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DESIGN OF PRODUCT SPECIFICATIONS FOR MULTI-CHARACTERISTIC INSPECTION*

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A product often requires inspection on more than one characteristic. The traditional method determines inspection specifications for each characteristic independently. This practice ignores the interactions among characteristics in determining the disposition of an item, and prohibits tradeoffs among the quality of characteristics. In this paper, two multi-characteristic screening (complete inspection) models are proposed with different information processing requirements. In both models, screening specifications are jointly determined by considering all the economic and stochastic factors associated with the characteristics of interest. However, in Model 1, each characteristic has separate screening specifications and the inspection results of conformance (acceptance or rejections) of all the characteristics are used to determine the disposition of an item. In the second model, a joint screening rule based on an aggregation of characteristics is used to allow direct tradeoffs among the quality of characteristics. To implement the second model, the exact measured values of all characteristics of an item have to be recorded and used for a decision on that item. These two models are formulated and the solution procedures are developed. A numerical study is used to compare the cost performance and other plan characteristics of the independently-determined single characteristic models and the two multi-characteristic models.

(TAGUCHI METHOD; PRODUCT SPECIFICATION; SCREENING; MULTI-CHARACTERISTICS)

1. Introduction

The field of quality control is radically different today from what it was just a few years ago. In particular, screening (100% inspection) becomes feasible and cost-effective due to the rapid growth in computer-controlled testing and inspection systems (Baird, Patel, Stitt and Mundel 1982). These automated inspection systems, which use laser, machine vision, pattern recognition, and other advanced techniques, offer not only accurate and consistent results at low operating costs but also better report preparation, automated calibration, and malfunction diagnostics (Pryor 1982, Stover 1984). As a matter of fact, in the modern self-correcting computerized numerical control (CNC) systems, inspection has become an inherent part for tracking and reporting incipient equipment or tool failures for corrective actions (Hill 1985, Stiles 1987). Moreover, some modern manufacturing systems, such as the flexible manufacturing system (FMS) and the just-in-time (JIT) system, have a trend toward smaller production lot sizes to reduce inventory costs, in which cases screening is more efficient than the traditional lot-by-lot sampling schemes in both controlling and providing early feedback of the quality of incoming materials and outgoing items. Consequently, design and implementation of screening procedures have drawn increased attention in the last several years.

It is well known that items produced by the same production process vary in performance due to some inevitable random variations in materials, machine operations and human operations. In a typical screening procedure, all the outgoing items are subject to acceptance inspection. If an item fails to conform to the predetermined screening

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specifications, it is rejected and subject to certain corrective actions. For example, a rejected item may be scrapped or reworked.

Recent papers (Bisgaard, Hunter and Pallesen 1984, Carlsson 1984, Hunter and Kartha 1977) and several earlier papers (Bettes 1962, Burr 1967, and Springer 1951) discussed the selection of the most profitable process mean for given product specifications. In their screening procedures, the accepted items are sold at the regular price, and the rejected items are sold at reduced prices. The manufacturing cost is a function of the performance variable. The most economic process mean is determined by maximizing the expected profit, which is a function of the expected selling price and the expected manufacturing cost. This model is typically applicable to the products with weight, volume, number or concentration as the most important quality characteristic.

For many industrial products, there are often target (ideal) values—for example, zero running error for watches and a specific output voltage for power circuits—and specification limits need to be set for inspection and control purposes. The payoff of using tight specifications is a high degree of consistency in product performance. However, this will be at the expense of high costs associated with the disposition of rejected items. Therefore the selection of the optimal screening specifications should be based on a balance of the outgoing quality level and the costs incurred by the corrective actions on the rejected items. Unfortunately, in practice, product specifications are often set with little or no critical consideration of the various factors involved (Grant and Leavenworth 1972).

Screening specifications can be designed on the basis of the performance variable of interest or a surrogate variable which is correlated with the performance variable. The practice of using a correlated variable in lieu of the performance variable has been widely found in electronics, machinery, food, medicine, and many other industries when measuring the performance variable is costly, time consuming or destructive. Statistically-based screening procedures select screening specifications to control the defective ratio (AOQ) of outgoing items. Owen, McIntire and Seymour (1975) developed useable tables under one-sided specifications and known distribution parameters. When the distribution parameters are unknown, the problem becomes an interesting but complicated one and often requires approximate solutions. There has been considerable research in this area, including the work by Owen and Boddie (1976), Owen and Su (1977), Thomas, Owen and Gunst (1977), Li and Owen (1979), Odeh and Owen (1980), Owen, Li and Chou (1981), Madsen (1982), and Wong, Meeker and Selwyn (1985).

However, the criterion AOQ used in these procedures has been shown to be a poor measure of quality level (Taguchi 1984, Kackar 1985, Leon, Shoemaker and Kackar 1987). In fact, as indicated by Landry (1976), statistical criteria are often determined by economic conditions. For example, a low AOQ is desired because the loss resulted from accepting defective items is large. Therefore it is more appropriate to directly consider economic factors in designing the screening specifications.

Tang (1987) developed a general economic model for one-sided screening procedure using a correlated variable. The cost components of the model include the cost incurred by the disposition of rejected items and the cost incurred by accepting imperfect items. This work has been extended to a two-sided screening procedure (Tang 1988c), a two-stage screening procedure (Tang 1988d), and a one-sided screening procedure using more than one correlated variable (Tang and Tang 1987).

