed statistically independent (Ailor et al. 1975, Moskowitz et al. 1984, Schmidt and Bennett 1972). Based on the disposition of rejected lots two classes of attributes are assumed:

A. *Scrappable Attributes*. Rejection results in scrapping an entire lot or returning an entire lot to the supplier.

B. *Screenable Attributes*. Rejection results in sorting the uninspected items or downgrading the value of a lot.

If the lot is rejected on one or more scrappable attributes, the lot is scrapped or returned to the supplier. Otherwise, corrective actions, if any, are in effect for the screenable attributes on which lot rejections are called. In the next section we discuss the impact of this classification scheme on single attribute models and then on multiattribute models.

3. Single Attribute Models

We first discuss the costs inherent in a sampling plan for both scrappable and screenable attributes. We then develop the single attribute models for both classes of attributes. Properties of these single attribute models, such as cost sensitivity, are investigated. These properties will prove useful in our development and examination of the multiattribute models.

3.1. *Cost*

The economic consequences associated with a sampling plan essentially consist of three costs; inspection, lot acceptance and lot rejection (Hald 1960, Moskowitz et al. 1984). Inspection cost is that incurred by the examination of a random sample and is proportional to the size of the sample. The cost of acceptance is that caused by defective items encountered in accepted lots. Acceptance cost, then, is directly associated with the number of defective items in an accepted lot. The cost of rejecting a lot depends on the corrective action taken on a rejected lot. For screenable attributes, the cost is associated with identifying defective items in the uninspected portion of the lot. Therefore, the cost is proportional to the number of uninspected items.

If a lot is downgraded or sold at a reduced price, the loss, generally, depends on the quality level of the lot. To keep the model simple, the per-item cost due to downgrading or price reduction is assumed to be constant. Since inspected samples are free of defectives (no inspection error is assumed), the cost due to downgrading or price reduction is proportional to the number of uninspected items. As a result, attributes which, upon rejection, result in downgrading the value of a lot are considered to be screenable attributes. Finally, for scrappable attributes the cost is the investment on the entire lot, which is the maximum possible loss for attribute rejection. This cost is proportional to the lot size. We now incorporate these costs in the development of the acceptance sampling model.

3.2. *The Model*

An inspection lot of size N is drawn from a production process having a quality level p where the variation of p from lot to lot is described by a density function $f(p)$. It is assumed that the production process is under binomial control, such that the distribution of the number of lot defectives X is

$$h(X) = \int_0^1 \binom{N}{X} p^X (1-p)^{N-X} f(p)\,dp. \tag{1}$$

The number of sample defectives x is a random variable described by the conditional distribution $t(x \mid X)$, such that the marginal distribution of x can be obtained by

$$g(x) = \sum_{X=0}^{N} t(x \mid X) h(X). \tag{2}$$

Various forms of prior distributions of p have been proposed (Chin and Wetherill 1975, Hald 1960). Throughout this study a beta distribution Beta (α, β) is used as $f(p)$. The selection of the conditional distribution $t(x \mid X)$ is primarily based on probabilistic and computational considerations. Three commonly-used distributions are the hypergeometric, binomial and poisson mass functions. We advantageously use the hypergeometric distribution as the conditional distribution of x, since the distributions of X and x then fall into a well-known family of "reproducible distributions" (Hald 1960) in which X and x have the same distribution but with different parameter values. Specifically, $h(X)$ is Beta-Binomial with parameters N, α and β while $g(x)$ is also Beta-Binomial but with parameters n, α and β. Other distributions with this property can be found in Hald (1960).

