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# 投票困境的解決方式
# Toward A Solution to Voting Dilemma

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# 投票困境的解決方式

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## 中文摘要

集體選擇的投票問題之一在於一個尋求個人投票以及制度設計間平衡的困境。為了解決這個困境，本文提出一 AHP 法的變型，稱之為 AHP'法，其跨越了 MAUT 及 AHP 的界限並為有效的。AHP'可以用來解決隱藏在 AHP 的排序逆轉現象中因候選人退出所造成的投票困境。這個方法並不需要造成額外行政成本的新的制度設計，故而在真實選舉的情形中是可行的。

**關鍵詞**：投票困境、多屬性決策、排序逆轉

## 1. Introduction

Policies or plans for urban development are usually made collectively in a democratic society (Hopkins, 2001). As a result, environmental management can only be carried out effectively through a reasonable institutional design that aggregates individual preferences in a representative government (Haefele, 1973). For the theoretical part, multi-attribute decision making techniques are widely applied in many fields as a way of aggregating preferences (c. f., Yoon and Hwang (1995) for an introduction and overview), partly because the decision maker needs to make tradeoff judgments frequently among conflicting attributes or objectives (Keeney and Raiffa, 1993). Most of the applications of such techniques have been introduced to help individual decision makers to make choices among given alternatives (e. g., Hammond et al., 2002). It is relatively less known that multi-attribute decision making techniques have profound implications for group decision making in general (Sarin and Dyer, 1979), and social choice in particular (Arrow and Raynaud, 1986). In addition, Laukkanen et al. (2002) apply voting theory in natural resource management in terms of multi-criteria group decision making. We focus here on specific preference aggregation methods that are commonly applied, namely the analytic hierarchy process (AHP) and multi-attribute utility theory (MAUT). Multi-attribute decision making and collective choice share a common logic of preference aggregation, except that the former deals with individual decision making in relation to attributes while the latter focuses on group decision making in relation to individuals. The present paper addresses the issues of collective choice through multi-attribute decision making methods.

Two commonly applied multi-attribute decision making techniques are multi-attribute utility theory (MAUT) (Keeney and Raiffa, 1993) and the analytic hierarchy process (AHP) (Saaty, 1986). Though developed independently, the two techniques mean implicitly the same thing. Lai (1995) has shown that, if applied in an appropriate way, the decision rules in the two techniques are mutually permissibly transformable. That is, the weights of attributes and value functions of these attribute levels are mutually transformable from one technique to another so that the relative worths of alternatives are retained. In particular, Lai and Hopkins (1995) designed a variant scaling procedure of AHP, AHP', that combines the merits of MAUT and AHP and simplifies further the elicitation questions for weights and value functions by making them more concrete and meaningful. The formal proof for the validity of AHP' is given by Lai (1995). There is a large literature on the relationship between MAUT and AHP which we do not intend to delve into in the present paper. But one particular piece of work we want to single out for our purposes here is Pérez's (1995) demonstration on how the multi-district proportional elections can be interpreted in terms of MAUT and AHP.

The voting dilemma was well formulated by Pérez's (1995) and will be introduced in detail in Section 2. In essence, based on a consistent multidistrict proportional election mechanism, the dilemma implies that the election outcome would be different if a candidate decided to abstain before the voting took place, and we will show that this inconsistency can be resolved by the proposed AHP' preference aggregation method. Multidistrict proportional elections are widely applied in many countries, in particular parliamentary elections. For example, in Taiwan the election of legislators in the Legislative Yuan adopts a multidistrict proportional election in that each district shares a fixed number of the total seats competed by more than one candidate from different political parties.

We shall first review Pérez's voting model in the context of multidistrict proportional elections, pinpointing how the rank reversal phenomenon in AHP renders any universal election procedure as impossible. We shall then demonstrate how AHP' developed earlier can help resolve this dilemma and finally discuss its implications.

## 2. The Dilemma

According to Pérez (1995), consider $n$ divisions (attributes) ($D_1$, $D_2$, ..., $D_n$) wanting to set up a procedure for the election of an assembly of 200 representatives from m parties (alternatives) ($A_1$, $A_2$, ..., $A_m$). In addition, let

$v_{ji}$ = the number of votes obtained by candidature $A_j$ in division $D_i$,

$V_j$ = the total number of votes obtained by candidature $A_j$,

$W_i$ = the total number of votes cast in division $D_i$,

$E_i$ = the electorate (potential votes) of division $D_i$,

$E$, $V$ = the total electorate and the total number of votes cast respectively,

$s_{ji}$ = the number of seats obtained by candidature $A_j$ in division $D_i$, and

$S_j$ = the total number of seats obtained by candidature $A_j$.