All the previous studies are limited to single-characteristic inspection. However, it is more often the case that an item requires inspection on more than one characteristic, such as weight, dimensions and color. The traditional method selects screening specifications for each characteristic independently and then determines the disposition of the items using the inspection results of all the characteristic of interest. The two main drawbacks of this approach are that the interactions among the characteristics in determining the disposition of an item are ignored, and tradeoffs among the quality of the characteristics are prohibited. In this paper, two multi-characteristic models are proposed to overcome these problems. In both models, screening specifications are jointly determined by considering all the economic and stochastic factors associated with the characteristics of interest. However, in Model 1, each characteristic has separate screening specifications and only the inspection results of conformance (acceptance or rejection) of all the characteristics are needed to determine the disposition of an item. In Model 2, a joint screening rule based on an aggregation of characteristics is used to allow direct tradeoffs among the quality of the characteristics. To implement Model 2, the exact measured values of all characteristics of an item have to be recorded and used to determine the disposition of that item.

This paper assumes inspection is based on the performance variables. Nevertheless, the results of this paper can be extended to the situation where inspection is based on correlated variables. This paper is organized as follows: In the next section, a single-characteristic model is presented and the solution and the properties of the model are studied. In §3, Model 1 is formulated and the interactions among characteristics in determining the optimal screening specifications are analyzed. The results of the interaction analysis provide not only important insights of the model but also a basis for developing an iterative solution algorithm. In §4, Model 2 is formulated on the basis of a joint screening rule and the solution of the model is obtained. In §5, a numerical study is used to compare the independently-determined single characteristic models and the two proposed multi-characteristic models.

2. Single-Characteristic Model

If screening is directly based on the performance variable of interest, screening specifications of a single-characteristic model are easy to determine. There are two costs considered in this decision (Tang 1988a). The first cost is incurred by corrective actions taken on the rejected items. Typical corrective actions are repairing, scrapping or returning the items to the supplier. This cost is usually called the rejection cost. The second cost is the acceptance cost caused by imperfect quality when no corrective action is taken. This may include loss in sales and goodwill, warranty costs, handling costs, etc. Clearly, the rejection cost is more objective and easier to determine, but the acceptance cost which relates to consumer's perception of quality is difficult to assess. As a matter of fact, this cost is often determined conceptually.

In traditional inspection by attributes, an item is classified as defective if it fails to meet predetermined product specifications; otherwise, it is classified as a nondefective item. This practice implicitly assumes that consumers remain equally satisfied with the items conforming to the specifications and become completely dissatisfied with the items which do not conform to specifications. This assumption can be described by the step-loss function in Figure 1(a), and is actually embedded in many attribute sampling and process control methods, such as lot-by-lot attribute acceptance sampling (Dodge and Romig 1929, Hald 1960, Chiu and Wetherwill 1975, Tang, Plante and Moskowitz 1986), p-charts and np-charts (Duncan 1986).

Taguchi (1984) used empirical evidences to show that the step-loss function does not adequately reflect consumer's perception of quality. Instead, he suggests that for a quality characteristic, is a target (ideal) value and only at this value consumer is completely satisfied (Loss is zero). Any deviation from this target value would cause consumer dissatisfaction and result in an economic loss described by the following quadratic function (see Figure 1(b)):

$$L(y) = ky^2, \tag{1}$$

where k is a positive constant and y is the measured deviation of the characteristic from the target value. As explained by Scherkenback (1986), "The issue is not whether the

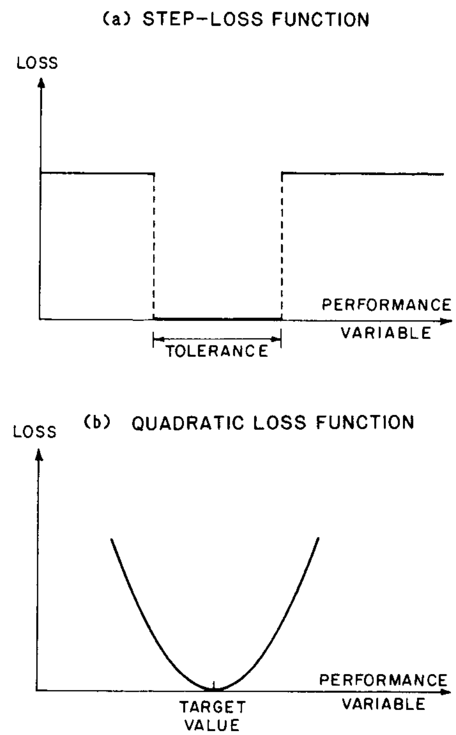

FIGURE 1. Two Loss Functions.

loss is exactly quadratic. The issue is that the quadratic model is a lot closer to the real world than the step-function." Nevertheless, it should be pointed out that using this function assumes that the consumer has decreasing risk aversion toward the quality deviation (Pratt 1964). This quality loss function, now, has been widely accepted and used in product design, process design and control, and many other areas (Kackar 1985, Leon, Shoemaker and Kackar 1987, Tang 1988b).

Let $[-\delta, \delta]$ be the acceptance region and an item is accepted if its y is in this interval. The per-item expected acceptance cost due to quality deviation is obtained by

$$EA = \int_{-\delta}^{\delta} ky^2 f(y)dy, \tag{2}$$

where $f(y)$ is the probability density function of y. It is assumed that y follows a normal distribution with mean 0 and variance σ^2. It can be shown that (Tang 1988d)

$$EA = k\{\sigma^2[2\Phi(\delta/\sigma) - 1] - 2\sigma\delta\phi(\delta/\sigma)\}, \tag{3}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the normal distribution and density functions, respectively.