An important characteristic of a sampling plan (n, c), the probability of acceptance, can now be presented as

$$p(n,c) = \sum_{x=0}^{c} g(x). \tag{3}$$

The cost of acceptance is determined by the product of the per-item cost of acceptance A and the number of defective items in the uninspected portion of the lot. Averaging this cost over all possible values of X and $x \leqslant c$, the expected cost of acceptance is

$$EA(n,c) = \sum_{x=0}^{c} \sum_{X=0}^{N} A(X-x) t(x \mid X) h(X). \tag{4}$$

The cost of rejection is the product of the per-item cost of rejection R and the number of uninspected items. Hence, we obtain the expected cost of rejection by

$$ER(n,c) = \sum_{x=c+1}^{n} \sum_{X=0}^{N} R(N-n)t(x \mid X)h(X)$$

$$= R(N-n) \sum_{x=c+1}^{n} g(x)$$

$$= R(N-n)(1-p(n,c)). \tag{5}$$

The cost of inspection is the product of sample size and the per-item cost of inspection S, that is nS. The expected total cost of the model for a screenable attribute is then

$$ETC = EA(n,c) + ER(n,c) + nS. \tag{6}$$

The only difference between the model for a scrappable attribute and that for a screenable attribute is the cost of rejection. The cost of rejection for a scrappable attribute is proportional to the lot size. Consequently, the expected total cost for a scrappable attribute can be obtained by substituting the following expected cost of rejection in (6)

$$ER(n,c) = RN(1-p(n,c)) \tag{7}$$

where R denotes the per-item rejection cost for scrapping or returning a lot item.

3.3. *Cost Sensitivity of an Optimal Single Attribute Sampling Plan*

We state three important properties that characterize the sensitivity of an optimal single attribute sampling plan with respect to the cost parameters, S and R, which will prove useful in analyzing the multiattribute model.

We first state the following two intuitively plausible properties of an optimal single attribute sampling plan (proofs in Appendix I):

Property 1. An optimal sample size is nonincreasing in S for both scrappable and screenable attributes. This simply means that as sampling costs increase, the number of items sampled decreases.

Property 2. The probability of acceptance of an optimal sampling plan is nondecreasing in R for scrappable attributes. This means that as the cost of scrapping a lot increases, the probability of accepting the lot increases.

A simultaneous change in R and S does not imply that a combined result of Property 1 and Property 2 must apply. If we increase the per-item cost of inspection and reduce that of rejection, it is not necessarily economical to take a smaller sample and also reduce the likelihood of accepting the lot. In fact, it is sometimes even impossible to make such a change from the original sampling plan. For example, suppose that the original sample plan calls for the inspection of two items and lot acceptance if none of the items are found defective. In such a case, we cannot reduce the sample size and acceptance probability simultaneously. However, it is economical to reduce the sample size or the acceptance probability or both in response to a simultaneous change in S and R (proofs in Appendix II). Hence we have:

Property 3. For a simultaneous change in S and R, at least one of the following statements is true for scrappable attributes:

1. The optimal sample size is nonincreasing in S.
2. The probability of acceptance associated with an optimal sampling plan is nondecreasing in R.

We have shown the qualitative (directional) sensitivity of an optimal single attribute acceptance sampling plan to changes in the cost parameters R and S for scrappable attributes. However, due to the discrete nature of n and c an optimal sampling plan is very robust with respect to changes in the parameters R and S (Tang 1984). This robustness property becomes important in analyzing the response surface of the multiattribute model developed subsequently.

We have completed the development and analysis of the single attribute models for both scrappable and screenable attributes. The models and the results are now used as the basis for developing and analyzing the multiattribute model in the following sections.

4. Multiattribute Acceptance Sampling Plan

We first develop a multiattribute acceptance sampling model. We then establish, through an analysis of attribute interactions, weak necessary conditions for an optimal multiattribute acceptance sampling plan.