There are two extreme solutions for this voting procedure problem: proportional election and multi-district proportional election. In the proportional election, each candidature $A_j$ obtains a number of seats $S_j$ proportional to the total number $V_j$ of votes obtained, that is,

$$S_j = 200\frac{V_j}{V_1 + V_2 + \ldots + V_m} = 200\frac{V_j}{V} \tag{1}$$

In the multi-district proportional election, each division $D_i$ is assigned a fixed number of seats, $r_i$, proportional to its electorate $E_i$. Thus, $r_i = 200\frac{E_i}{E}$. For each division $D_i$, each candidature $A_j$ obtains a number of seats, $s_{ji}$, proportional to the number $v_{ji}$ of his or her votes in $D_i$, that is,

$$S_j = \sum_{i=1}^{n} s_{ji} = \sum_{i=1}^{n} r_i \frac{v_{ji}}{W_i} = \sum_{i=1}^{n} 200\frac{v_{ji}}{W_i}\frac{E_i}{E} \tag{2}$$

Pérez (1995) proposed a more general solution to this voting problem: Let $\beta$ be a parameter with values in the interval [0, 1]. Each division $D_i$ is assigned a variable number of seats, $r_i^{\beta}$, inside the interval [0, 200], allocating the 200 seats among the divisions in proportion to the coefficients $c_i = \beta W_i + (1-\beta)E_i$. Thus,

$$\begin{aligned} r_i^{\beta} &= 200\frac{c_i}{c_1 + c_2 + \ldots + c_n} \\ &= 200\frac{\beta W_i + (1-\beta)E_i}{\beta(W_1 + W_2 + \ldots + W_n) + (1-\beta)(E_1 + E_2 + \ldots + E_n)} \\ &= 200\frac{\beta W_i + (1-\beta)E_i}{\beta V + (1-\beta)E} \end{aligned} \tag{3}$$

In addition, for each division $D_i$, each candidature $A_j$ obtains a number of seats proportional to the number of his or her votes in $D_i$ as in Equation (2), and we have

$$s_{ji} = r_i^{\beta} \frac{v_{ji}}{W_i} = 200 \frac{\beta W_i + (1-\beta)E_i}{W_i(\beta V(1-\beta)E)} v_{ji} \tag{4}$$

It can be easily shown that when $\beta = 0$ and 1, the general solution is reduced to the multi-district proportional election as in Equation (2) and the proportional election as in Equation (1), respectively.

What is remarkable is that the voting problem of collective choice corresponds to MAUT and AHP of multi-attribute decision making when $\beta$ is equal to 1 and 0, respectively. If we interpret candidatures as alternatives and divisions as attributes, "the evaluation of global election results is a simple but proper multicriteria decision problem." (Pérez, 1995, p.1093) Consider MAUT first in the context of the voting problem. Let $v_i^*$ be the best level of attribute $i$ across all alternatives, or $Max_{j=1,2,...,m}\{v_{ji}\}$, and $v_{*i}$ the worst level of that attribute across all alternatives, or $Min_{j=1,2,...,m}\{v_{ji}\}$. Since each vote is treated as equally important, $a_{ji} = \frac{v_{ji} - v_{*i}}{v_i^* - v_{*i}}$ evaluates the value obtained by $A_j$ in $D_i$ in an interval scale and $w_i = \frac{v_i^* - v_{*i}}{I^*}$ is the weight of $D_i$, where $I^* = \sum_{i=1}^{n} (v_i^* - v_{*i})$. Thus, the aggregate evaluation of $A_j$ is

$$S_j^{MAUT} = \sum_{i=1}^{n} a_{ji} w_i = \sum_{i=1}^{n} \frac{v_{ji} - v_{*i}}{I^*} = \frac{\sum_{i=1}^{n} v_{ji} - \sum_{i=1}^{n} v_{*i}}{I^*} = \frac{V_j - \sum_{i=1}^{n} v_{*i}}{I^*} \tag{5}$$

Equation (5) is the same as Equation (1), up to an admissible transformation, meaning that the extreme solution in (1) or $\beta = 1$ in (4) is nothing but the application of MAUT to the voting problem. Put differently, Equation (5) transforms the vote counts into an MAUT scale of multi-attribute utility.