The probability of rejection is

$$p(\text{rej}) = 1 - \int_{-\delta}^{\delta} f(y)dy$$
$$= 2[1 - \Phi(\delta/\sigma)]. \tag{4}$$

Let the per-item cost of rejection be r, then the expected per-item cost of rejection is

$$ER = rp(\text{rej}). \tag{5}$$

The total per-item expected cost is

$$ETC = EA + ER. \tag{6}$$

It can be shown (Tang 1988a) that the optimum specification limit δ^* is obtained by solving

$$L(\delta) = r. \tag{7}$$

In other words,

$$\delta^* = \sqrt{r/k}. \tag{8}$$

The result can be illustrated by Figure 2. Consider a given value of the deviation, say y_1, the vertical distance between the quadratic function and the rejection cost is the benefit of rejecting an item when the deviation is equal to y_1. Clearly, it is not economical to reject an item when y is between $-\delta^*$ and δ^* but it is economical to reject the items outside $[-\delta^*, \delta^*]$. Consequently, the optimal screening rule is to reject when $L(y) > r$, and $-\delta^*$ and δ^* are the specification limits for separating the acceptance and rejection regions. Notice that *the optimal specification limits are independent of the distribution of the performance variable*. Furthermore, as intuitively expected, when the step-loss function is used the optimal screening rule is simply to reject "defective" items and accept "nondefective" items. The following obvious result is very important to the analysis of the proposed multi-characteristic models:

Result 1. $p(\text{rej})$ is decreasing in r.

3. Multi-Characteristic Model 1

3.1. *Assumptions*

In this multi-characteristic model each characteristic has separate specifications $[-\delta_i, \delta_i]$. Based on these specifications, it is determined whether or not an item is rejected on that characteristic. According to the disposition of the rejected items, quality characteristics can be generally classified into two classes (Ailor, Schmidt and Bennett 1975, Tang, Plante and Moskowitz 1986):

(1) Scrappable characteristics. Rejection results in scrapping the item.

(2) Reworkable characteristics. Rejection results in reworking the item on the characteristics so that their values are exactly equal to the target values.

The item is scrapped if it is rejected on one or more scrappable characteristics; otherwise, it is reworked according to the inspection results of the reworkable characteristics. For convenience, let S and W denote the index-sets of the scrappable characteristics and the reworkable characteristics, respectively. For simplicity, the values of the characteristics associated with an item are assumed to be statistically independent. Furthermore, the

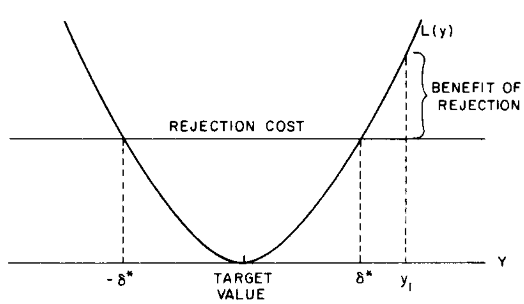

FIGURE 2. Single Characteristic Model.

total loss due to the quality deviations of the characteristics of interest $y_1, y_2 \cdots, y_n$ is assumed to be additive, i.e. the sum of the single characteristic (marginal) loss functions:

$$L_n(y_1, y_2 \cdots, y_n) = k_1 y_1^2 + k_2 y_2^2 + \cdots + k_n y_n^2. \tag{9}$$

3.2. *The Model*

In formulating the model, the outcomes of inspection can be aggregated into the following mutually exclusive events:

(A) One or more rejections are called on the scrappable characteristics.

(B) No rejection is called on the scrappable characteristics.

Due to the independence assumption, the probability of accepting all the scrappable characteristics is determined by

$$P(S) = \prod_{i \in S} p_i, \tag{10}$$

where p_i is the probability of accepting the ith characteristic. For event (A), since the item is scrapped the decision cost is the total investment on the item. The per-item expected cost associated with event (A) is

$$ER(S) = R[1 - P(S)], \tag{11}$$

where R is the per-item cost of scrapping an item.

In event (B), the item is accepted for shipment after rework is performed on the rejected reworkable characteristics. Three cost components should be considered: acceptance cost associated with the scrappable characteristics, acceptance cost associated with the accepted reworkable characteristics, and rework costs associated with the rejected reworkable characteristics. The expected acceptance cost associated with the scrappable characteristics is given by

$$EA(S) = \underset{\substack{y_j \in [-\delta_j, \delta_j] \\ j \in S}}{\int\int \cdots\cdots \int} [\sum_{i \in S} k_i y_i^2] \prod_{j \in S} f_j(y_j) dy_j, \tag{12}$$

where $f_j(y_j)$ is the p.d.f. of y_j. It can be shown that

$$EA(S) = \sum_{i \in S} [EA_i \prod_{j \in S, j \neq i} p_j], \tag{13}$$

where EA_i is the per-item expected acceptance cost of the ith scrappable characteristic in the single-characteristic model. Since the cost of acceptance associated with the reworkable characteristic is incurred only when all the scrappable characteristics are accepted, the expected acceptance cost associated with the reworkable characteristics is

$$EA(W) = (\sum_{i \in W} EA_i) P(S). \tag{14}$$

Similarly, the per-item expected rework cost associated with the reworkable characteristics is

$$ER(W) = [\sum_{i \in W} r_i (1 - p_i)] P(S), \tag{15}$$

where r_i is the per-item rework cost for the ith reworkable characteristic. Consequently, the per-item expected total cost of this model is

$$ETC = EA(S) + ER(S) + EA(W) + ER(W). \tag{16}$$

3.3. *Analysis of Attribute Interactions*

The focus of the interactions analysis is on the direction of changes of a multi-characteristic procedure as compared to the independently-determined single-characteristic procedures. The result of the analysis provides not only some important insights of the model but also the basis for developing the solution procedure.