4.1. *The Multiattribute Model*

For convenience, let Ω and Φ denote the index-sets of scrappable attributes and screenable attributes respectively. Due to the independence assumption about the p_i's, the joint distribution of the X_i's is simply the product of the $h_i(X_i)$'s. Similarly, the joint distribution of the X_i's and x_i's is

$$\prod_i P_i = \prod_i t_i(x_i \mid X_i) h_i(X_i), \qquad \text{where} \tag{8}$$

$$P_i = \text{the joint marginal probability of } x_i \text{ and } X_i .$$

The probability of accepting the lot on all the scrappable attributes $P(\Omega)$, an important characteristic of the model, is then given by

$$P(\Omega) = \prod_{i \in \Omega} \sum_{x_i = 0}^{c_i} \sum_{X_i} P_i = \prod_{i \in \Omega} p_i(n_i, c_i). \tag{9}$$

In determining the total cost of the model, the outcomes of inspection can be aggregated into the following two mutually exclusive events:

(A) One or more rejections are called on the scrappable attributes.

(B) No rejection is called on the scrappable attributes.

For event (A), the decision cost is the total investment on the lot, that is NR. Hence the total expected cost associated with (A), $ER(\Omega)$, is

$$ER(\Omega) = RN[1 - P(\Omega)]. \tag{10}$$

In event (B), the lot is accepted on all the scrappable attributes, and corrective actions are carried out for those screenable attributes on which lot rejections are called. Thus, the costs in (B) are the cost of acceptance on the scrappable attributes and the cost of acceptance or rejection on the screenable attributes.

The cost of acceptance on the ith scrappable attribute is determined by the product of the per-item cost of acceptance A_i and the number of items possessing the ith attribute in the uninspected portion of the lot $X_i - x_i$. Notice that this cost is incurred only when the lot is accepted on all the scrappable attributes. Therefore the expected cost of accepting the lot on the ith scrappable attribute $EA_i(\Omega)$ is given by

$$EA_i(\Omega) = \sum_{j \in \Omega} \sum_{x_j = 0}^{c_j} \sum_{X_j} A_i(X_i - x_i) \prod_{k \in \Omega} P_k$$

$$= EA_i(n_i, c_i) \prod_{j \in \Omega, j \neq i} p_j(n_j, c_j). \tag{11}$$

Hence, the total expected cost of acceptance for the scrappable attributes $EA(\Omega)$ is

$$EA(\Omega) = \sum_{i \in \Omega} EA_i(n_i, c_i) \prod_{j \in \Omega, j \neq i} p_j(n_j, c_j). \tag{12}$$

Since the cost of accepting the lot on a screenable attribute is incurred only when the lot is accepted on all the scrappable attributes, the expected cost of acceptance on the ith screenable attribute $EA_i(\Phi)$ is

$$\begin{aligned} EA_i(\Phi) &= \sum_{j \in \Omega} \sum_{x_j=0}^{c_j} \sum_{X_j} \sum_{x_i=0}^{c_i} \sum_{X_i} A_i(X_i - x_i) P_i \prod_{k \in \Omega} P_k \\ &= EA_i(n_i, c_i) P(\Omega). \end{aligned} \tag{13}$$

Then, the total expected cost of acceptance for the screenable attributes $EA(\Phi)$ is

$$EA(\Phi) = \sum_{i \in \Phi} EA_i(n_i, c_i) P(\Omega). \tag{14}$$

Similarly, the total expected cost of rejection for the screenable attributes $ER(\Phi)$ is given by

$$ER(\Phi) = \sum_{i \in \Phi} ER_i(n_i, c_i) P(\Omega). \tag{15}$$

The last cost component to be considered is the total cost of inspection. Since inspection on the sample selected for each attribute is carried to completion, the total cost of inspection I is

$$I = \sum_i n_i S_i. \tag{16}$$

Consequently, the expected total cost of the multiattribute model is the sum of equations (10), (12), (14), (15) and (16), i.e.,

$$ETC = EA(\Omega) + ER(\Omega) + EA(\Phi) + ER(\Phi) + I. \tag{17}$$

4.2. *Analysis of Attribute Interactions*

Three types of interactions occur among attributes that affect an optimal multiattribute sampling plan: (1) the interactions among screenable attributes, (2) the interaction between classes of scrappable attributes and screenable attributes, and (3) the interaction among scrappable attributes. As previously mentioned the investigation of these interactions results in weak necessary conditions for an optimal multiattribute sampling plan.