Now, consider AHP in the same voting problem of collective choice. Let $a_{ji} =$

$\frac{v_{ij}}{W_i}$, which evaluates the values of $A_j$ for $D_i$, but in a ratio scale. The fixed number $r_i = \frac{E_i}{E}$ is the weight of $D_i$. Thus, the aggregate evaluation of $A_j$ becomes

$$S_j^{AHP} = \sum_{i=1}^{n} a_{ji} \frac{E_i}{E} = \sum_{i=1}^{n} \frac{E_i v_{ji}}{W_i E} \qquad (6)$$

Equation (6) is the same as Equation (2), up to an admissible transformation, meaning that the extreme solution in (2) or $\beta = 0$ in (4) is nothing but the application of AHP to the voting problem. Put differently, Equation (6) transforms the vote counts into an AHP scale of multi-attribute score. Ideally, MAUT and AHP would reach the same election outcome when no candidate withdraws. However, when a candidate abandoned the election and all the followers of that candidate abstained, MAUT and AHP would come up with different election outcomes. This inconsistency between MAUT and AHP is equivalent to the rank reversal debate that occurred in the 1990's due to the deletion or addition of alternatives (e. g., Dyer, 1990a; 1990b; Harker and Vargas, 1990), but Pérez (1995) was able to present it in the context of voting problem.

For concreteness and following Pérez (1995), consider the following voting matrix with $m$ = 3 candidates, $n$ = 2 divisions, with the votes cast and potential electors given as shown in Table 1.

■ Table 1. A hypothetical voting matrix

| | $D_1$ | $D_2$ | **Total** |
|---|---|---|---|
| $A_1$ | 500 | 520 | 1,020 ($V_1$) |
| $A_2$ | 260 | 745 | 1,005 ($V_2$) |
| $A_3$ | 240 | 735 | 975 ($V_3$) |
| Total Votes Cast | 1,000 ($W_1$) | 2,000 ($W_2$) | 3,000 ($V$) |
| Potential Votes | 1,000 ($E_1$) | 2,000 ($E_2$) | 3,000 ($E$) |

Applying Equation (4) and since $W_i = E_i$ for $i = 1, 2$, a closer examination will show that $S_1 = 68$, $S_2 = 67$, and $S_3 = 65$ and $A_1$ wins. MAUT and AHP agree. However, if $A_3$ withdrew before the election took place and all his or her followers abstained, then when $\beta = 1$ (i. e., the adoption of the MAUT procedure), $A_1$ would obtain approximately 101 seats and win, but when $\beta = 0$ (i. e., the adoption of the AHP procedure), $A_2$ would obtain approximately 101 seats and win, and, as a result, the voting dilemma of rank reversal occurs. The voting dilemma is a general phenomenon caused by the different aggregation procedures as manifested by MAUT and AHP as noted in the literature on rank reversal (e. g., Dyer, 1990a; 1990b; Harker and Vargas, 1990).

**3. A Solution**

As argued by Lai (1995), the rank reversal phenomenon of AHP is caused by the decision maker applying the wrong weights to attribute levels or values, both being measured in different scales. One way to resolve this phenomenon is to rescale the weights or attribute levels so that the two values are measured in a consistent scale. In particular, Lai and Hopkins (1995) proposed a variant scaling procedure of AHP, AHP', and later proved formally as valid by Lai (1995), that requires the decision maker to make interval judgments between attributes to derive attribute weights in MAUT, make ratio judgments within attributes to derive attribute values in AHP, and then rescale the attribute values in AHP proportionally so that the best attribute value within an attribute across all alternatives is equal to one. The resulting evaluation outcome should be consistent with either MAUT or AHP, if applied in an appropriate way.

More formally, using Pérez's (1995) language and referring to Equation (5), we have the attribute weights $w_i$ for MAUT as $\frac{v_i^* - v_{*i}}{I^*} = \frac{v_i^* - v_{*i}}{\sum_{i=1}^{n}(v_i^* - v_{*i})}$; referring Equation (6), we have the attribute values $a_{ji} = \frac{v_{ji}}{W_i}$. Rescaling $a_{ji}$ so that $a_{ji} = \frac{W_i}{v_i^* - v_{*i}} \frac{v_{ji}}{W_i} = \frac{v_{ji}}{v_i^* - v_{*i}}$, and multiplying the rescaled attribute values with the associated MAUT weights, we have

$$S_j^{AHP'} = \sum_{i=1}^{n} \frac{v_{ji}}{v_i^* - v_{*i}} \frac{v_i^* - v_{*i}}{I^*} = \frac{\sum_{i=1}^{n} v_{ji}}{I^*} = \frac{V_j}{I^*} \tag{7}$$