The model (16) can be written as

$$ETC = EA(S) + ER(S) + [\sum_{i \in W} ETC_i]P(S), \tag{17}$$

where ETC_i is the expected total cost (6) associated with the ith reworkable characteristic in the single-characteristic model. Consequently, the optimal screening specifications of the reworkable characteristics are obtained by solving the single-characteristic models independently. In other words, the determination of the optimal screening specifications for a reworkable characteristic is independent of the presence of other characteristics. Of course, if all the characteristics of interest are the reworkable characteristics, the screening specifications are obtained by considering the characteristics independently.

Using $P(S) = 1 - [1 - P(S)]$, ETC can be expressed as

$$ETC = EA(S) + [R - \sum_{i \in W} ETC_i](1 - P(S)) + \sum_{i \in W} ETC_i. \tag{18}$$

Since ETC_i's are positive, the last expression indicates that the presence of the reworkable characteristics reduces the cost of rejecting the scrappable characteristics relative to the cost of acceptance. Let

$$R' = R - \sum_{i \in W} ETC_i. \tag{19}$$

For the ith scrappable characteristic, we use

$$1 - \prod_{j \in S} p_j = (1 - p_i)(\prod_{j \in S, j \neq i} p_j) + (1 - \prod_{j \in S, j \neq i} p_j) \tag{20}$$

to rewrite (18) as

$$ETC = \{EA_i + [R' - \sum_{j \in S, j \neq i} EA_j/p_j](1 - p_i)\} \prod_{l \in S, l \neq i} p_l$$
$$+ \sum_{j \in S, j \neq i} EA_j \prod_{l \in S, l \neq i, j} p_l + R'(1 - \prod_{j \in S, j \neq i} p_j) + \sum_{j \in W} ETC_j. \tag{21}$$

The second line of the last expression is actually a multi-characteristic model without the ith scrappable characteristic and the expression in the brackets { } of the first line is a single-characteristic model for the ith scrappable characteristic with a "revised" cost of rejection. This revised cost describes the effects of other characteristics on the ith scrappable characteristic. Based on Result 1, it is clear that the interaction among the scrappable characteristics is an increase in the probability of rejection (scrapping). Using Result 1 and (18), the effect of the reworkable characteristics on the scrappable characteristics is found to be also an increase in the probability of rejection.

We define the following single characteristic model as the subproblem for the ith characteristic:

$$\mathrm{Sub}_i = EA_i + [R' - \sum_{j \in S, j \neq i} EA_j/p_j](1 - p_i). \tag{22}$$

Note that the exact values of the characteristics need not be recorded in this method. Rather, only the results of acceptance inspection of the characteristics are used to determine the disposition of an item. It is interesting to point out that, as a result of this information structure, the effects of other characteristics on a characteristic are in terms of expected values (see (17), (18) and (21)) not the exact values of the characteristics. More specifically, the solution to the subproblem (22) is to scrap an item when

$$k_i y_i^2 + \sum_{j \in S, j \neq i} EA_j/p_j > R', \tag{23}$$

where EA_j/p_j is the per-item conditional expected acceptance cost $E[k_j y_j^2 \mid -\delta_j \leq y_j \leq \delta_j]$ associated with the jth scrappable characteristic. This optimal decision rule scraps an item when the sum of the acceptance cost caused by y_i and the total *expected* cost of accepting other characteristics is larger than the rejection cost R'. Since EA_j/p_j is non-negative, $\sqrt{R'/k_i}$ actually provides an upper bound for the optimal specification limit for the ith scrappable characteristic δ_i^*.

3.4. *Solution Procedure*

Based on the discussion in the last section, the solution obtained by the following iterative search algorithm will satisfy the first partial derivative condition for optimality:

Step 1. Obtain δ_i^* by using (8) for the reworkable characteristics.

Step 2. Evaluate ETC_i associated with the reworkable characteristics by using (6) and obtain R'.

Step 3. Iteratively solve and update the subproblem (22) for each scrappable characteristic until the solution converges.

The initial condition of Step 3 can be obtained by solving single-characteristic models independently using R' as the rejection cost (i.e., $\delta_i = \sqrt{R'/k_i}$). Since the solution of each subproblem in each iteration is actually given by (8) using "updated" EA_j/p_j, the algorithm consumes very little computing time even in a very large-sized problem. As mentioned, this algorithm provides approximate solutions which satisfy the first derivative conditions for optimality. The Hessian matrix can be derived by using (21) and evaluated numerically. The diagonal elements of the Hessian matrix are given by

$$\partial^2 ETC/\partial\delta_i^2 = 4 \prod_{\substack{l \in S \\ l \neq i}} p_l [k_i \delta_i f_i(\delta_i)], \tag{24}$$

and the off-diagonal elements of the matrix are

$$\partial^2 ETC/\partial\delta_i \partial\delta_j = 4 \prod_{\substack{l \in S \\ l \neq i,j}} p_l [(k_j \delta_j^2 - EA_j/p_j) f_i(\delta_i) f_j(\delta_j)]. \tag{25}$$

However, it is difficult to analytically show that the matrix is positive definite. Our extensive computational experience indicates that this algorithm does provide the optimal solutions and converges very fast in many problems we tested.

In the following illustrative example, various expected costs and other plan characteristics are evaluated by using FORTRAN and IMSL (International Mathematical and Statistical Libraries) subroutines in double precision on an IBM 3081 computer. The total computing time is only a fraction of a second.

3.5. *An Illustrative Example*

Consider a five-characteristic problem with the parameters given in Table 1, where δ_i and p_i are obtained by treating the characteristics independently. For simplicity, the variances of all the characteristics are assumed to be 1.0. The total expected cost associated

TABLE 1

Model Parameters Used in the Example

Characteristic	Type	k_i	R or r_i[a]	δ_i[b]	p_i[b]
1	Scrappable	1.5	6	2.00	0.955
2	Scrappable	1.5	6	2.00	0.955
3	Scrappable	1.5	6	2.00	0.955
4	Reworkable	1.2	2	1.29	0.803
5	Reworkable	1.2	2	1.29	0.803

[a] Scrapping cost R for Characteristics 1, 2, and 3; Rework cost r_i for Characteristics 4 and 5.