4.2.1. *Interaction Among Screenable Attributes.* Consider the following model which consists entirely of screenable attributes:

$$ETC = \sum_{i \in \Phi} \left[EA_i(n_i, c_i) + ER_i(n_i, c_i) + (n_i S_i) \right]. \tag{18}$$

Clearly, the model can be separated into a set of independent single-attribute models. An optimal multiattribute sampling plan can thus be obtained by determining an optimal sampling plan on each of the attributes independently (Moskowitz et al. 1984). Hence, for the case described above no interaction exists among screenable attributes in determining an optimal multiattribute sampling plan.

4.2.2. *Interaction Between Scrappable Attributes and Screenable Attributes.* For purposes of illustration and without loss of generality, we will consider the following two-attribute model in which the first attribute is scrappable and the second attribute is screenable. From (17) we have

$$ETC = EA_1(n_1, c_1) + ER_1(n_1, c_1) + n_1 S_1 + \left[EA_2(n_2, c_2) + ER_2(n_2, c_2)\right] p_1(n_1, c_1) + n_2 S_2. \tag{19}$$

The following equivalent expression shows the influence of the scrappable attribute on the selection of the sampling plan for the screenable attribute.

$$ETC = EA_1(n_1, c_1) + ER_1(n_1, c_1) + n_1 S_1 + p_1(n_1, c_1)\left[EA_2(n_2, c_2) + ER_2(n_2, c_2) + n_2 S_2 / p_1(n_1, c_1)\right]. \tag{20}$$

Equation (20) indicates that the probability of accepting the scrappable attribute becomes a factor that essentially increases the "effective" cost of inspection of the screenable attribute ($n_2 S_2 / p_1(n_1, c_1) \geqslant n_2 S_2$). It is apparent that, as the probability of accepting the scrappable attribute becomes smaller, inspection on the screenable attribute becomes more costly than the corresponding expected decision losses for that attribute $[EA_2(n_2, c_2) + ER_2(n_2, c_2)]$. Per Property 1 (§3.3) this results in a smaller optimal sample size for the screenable attribute. Thus the economic sample sizes for screenable attributes in an optimal multiattribute sampling plan which includes scrappable attributes are equal to or less than those which do not.

On the other hand, expression (19) indicates that the decision costs of the screenable attribute are a multiplier of $p_1(n_1, c_1)$ and consequently increase the expected cost associated with the decision to accept the lot on the scrappable attribute. This becomes clearer if we express $p_1(n_1, c_1)$ in (19) as $1 - [1 - p_1(n_1, c_1)]$ and rewrite (19) as:

$$ETC = EA_1(n_1, c_1) + N\left\{R - (1/N)\left[EA_2(n_2, c_2) + ER_2(n_2, c_2)\right]\right\} \times \left[1 - p_1(n_1, c_1)\right] + n_1 S_1 + EA_2(n_2, c_2) + ER_2(n_2, c_2) + n_2 S_2. \tag{21}$$

Equation (21) suggests that the presence of the screenable attribute reduces the per-item rejection cost of the scrappable attribute relative to the acceptance cost. Per Property 2, this results in a lower likelihood of accepting the lot on the scrappable attribute. It also implies that the probability of acceptance of a scrappable attribute will be further reduced by the introduction of new screenable attributes to the model. Furthermore (20) and (21) indicate that expected total cost increases as each additional attribute is introduced into the model.