Note that, in the voting dilemma case, the rescaling factor $\frac{v_{ji}}{v_i^* - v_{*i}}$ would not make the best attribute value within an attribute across all alternatives equal to one; it simply restores the MAUT interval scale from the AHP ratio scale so that both the weights and attribute values are expressed in the same scale. Apparently, $S_j^{AHP'}$ is the same, up to an admissible transformation, as the scale $S_j$ of Equation (1). For concreteness, returning to our voting matrix, if $A_3$ withdraw before the election took place and all his or her followers abstained, we have the following revised voting matrix as shown in Table 2:

■ Table 2. The hypothetical voting matrix if $A_3$ withdrew

| | $D_1$ | $D_2$ | Total |
|---|---|---|---|
| $A_1$ | 500 | 520 | 1,020 ($V_1$) |
| A2 | 260 | 745 | 1,005 ($V_2$) |
| Total Votes Cast | 760 ($W_1$) | 1,265 ($W_2$) | 2,025 ($V$) |
| Potential Votes | 1,000 ($E_1$) | 2,000 ($E_2$) | 3,000 ($E$) |

$A_1$ obtains the proportionality of $c_1$; applying the AHP' procedure and since $I^* = 240 + 225 = 465$, we have

$$c_1 = \frac{s_2^{AHP'}}{s_1^{AHP'} + s_2^{AHP'}} = \frac{500}{500-260} \times \frac{500-260}{465} + \frac{520}{745-520} \times \frac{745-250}{465} \approx 2.193.$$

$A_2$ obtains the proportionality of

$$c_2 = \frac{s_2^{AHP'}}{s_1^{AHP'} + s_2^{AHP'}} = \frac{260}{500-260} \times \frac{500-260}{465} + \frac{745}{745-520} \times \frac{745-250}{465} \approx 2.161.$$

Thus, $A_1$ obtains $200 \times \frac{2.193}{2.193+2.161} \approx 101$ seats and $A_2$ obtains $200 \times \frac{2.161}{2.193+2.161} \approx 99$ seats. Note that, compared to the situation when A3 participates, not only the rank but also the proportionality among the seats obtained from the candidatures are preserved in the application of the AHP' election procedure.

**4. Conclusions**

Derived from the detailed exposition, Pérez (1995) was only partially correct by arguing that no general preference "aggregation method is expected to be suited for every situation." (p. 1095) In our view, this claim is a manifestation of Arrow's (1963) Impossibility Theorem that no preference aggregation method exists for at least three alternatives (or candidatures in the voting context), that simultaneously satisfies four conditions: non-dictatorship, Pareto principle, unrestricted scope, and independence of irrelevant alternatives (MacKay, 1980). However, we have shown that a variant scaling procedure of AHP, AHP', can resolve partially this voting dilemma or rank reversal in the context of AHP and MAUT as framed by Pérez (1995). It does not require a priori an institutional design that might impose additional administrative costs as usually perceived by selecting the parameter $\beta$. All AHP' requires is to count the votes and apply the aggregation procedure as shown in Equation (7) and in the numerical example, regardless of the withdrawal or addition of candidatures. To simplify, we assume strictly in our analysis that when a candidature withdraws, all his/her followers abstain. In order to retain some realism, it is possible to extend the current formulation to allow for shifts in voting when this situation occurs. Our focus here is however to demonstrate the logic of voting dilemma and propose a possible solution. Environmental management in a democratic society in general, and a representative government in particular, would be more effective by adopting reasonable preference aggregation procedures as presented here.

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# Toward A Solution to Voting Dilemma


## Abstract

One of the voting problems in collective choice is a dilemma in which a balance must be struck between individual votes and institutional designs. In the present paper, we approach the dilemma by proposing a variant scaling technique of AHP, AHP', that is valid across MAUT and AHP. The voting dilemma due to abstention of candidates embedded in the rank reversal phenomenon of AHP can be resolved by AHP'. The technique does not require institutional design that may impose additional administrative costs, and thus is feasible in real election situations.


**Keywords**: voting dilemma, multi-attribute decision making, rank reversal