[b] Obtained by independent single characteristic models.

with the single-characteristic models is 5.236, the probability of scrapping an item is 13.04%, and the expected acceptance cost associated with the scrappable characteristics $EA(S)$ is 3.028. Using (19), R' is found to be 4.360. The search procedure stops when the differences of δ_i's between two successive iterations of all the scrappable characteristics are smaller than 0.0001. The search process and the results are shown as follows:

Iteration	δ_1	p_1	δ_2	p_2	δ_3	p_3
1	1.1657	0.617	1.3252	0.815	1.4384	0.850
2	1.3877	0.835	1.3758	0.831	1.3876	0.835
3	1.3876	0.835	1.3854	0.834	1.3858	0.834
4	1.3862	0.834	1.3863	0.834	1.3860	0.834
5	1.3860	0.834	1.3860	0.834	1.3860	0.834
6	1.3860	0.834	1.3860	0.834	1.3860	0.834

The algorithm stops after only six iterations. As expected, the probabilities of acceptance of the scrappable characteristics are, respectively, lower than those in the single-characteristic models. The percentage of scrapped items is 41.9% and the expected per-item acceptance cost associated with scrappable characteristics is 1.2876. The expected total cost associated with the optimal solution is 4.756, which is about 10.11% lower than that of the single-characteristic models. It is obvious that the higher cost of the single-characteristic models is caused by accepting too many poor quality items.

4. Multi-Characteristic Model 2

Model 1 essentially prohibits the direct tradeoffs among characteristics. For example, an item is scrapped if it is rejected on one scrappable characteristic even though the values of other characteristics are perfectly equal to the target values. Consequently, to allow possible tradeoffs among characteristics, joint specifications based on an aggregation of all the characteristics is desired.

For a given method of aggregating all the characteristics, the problem is reduced to a single characteristic problem. As shown in §2, the optimal screening specifications of a single-characteristic problem are determined by equating the loss caused by quality deviations and the cost of rejection, and are independent of the distribution of the aggregated variable. Consequently, a reasonable and promising method of aggregating the characteristics is based on the consumer's quality loss function of the characteristics. For illustration, we consider a two-characteristic problem with the following loss function

$$L_2(y_1, y_2) = k_1 y_1^2 + k_2 y_2^2. \quad (26)$$

Assume that both characteristics are scrappable characteristics, then the joint screening rule is simply to

$$\begin{aligned} &\text{reject} \quad \text{if} \quad L_2(y_1, y_2) > R, \\ &\text{accept} \quad \text{if} \quad L_2(y_1, y_2) \leqq R. \end{aligned} \quad (27)$$

Let the ellipse in Figure 3 be the contour of $L_2(y_1, y_2) = R$. According to the screening rule, all the items outside the ellipse should be rejected. It is interesting to compare this screening rule with that of Model 1. In the same figure, the inner rectangular area represents the acceptance region of Model 1. The acceptance cost at point "A" is lower than that at point "B". However, in Model 1, "A" is in the rejection region and "B" is in the acceptance region. Consequently, the proposed method should have some definite economic advantages over Model 1. Note that the area bounded by the outer rectangle is the acceptance region of the single-characteristic models.

Since the decision as to whether or not an item should be reworked has to be made

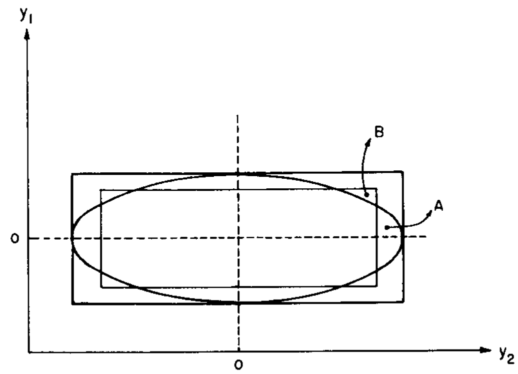

FIGURE 3. Acceptance Regions under Additive Loss Function.

separately for each of the reworkable characteristics, we still use separate screening specifications for the reworkable characteristics and use the joint screening rule for the scrappable characteristics. In other words, we determine whether or not an item is scrapped by the joint screening rule and if the item is not scrapped then the item is reworked, if necessary, according to the inspection outcomes of the reworkable characteristics. The procedure is as follows:

1. Determine the optimal screening specifications for the reworkable characteristics independently.
2. Compute R' by using (19).
3. Scrap the item if

$$q = \sum_{i \in S} k_i y_i^2 > R'. \tag{28}$$

4. If the item is not scrapped, rework the item on the rejected reworkable characteristics.

Notice that equations (23) and (28) describe the difference between Models 1 and 2: Model 2 uses the exact values of the scrappable characteristics to make a decision on scrapping, and Model 1 makes separate decision on each of the scrappable characteristics by considering the *expected* acceptance cost of other characteristics.

In Model 2, the probability of accepting all the scrappable characteristics is

$$P(S) = \int_0^{R'} h(q)dq, \tag{29}$$

where $h(q)$ is the p.d.f. of q. $EA(S)$ and $ER(S)$ are given, respectively, by

$$EA(S) = \int_0^{R'} qh(q)dq \qquad \text{and} \tag{30}$$

$$ER(S) = R[1 - P(S)]. \tag{31}$$

$EA(W)$ and $ER(W)$ of this model are identical to those in Model 1 ((14) and (15)). The expected total cost of this model then is obtained by (16). It can be proved, following Robbins (1948), that $P(S)$ can be written as a series of chi-square distributions with coefficients being calculated iteratively. The result is given in the following theorem where, without loss of generality, we assume that the variances of the deviations of all the scrappable characteristics are 1.0 and k_1 is the smallest value among k_i's of the scrappable characteristics by rearranging the indices if necessary.