4.2.3. *Interaction Among Scrappable Attributes.* Consider the following model consisting of two scrappable attributes.

$$ETC = EA_1(n_1, c_1) p_2(n_2, c_2) + EA_2(n_2, c_2) p_1(n_1, c_1) + RN\left[1 - p_1(n_1, c_1) p_2(n_2, c_2)\right] + n_1 S_1 + n_2 S_2. \tag{22}$$

Rewriting $1 - p_1(n_1, c_1) p_2(n_2, c_2)$ in (22) as $1 - p_2(n_2, c_2) + p_2(n_2, c_2)[1 - p_1(n_1, c_1)]$ gives

$$ETC = p_2(n_2, c_2)\Big[EA_1(n_1, c_1) + N\left\{R - EA_2(n_2, c_2)/\left[N p_2(n_2, c_2)\right]\right\}\left[1 - p_1(n_1, c_1)\right] + n_1 S_1 / p_2(n_2, c_2)\Big] + EA_2(n_2, c_2) + ER_2(n_2, c_2) + n_2 S_2. \tag{23}$$

Note that the influence of one scrappable attribute on another is, in fact, a combined influence of a scrappable attribute on a screenable attribute and a screenable attribute on a scrappable attribute; e.g., an increase in inspection cost and a decrease in rejection cost. Consequently, according to Property 3, either a smaller sample size, a tighter sampling plan, or both are taken for scrappable attributes.

Summarizing, we state two weak necessary conditions for an optimal multiattribute sampling plan:

1. An optimal sample size of a screenable attribute in a multiattribute model is equal to or less than that in a single attribute model for the same attribute.

2. A smaller sample size, a lower probability of acceptance, or both should be taken for a scrappable attribute in a multiattribute model than that in a single-attribute model for the same attribute.

5. Subproblem Heuristic

We first develop a heuristic solution procedure to obtain near optimal multiattribute acceptance sampling plans. We then illustrate the implementation of this procedure through use of a 4-attribute example problem.

5.1. *Algorithmic Development*

Due to the complexity of the model, there is no known algorithm, other than complete enumeration, that can be used in obtaining an optimal sampling plan. However, the response surface of the model in the neighborhood of an optimal solution is extremely flat with respect to ETC; thus it allows us to make use of a good heuristic algorithm to obtain near optimum solutions. The algorithm's concept is simple. The notion is to improve the objective function by changing the sampling plan of one attribute at a time until no improvement is possible. The solution provided in *each step* of the algorithm satisfies the weak necessary conditions for optimality discussed in the previous section.

If we seek an improvement on the objective function (17) by changing the sampling plan of a scrappable attribute, we have the following single-attribute problem (subproblem), a generalization of (21) and (23), for the ith scrappable attribute,

$$\mathrm{SUB}_i(\Omega) = EA_i(n_i, c_i) + N\left\{R - (1/N)\left[\sum_{j\in\Omega, j\neq i} EA_j(n_j, c_j)/p_j(n_j, c_j) + \sum_{k\in\Phi} EA_k(n_k, c_k) + ER_k(n_k, c_k)\right]\right\} \times\left[1 - p_i(n_i, c_i)\right] + n_i S_i \Big/ \prod_{j\in\Omega, j\neq i} p_j(n_j, c_j). \tag{24}$$

Similarly, the subproblem for the jth screenable attribute $\mathrm{SUB}_j(\Phi)$ is defined as

$$\mathrm{SUB}_j(\Phi) = EA_j(n_j, c_j) + ER_j(n_j, c_j) + n_j\left[S_j \Big/ \prod_{k\in\Omega} p_k(n_k, c_k)\right]. \tag{25}$$

The algorithm iteratively solves and updates the subproblem of each attribute until the solution converges. The initial conditions of the algorithm are established by letting all the cost components in the model be zero and letting the probability of each attribute's acceptance be equal to one. In the first iteration, initiated by solving a single attribute model, attributes are successively introduced. Once all attributes have been introduced, the algorithm recursively updates and solves the subproblem associated with each attribute until two successive iterations yield the same sampling plan for each attribute.