THEOREM. *Let m be the number of the scrappable characteristics of interest. Then*

$$P(S) = \sum_{j=0}^{\infty} c_j F_{m+2j}(R'/k_1), \qquad \textit{where} \tag{32}$$

(1) $F_{m+2j}(\cdot)$ *is the Chi-square distribution function with* $m + 2j$ *degrees of freedom.*
(2) c_j *is obtained by*

$$c_j = \beta_j[\prod_{i=2}^{m} (k_i/k_1)^{-1/2}], \qquad j = 0, 1, 2, \ldots, \qquad \textit{with}$$

(i) $\alpha_h = \sum_{j=2}^{m} (1 - k_1/k_j)^h/2h,\ h = 1, 2, 3 \cdots,$
(ii) $\beta_0 = 1,$
(iii) $\beta_j = (1/j) \sum_{h=1}^{j} h\alpha_h\beta_{j-h}$ *for* $j = 1, 2, \ldots.$

It has been shown (Robbins 1948) that if the infinite series in (32) is truncated at the $t + 1$th term, the error is bounded by

$$0 \leqq P(S) - \sum_{j=0}^{t} c_j F_{m+2j}(R'/k_1) \leqq 1 - \sum_{j=0}^{t} c_j. \tag{33}$$

Note that for a given value of error, the smaller k_1 is the more terms (larger t) are required. Also note that in Model 1 when k_1 is very small, it is likely that most of the items will be accepted on the characteristic. In this situation, if we drop the first characteristic from Model 2 by subtracting the expected acceptance cost k_1 from both sides of (26), then the number of terms needed is significantly reduced and the result is very close to the optimal solution. However, this approach is not needed for most problems. Using the fact that $zf_n(z) = nf_{n+2}(z)$, where $f_n(z) = F'_n(z)$ is the probability density function of the Chi-square distribution with n degrees of freedom, we can show that

$$EA(S) = \sum_{j=0}^{\infty} \{k_1 c_j(m + 2j)\} F_{m+2j+2}(R'/k_1). \tag{34}$$

A computer program has been developed for computing $P(S)$ and $EA(S)$ and the error bound used in this paper is 10^{-5}. We use the same example in the last section and obtain $P(S) = 0.594$ and $EA(S) = 1.2852$. It is interesting to note that in this example Model 2 accepts a larger proportion of items but has a lower expected acceptance cost. However, the per-item cost difference is not very substantial. The expected total cost of Model 2 in this example is 4.6967, which is approximately 1.3% lower than that of Model 1. Nevertheless, the actual cost savings of using Model 2 may be substantial in high volume production and/or where the costs involved are significant.

5. Numerical Results

In this section, a numerical study is carried out to investigate the comparative performance of the single-characteristic models and the two multi-characteristic models. In particular, we study the effects of the variation in k_i's of the scrappable characteristics on the performance of the three models. We use the same parameter values of the example in the last two sections with the exception of k_i's associated with the scrappable characteristics. We randomly generate k_i's of the scrappable characteristics by a uniform distribution over [0, 1]. Then these values are multiplied by an appropriate constant so that their sum is equal to 6.0. The reason is to allow a fair comparison among different testing problems. A total of 100 testing problems are generated and the solutions and plan statistics are obtained for the three models. For each problem, we compute the percent cost difference between the single-characteristic models and Model 1

$$\epsilon_1 = [ETC(0) - ETC(1)]/ETC(1), \tag{35}$$

and that between Models 1 and 2

$$\epsilon_2 = [ETC(1) - ETC(2)]/ETC(2), \tag{36}$$

where $ETC(0)$, $ETC(1)$, and $ETC(2)$ are the total expected costs associated with the single-characteristic models, Models 1 and 2, respectively. In addition, the standard deviation of k_i's denoted by σ_k, $P(S)$ and EA(S) associated with three models are also recorded. Note that σ_k measures the variation in k_i's in determining the multi-characteristic loss function. In other words, σ_k provides an index of the variation of the importance among the scrappable characteristics. The correlations among these parameters in the testing problems are given in Table 2. All the correlations in the table are statistically significant at 0.1% level.

It is clear from the results that when σ_k is large the advantages of Model 1 over the single-characteristic models and Model 2 over Model 1 reduce. In fact, both ϵ_1 and ϵ_2 in the illustrative example (where $\sigma_k = 0$) are larger than those in all the test problems. Note that the averages of ϵ_1 and ϵ_2 are 7.5% and 1.0%, respectively. It is also found that when σ_k increases $P(S)$ of the single-characteristic models decreases but $P(S)$ of Models 1 and 2 increase. However, although $P(S)$ moves in different directions, $EA(S)$ decreases not only for the single-characteristic models but for all of the three models. This phenomenon is also indicated by the negative correlations between $P(S)$ and $EA(S)$ in the multi-characteristic models. As a result, ETC decreases as σ_k increases. The strong correlation between ϵ_1 and ϵ_2 indicates that when Model 1 has more cost advantage over the single-characteristic models, Model 2 also tends to have more cost advantage over Model 1. The results also suggest that when $P(S)$ is high the multi-characteristic models lose cost advantage. This is expected, since one of the main disadvantages of the single-characteristic models is the ignorance of the dominance of the scrappable characteristics over the reworkable characteristics in determining the disposition of an item. If $P(S)$ is high the effect of the dominance becomes less significant and consequently the cost advantage of the multi-characteristic models becomes small.