Each subproblem is solved using an extended discrete pattern search, which essentially adds a third exploratory search to the usual pattern search algorithm (Himmelblau 1972, Hook and Jeeves 1961) and maintains an integer step size at each iteration of the search procedure (Moskowitz et al. 1984). The sequence of introducing attributes affects the efficiency of the algorithm, particularly when the pattern search is used in solving the subproblems. Our experience is that it is more difficult to introduce scrappable attributes before screenable attributes when the origin is used as the starting point in each subproblem. The main reason for this is that the effort to find optimal sample sizes (and acceptance numbers) for screenable attributes is reduced in the first iteration. Furthermore, the sampling plan obtained in the previous iteration is used as the starting point for the current iteration. An example is now used to illustrate the procedure.

5.2. *An Illustrative Example*

Consider a lot with 100 items for four-attribute inspection. The model parameters are listed as follows:

Attribute	Type	α_i	β_i	S_i	A_i	R_i
1	Scrappable	1	9	$1.0	$10.0	$2.0
2	Scrappable	1	9	1.0	10.0	2.0
3	Screenable	1	7	0.2	2.0	0.3
4	Screenable	1	7	0.2	2.0	0.3

The initial conditions of the algorithm are set by letting all the cost components in the model be equal to zero and the probability of acceptance for each of the four attributes be equal to one. In the initial iteration of the algorithm, subproblem (24) for the first scrappable attribute is solved. The resulting sampling plan is (11, 3); the probability of acceptance is 0.932; and the costs of acceptance and rejection are $72.55 and $13.62 respectively. Subproblem (24) for the second scrappable attribute is updated by these results and then solved. The acceptance probabilities of the first attribute and the second attribute are used to update each of the subproblems (25) for the two screenable attributes. The procedure solves these two subproblems independently for the screenable attributes and starts the second iteration by considering again the first scrappable attribute. The procedure is repeated until two successive iterations yield the same sampling plans for all the attributes.

For this example, the proposed algorithm reached the optimal solution during the *second* iteration and terminated at the end of the third iteration. The search process and the results are shown as follows:

Iteration	n_1	c_1	p_1	n_2	c_2	p_2	n_3	c_3	p_3	n_4	c_4	p_4	ETC
1	11	3	0.932	9	1	0.765	15	2	0.705	15	2	0.705	$208.80
2	5	0	0.643	5	0	0.643	3	0	0.700	3	0	0.700	198.31
3	5	0	0.643	5	0	0.643	3	0	0.700	3	0	0.700	198.31

In a single attribute model, optimal sampling plans for the first and second scrappable attributes are identical, with a sampling plan of (11, 3) and an acceptance probability of 0.932. In the multiattribute model, however, the sample sizes and acceptance probabilities of the two attributes are 5 and 0.643 respectively (both are reduced). For screenable attributes, the sample sizes are reduced significantly from those of a single attribute model, but the probabilities of acceptance stay almost unchanged. It is interesting to point out that if the sampling plans for each attribute are obtained independently, the total resulting cost is \$219.0 in the multiattribute model, more than that obtained by solving for all attributes simultaneously (\$198.31).

6. Computational Results

A sample of 16 problems was used to evaluate the efficiency of the proposed algorithm. The results were concerned with the cost performance as well as the computational time required to reach final solutions. All the problems in the study were based on a four-attribute model whose parameters are listed in Table 1. Two levels of per-item inspection cost were set for each type of attribute. Through different combinations of these inspection costs 16 problems were constructed. In addition, for comparison purposes, the optimal single attribute acceptance sampling plan for each of the attributes is also listed. These single attribute plans were obtained using the difference algorithm in Moskowitz and Plante (1984) with CPU times ranging from 0.50 to 1.10 seconds.

The efficiency of the proposed subproblem algorithm was examined by using a multidimensional pattern search (Moskowitz et al. 1984) to also solve each of the 16 problems. Efficiency was measured by comparing the solution and the solution time of both algorithms.