6. Discussion

One of the assumptions used in developing Models 1 and 2 is that the multi-characteristic quality loss function is additive. The acceptance region of Model 1 actually is not much different from that of Model 2 (see Figure 3). As a result, the cost difference is not substantial. This is especially true when the probability density in the nonoverlapping area is small. It is well known that there are several important conditions for using an additive loss function (see Keeney and Raiffa 1976). When these conditions are not met,

TABLE 2

Correlations among the Plan Characteristics in the Numerical Study

	σ_k	ϵ_1	ϵ_2	$P(0)$	$P(1)$	$P(2)$	$EA(0)$	$EA(1)$	$EA(2)$	$ETC(0)$	$ETC(1)$
ϵ_1	−0.993										
ϵ_2	−0.952	0.972									
$P(0)$	−0.958	0.976	0.991								
$P(1)$	0.978	−0.976	−0.921	−0.909							
$P(2)$	0.977	−0.973	−0.920	−0.907	0.999						
$EA(0)$	−0.981	0.993	0.991	0.991	−0.954	−0.953					
$EA(1)$	−0.738	0.767	0.882	0.874	−0.656	−0.665	0.839				
$EA(2)$	−0.822	0.848	0.937	0.930	−0.754	−0.760	0.906	0.990			
$ETC(0)$	−0.986	0.994	0.986	0.982	−0.968	−0.968	0.998	0.819	0.890		
$ETC(1)$	−0.973	0.982	0.988	0.977	−0.957	−0.960	0.994	0.847	0.912	0.997	
$ETC(2)$	−0.973	0.982	0.985	0.974	−0.959	−0.962	0.993	0.844	0.909	0.996	0.999

1. All the correlations are statistically significant at 0.1% level.
2. $P(0)$, $P(1)$ and $P(2)$ are $P(S)$ of the single-characteristic models, Models 1 and 2, respectively.
3. $EA(0)$, $EA(1)$, and $EA(2)$ are $EA(S)$ of the single-characteristic models, Models 1 and 2, respectively.

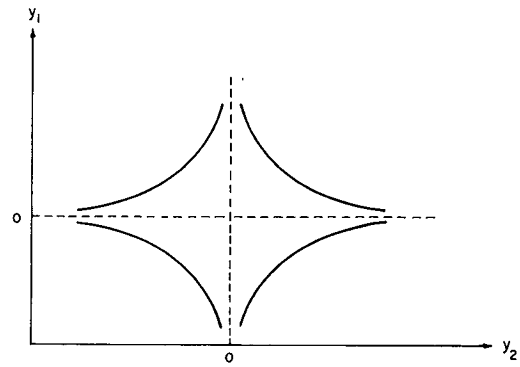

FIGURE 4. Acceptance Region under Multiplicative Loss Function.

other forms of loss function should be considered. Another possible loss function is of multiplicative form:

$$L_n(y_1, y_2, \ldots, y_n) = K(k_1 y_1^2)(k_2 y_2^2)\cdots(k_n y_n^2),$$

where K is a scaling constant. This loss function may not be theoretically adequate for use in this paper since the loss is zero if an item is reworked on any one of the reworkable characteristics. This function can be modified to overcome this problem. We use this function here only for illustrative purposes. Consider a model of two scrappable characteristics. Based on the discussion in §4, the joint screening rule of Model 2 is to reject an item when $K(k_1 y_1^2)(k_2 y_2^2) > R$. The acceptance region is shown in Figure 4, which is almost completely different from that of Model 1. Consequently, when a multiplicative loss function is used Model 2 is expected to perform much better than Model 1. It can be verified that if the step-loss function is used the results of the multi-characteristic models will be either identical to that of the single-characteristic models (accept non-defective items and reject defective items) or just scrapping all the items.

The above discussion suggests that a correct use of the loss function is very important to a multi-characteristic problem. Although some general procedures for assessing multi-characteristic loss (utility) functions are available (Keeney and Raiffa 1976), specialized methods in quality control context may be worth the effort of developing. The result of this paper may draw the attention of the researchers and the practitioners in the quality control area to the research and the recent development of the decision analysis area.

One rather restrictive assumption used in this paper is the independence assumption of the characteristics. To relax this assumption requires extensive research to derive the solution and evaluate the cost components of Model 1. Of course, these can be done by using numerical search methods. However, the exact solutions of Model 2 are not difficult to obtain.[1]

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References

AILOR, R. H., J. W. SCHMIDT AND G. K. BENNETT (1975), "The Design of Economic Acceptance Sampling Plans for a Mixture of Variables and Attributes," *AIIE Trans.*, 7, 370–378.

BAIRD, P., H. K. PATEL, B. STITT AND A. B. MUNDEL, "Acceptance Inspection Techniques, a New Direction in the 80's," *ASQC Quality Congress Trans.—Detroit*, 1982, 696.

BETTES, D. C., "Finding an Optimum Target Value in Relation to a Fixed Lower Limit and an Arbitrary Upper Limit," *Appl. Statist.*, 11 (1962), 202–210.

BISGAARD, S., W. G. HUNTER AND L. PALLESEN, "Economic Selection of Quality of Manufactured Product," *Technometrics*, 26, 1 (1984), 9–18.

BURR, I. W., "Specifying the Desired Distribution Rather Than Maximum and Minimum Limits," *Industrial Quality Control*, 24 (1967), 94–101.

CARLSSON, O., "Determining the Most Profitable Process Level for a Production Process under Different Sales Conditions," *J. Quality Technology*, 16 (1984), 44–49.