Table 2 shows the results obtained by the two algorithms. The computer programs for both algorithms were written in FORTRAN IV and run on a CDC 6500/6600. The time required to solve each problem is reported in seconds of CPU time. The proposed algorithm obtained a smaller ETC in 13 out of the 16 cases, and an identical ETC in the remaining problems. The largest relative cost difference was as much as 9%. Moreover, the proposed algorithm generally needed less run time to reach final solutions when cost performances of the two algorithms were close. Also in three cases the pattern search obtained less economical solutions after consuming considerably more time. Finally, the proposed algorithm showed a fast rate of convergence in each of the problems. The required number of iterations was between two and four. A complete study of the convergence properties of the proposed algorithm is presented in Tang (1984) as well as the near optimal properties of solutions obtained by the proposed algorithm.

7. Summary

A general Bayesian multiattribute acceptance sampling model which accommodates various dispositions of rejected lots was developed. We also discuss a classification scheme of scrappable and screenable attributes which facilitates the design of such models. The analysis of interactions between and within these two classes of attributes

TABLE 1

Characteristics of a Four-Attribute Model (Lot Size = 100)

Attribute	Prior Parameters		Costs			Optimal Single Attribute Plans	
i	α_i	β_i	S_i	A_i	R_i	n_i	c_i
1	1	9	\$0.50	\$10.0	\$2.0	100	100
1	1	9	1.00	10.0	2.0	11	3
2	1	10	0.50	10.0	2.0	100	100
2	1	10	1.00	10.0	2.0	6	2
3	1	7	0.18	2.0	0.3	100	100
3	1	7	0.20	2.0	0.3	36	5
4	1	8	0.18	2.0	0.3	54	8
4	1	8	0.20	2.0	0.3	28	4

TABLE 2

A Comparison of Multiattribute Acceptance Sampling Plans for Two Search Algorithms

				Pattern Search										Subproblem Heuristic									
S_1	S_2	S_3	S_4	n_1	n_2	n_3	n_4	c_1	c_2	c_3	c_4	ETC ($)	CPU (Sec)	n_1	n_2	n_3	n_4	c_1	c_2	c_3	c_4	ETC ($)	CPU (Sec)
1.0	1.0	0.20	0.20	5	5	3	2	0	0	0	0	195.5	10.7	5	4	8	3	0	0	1	0	195.5	18.0
1.0	0.5	0.20	0.20	4	13	8	7	0	1	1	0	192.7	41.7	13	100	22	21	3	100	3	3	182.9	61.9
0.5	1.0	0.20	0.20	18	4	8	7	2	0	1	1	191.4	50.1	100	8	28	21	100	2	4	3	176.6	59.3
0.5	0.5	0.20	0.20	100	100	100	28	100	100	100	4	138.6	920.1	100	100	36	28	100	100	5	4	137.9	92.2
1.0	1.0	0.20	0.18	5	4	8	7	0	0	1	1	195.4	24.2	5	4	8	7	0	0	1	1	195.4	16.7
1.0	0.5	0.20	0.18	4	13	8	7	0	1	1	1	191.9	35.6	13	100	22	28	3	100	3	4	182.5	79.6
0.5	1.0	0.20	0.18	18	4	8	8	2	0	1	1	191.3	48.7	100	8	28	34	100	2	4	5	176.1	63.1
0.5	0.5	0.20	0.18	100	100	100	100	100	100	100	100	138.0	289.6	100	100	36	54	100	100	5	8	137.2	95.4
1.0	1.0	0.18	0.20	5	4	8	3	0	0	1	0	195.4	23.0	5	4	8	3	0	0	1	0	195.4	13.7
1.0	0.5	0.18	0.20	4	13	8	7	0	1	1	1	191.9	31.5	12	100	42	21	3	100	6	3	182.4	77.4
0.5	1.0	0.18	0.20	18	4	9	7	2	0	1	1	191.2	50.0	100	8	49	21	100	2	7	3	175.9	97.4
0.5	0.5	0.18	0.20	100	100	100	28	100	100	100	4	136.6	938.0	100	100	100	28	100	100	100	4	136.6	91.6
1.0	1.0	0.18	0.18	5	4	8	7	0	0	0	1	195.3	28.0	5	4	8	7	0	0	1	1	195.3	18.5
1.0	0.5	0.18	0.18	4	13	8	7	0	1	1	1	191.7	38.7	12	100	42	28	3	100	6	4	181.9	81.6
0.5	1.0	0.18	0.18	18	4	9	8	2	0	0	1	191.1	44.1	100	7	62	35	100	2	9	5	175.4	80.3
0.5	0.5	0.18	0.18	100	100	100	100	100	100	100	100	136.0	299.8	100	100	100	54	100	100	100	8	135.8	94.9