CHIU, W. K. AND G. B. WETHERILL, "A Review of Acceptance Sampling Schemes with Emphasis on the Economic Aspect," *Internat. Statist. Rev.*, 43 (1975), 191–210.

DODGE, H. F. AND H. G. ROMIG, "A Method for Sampling Inspection," *Bell Systems Technical J.*, 8 (1929), 613–631.

DUNCAN, A. J. *Quality Control and Industrial Statistics*, Fifth Ed., Richard D. Irwin, Inc., Homewood, IL, 1986.

GRANT, E. L. AND R. S. LEAVENWORTH, *Statistical Quality Control*, Fourth Ed., McGraw-Hill, Inc., New York, 1972.

HALD, A., "The Compound Hypergeometric Distribution and a System of Single Sampling Inspection Plans Based on Prior Distributions and Costs," *Technometrics*, 2 (1960), 275–340.

HILL, M. R., "FMS—The Scope for Further Research," *Internat. J. Production Management*, 5, 3 (1985), 5–20.

HUNTER, W. G. AND C. P. KARTHA, "Determining the Most Profitable Target Value for a Production Process," *J. Quality Technology*, 9, 4 (1977), 176–181.

INTERNATIONAL MATHEMATICAL AND STATISTICAL LIBRARIES, INC., *IMSL Library: Reference Manual*, Houston, TX, 1980.

KACKAR, R. N., "Off-Line Quality Control, Parameter Design, and the Taguchi Method," *J. Quality Technology*, 17, 4 (1985), 176–188.

KEENEY, R. AND H. RAIFFA, *Decision Making with Multiple Objectives*, Wiley, New York, 1976.

LANDRY, S., "A Practical Meaning of the Bayesian Decision Making Approach to Acceptance Sampling," *J. Quality Technology*, 8, 3 (1976), 127–132.

LEON, R. V., A. C. SHOEMAKER AND R. N. KACKER, "Performance Measures Independent of Adjustment: An Explanation and Extension of Taguchi's Signal-to-Noise Ratio," *Technometrics*, 29, 3 (1987), 253–265.

LI, L. AND D. B. OWEN, "Two-Sided Screening Procedures in the Bivariate Case," *Technometrics*, 21 (1979), 79–85.

MADSEN, R., "A Selection Procedure Using a Screening Variate," *Technometrics*, 24 (1982), 301–306.

ODEH, R. E. AND D. B. OWEN, *Tables for Normal Tolerance Limits, Sampling Plans and Screening*, Marcel Dekker, New York, 1980.

OWEN, D. B. AND J. W. BODDIE, "A Screening Method for Increasing Acceptable Product with Some Parameters Unknown," *Technometrics*, 18 (1976), 195–199.

———, L. LI AND Y. M. CHOU, "Prediction Intervals for Screening Using a Measured Correlated Variate," *Technometrics*, 23, 2 (1981), 165–170.

———, D. MCINTIRE AND E. SEYMOUR, "Tables Using One or Two Screening Variables to Increase Acceptance Product under One-Sided Specifications," *J. Quality Technology,* 7, 3 (1975), 127–138.

———, AND Y. H. SU, "Screening Based on Normal Variables," *Technometrics*, 19 (1977), 65–68.

PRATT, J. W., "Risk Aversion in the Small and in the Large," *Econometrica*, 32 (1964), 122–136.

PRYOR, T., "Optical Inspection and Machine Vision," *Conference Record 1982 Workshop on Industrial Applications on Machine Vision, IEEE Computer Science Society*, 1982, 3–20.

ROBBINS, H., "The Distribution of a Definite Quadratic Form," *Ann. Math. Statist.*, 20 (1948), 552–560.

SCHERKENBACH, W. W., *The Deming Route to Quality and Productivity*, Mercury Press, Rockville, MD, 1986.

SPRINGER, C. H., "A Method of Determining the Most Economic Position of a Process Mean," *Industrial Quality Control*, 8 (1951), 36–39.

STILES, E. M., "Engineering the 1990s Inspection Function," *Quality Progress*, 10, 11 (1987), 70–71.

STOVER, A. C., *ATE: Automatic Test Equipment*, McGraw-Hill, Inc., New York, 1984.

TAGUCHI, G., *Quality Evaluation for Quality Assurance*, American Supplier Institute, Romulus, MI, 1984.

TANG, K., "Economic Design of a One-sided Screening Procedure Using a Correlated Variable," *Technometrics*, 29, 4 (1987), 477–485.

——, "Economic Design of Product Specifications for a Complete Inspection Plan," *Internat. J. Production Res.*, 26, 2 (1988a), 203–217.

———, "An Economic Model for Vendor Selection," *J. Quality Technology*, 20, 2 (1988b), 81–89.

———, "Economic Design of a Two-sided Screening Procedure Using a Correlated Variable," *Appl. Statist.*, 32, 2 (1988c), 231–241.

———, "Design of a Two-stage Screening procedure Using a Correlated Variable: A Loss Function Approach," *Naval Res. Logist. Quart.*, 35, 5 (1988d), 513–533.

———, R. PLANTE AND H. MOSKOWITZ (1986), "Multiattribute Acceptance Sampling Plans Under Nondestructive Inspection," *Management Sci.*, 32, 6 (1986), 739–750.

TANG, J. AND K. TANG, "A Two-sided Screening Procedure Using Several Correlated Variables," (to appear) *IIE Trans.*, (1989).

THOMAS, J. G., D. B. OWEN AND R. F. GUNST, "Improving the Use of Educational Tests as Selection Tools," *J. Educational Statist.*, 2 (1977), 55–57.

WONG, A., J. B. MEEKER AND M. R. SELWYN, "Screening on Correlated Variables: A Bayesian Approach," *Technometrics*, 27 (1985), 423–431.