establishes optimal design principles of Bayesian multiattribute acceptance sampling plans. A subproblem heuristic algorithm was also developed and shown to be very efficient in obtaining near optimal multiattribute acceptance sampling plans for a larger number of attributes than was previously possible.[1]

Appendix I. Derivation of Properties 1 and 2

Let (n^1, c^1) be an optimal sampling plan for a critical attribute with $R = r$ and $S = s$. Further, let (n^2, c^2) be an optimal sampling plan for a scrappable attribute with $R = r - \Delta r$, $S = s + \Delta s$, where $\Delta r > 0$ and $\Delta s > 0$. We can now write

$$(n^1)s + EA(n^1, c^1) + Nr(1 - p(n^1, c^1)) \leqslant (n^2)s + EA(n^2, c^2) + Nr(1 - p(n^2, c^2)) \quad \text{and} \tag{I-1}$$

$$n^1(s + \Delta s) + EA(n^1, c^1) + N(r - \Delta r)(1 - p(n^1, c^1))$$
$$\geqslant n^2(s + \Delta s) + EA(n^2, c^2) + N(r - \Delta r)(1 - p(n^2, c^2)). \tag{I-2}$$

By subtracting each side of expression (I-1) from that of (I-2) we have

$$(n^1 - n^2)\Delta s \geqslant \Delta r(p(n^2, c^2) - p(n^1, c^1)). \tag{I-3}$$

Using (I-3) we can now state three important properties.

Letting $\Delta r = 0$ in (I-3), it is apparent that the optimal sampling plan for scrappable attributes is nonincreasing in S. This is also true for screenable attributes.

PROOF. Let (n^1, c^1) be the optimal sampling plan when the per-item inspection cost is s, and (n^2, c^2) be the optimal sampling plan when the per-item inspection cost is $s + \Delta s$. Then

$$n^1(s + \Delta s) + EA(n^1, c^1) + ER(n^1, c^1) \geqslant n^2(s + \Delta s) + EA(n^2, c^2) + ER(n^2, c^2). \tag{I-4}$$

Suppose $n^2 > n^1$; the inequality still holds if we delete the terms associated with Δs. We obtain

$$n^1 s + EA(n^1, c^1) + ER(n^1, c^1) \geqslant n^2 s + EA(n^2, c^2) + ER(n^2, c^2), \tag{I-5}$$

which contradicts the fact the sampling plan (n^1, c^1) is the optimal sampling plan when the per-item inspection cost is s. Therefore, we conclude n^2 must be equal to or less than n^1. Q.E.D.

Further, letting $\Delta s = 0$ in (I-3), it is evident that the acceptance probability of an optimal sampling plan for a scrappable attribute is nondecreasing in R.

Appendix II. Proof of Property 3

If $n^2 > n^1$, then according to (10), $p(n^2, c^2)$ must be less than $p(n^1, c^1)$. On the other hand, if $p(n^2, c^2) > p(n^1, c^1)$ then n^1 must be larger than n^2. In brief, both $n^2 > n^1$ and $p(n^2, c^2) > p(n^1, c^1)$ cannot be valid simultaneously in response to an increase in S and a decrease in R. Q.E.D.

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