THE UNDERSTANDING OF CHILDREN'S LEARNING OF MATHEMATICS
兒童數學能力的本質與學習現象

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摘　要

我們認為，普通智力的小孩，加以適宜的敎導，都有能力把數學學好。本文在積極方面，強調先要瞭解孩子的需要，在消極方面，則試要找出孩子們學不好數學的主因，並說明反常現象是怎樣發生的。我們分開五類來說明：

(1)孩子的本性，引申義大利教育學家蒙特梭里的見解，說明孩子們本性的自由發展。

(2)認知的發展：引瑞士心理敎育學家皮亞傑，對認知結構分期的研究來說明。

(3)學習條件與課業分析法。

(4)敎材綱目與學習順序。

(5)學習態度與學習動機，根據美國霍伯的二元因素動機論。

我們以為改進數學教育工作中，必須先求整體瞭解孩子的學習情況，否則課程研究成果，永遠只停留在表層現象。

INTRODUCTION

In this work we are to study the problems arising from children's learning of mathematics. The approach is like this: while keeping beside us the problems and their preliminary analysis most contiguous to the indigenous situations in the motherland, we are to make a survey of the western literature in an attempt to identify the corresponding problems in the western schools and to study the western way of interpreting and attacking the problems.

Our emphasis is laid on the CHILDREN: the CHILDREN's learning, the CHILDREN's mathematics learning, and the CHILDREN's learning environments. In other words, this is a CHILD-centered study. We are concerned about the CHILDREN's success and failure in the mathematics learning, their making progress or their encountering setbacks, their interest or their antipathy. We like to see the CHILDREN become knowledgeable and live away fom ignorance. CHILDREN are constructors of future world. (Montessouri, 1969)

It is our ideal that the education be directed to the full development of the human personality and to the strengthening of respect for human rights and fundamental freedoms(UN), it is our belief that a certain amount of mathematical activities devised appropriately for them in school is an integral part of the overall education that the society is obliged to provide.

However, despite of the fact that mathematics has been included as one of the mandatory subjects in the school curriculum and the years of learning mathematics extended, the promotion of mathematical abilities among the average school children has been in question. For the children, there is failure in mathematics lessons but there is hardly any 'lesson taken from failure', there is setback but there is almost no way to set themselves back to mathematics learning.

In fact, the situation is the same as incriminated by Whitehead(1916)

> ... ... with pathetic ignorance of human psychology, it has proceeded by some educational scheme to bind humanity afresh with inert ideas of its own fashioning.

although it has been repeated ever since Rouseau(1712-1778) that the child is endowed with an authentic activity of its own and that education can not succeed with truly employing this acitivity and extending it(Piaget, 1971).

Moreover, the ignorance of human psychology which results in the misunderstanding of the child has caused chaos and retroactions in the educational practice of the society: in the school, at home, and on the streets. The Naive Realism of Common Sense(Piaget, 1955) prevents people from seeing the ingenuity of the child, from knowing what they have done to their youngsters. Only the psychic maladies (Montessouri, 1929) are frequently discussed and referred, from which pessimistic view of mathematics education is drawed. School mathematics is condemned as a mere form of New-Latin.

It is therefore necessary to investigate the situations in which children are put to learn, more closely, to seek a real understanding, if we do not want to give up the faith that 'Human society is capable of improvement and it is the duty of responsible men and women to promote this.' (Mill, 1905), if we believe that the utmost goal of mathematics education is to awake the creative thinking (Khinchin, 1968).

We are to rediscover the hidden nature of the child.

We are to look into child's cognitive development.

We are to enquire the conditions of learning.

We are to examine mathematics syllabus and its learning method.

We are to fathom child's attitutes and find out motives of learning.

What we intend to prove is that the learning ability of average children is high enough to deserve a quality mathematics education, and the problems of children's learning mathematics should be treated in its cultural context.

It is not to innovate but to make afresh some good saying and to revive some noble examples. It is our hope to see through the problems of mathematics learning with the wisdom eyes of great educators,and acquire a comprehensive understanding.

This understanding will serve cleaning the soiled floor, clearing up the polluted air, flattening out the dinted wall, and a base for the 'NORMALIZATION OF MATHEMATICS EDUCATION'

FROM RESCUE TO UNDERSTANDING

As early as in 1951, Professor Kuan Kung-Tu (管公度), in concluding his article: "It's Time To Reform The Secondary School Mathematics"(中等數學教育的改造是時候了), urged the society to RESCUE the children:

救 救 孩 子 們 罷 ！

He pointed out the retroactions of mathematics education under 'theorism' and the resulting sickness. He impeached the only goal of the mathematics lessons bent for the examinations. He described how the children were suffering from the recondite problems by rote, without knowing what to learn and where to go. To him, 'Mathematics Education of such is a total failure!'

His proposition was not echoed by none. At least, five years after his death, the 'NEW MATH' reform, with the introduction of the terms of the curricular development, the adoption of the 'new math' syllabus, and the translation of the textbooks into chinese to be administered in schools, of American's School Mathematics Study Group (SMSG), was claimed to be an endeavor to follow his teachings and carry out the reform of school mathematics in memory of him in the mid-sixties. (The Advancement of Natural Science, 1964)

Unfortunately, this kind of graft in education was not successful, as also shown in almost all third world countries(Swetz, 1975), let alone the innovator of NEW MATH, SMSG, itself did not get around to develop and turned out bankrupt (Kline, 1973) in USA.

It is not only because the 'reform' had little to do with the inherited problems of mathematics teaching and examinations, but also because the 'reform' caused even more perplexity among the 'new elements'(see for example, Sherman, 1972) in the syllabus, which, unavoidably led to the illusion for more rote learning.

The situations became worse and somewhat absurd, for there were not only poor teaching, there came on false teaching in the semblance of academic authority.

...... with its fantasy, the NEW MATH came,

The examinations are always there. While a certain portion of the test items remains with their level of difficulty, varieties of deliberately devised items appear, in the name of rigor and formelism, without any mathmatical significance, much the same as described by Kline(1973).

To resolve the anomalies in the school mathematics community, a group of young mathematicians was formed voluntarily to make correction of the textbooks in the junior secondary level and worked out a revision of the syllabus in senior level. A set of experimental textbooks was published with experimental teaching in a high school, which results and experiences were compiled into a series of booklets: "Mathematics Classroom". Subsequently, several longitude studies of school mathematics curriculum were launched in the primary(1974), junior secondary (1976), and senior secondary(1977) levels.

However, since examinations are taken as the sole goal of the mathematics lessons, and the competition becomes keener and keener, more and more excuses for aberration in the classrooms or back at homes appears.

Children are still suffering, but they no longer wait for the rescue. In addition to, or in escape of, the anomalies, there has developed in them a kind of resistance towerds mathematics. Some rather give up than try.

It is also rumored that:

... the creative children can't stand the mathematics lessons!

... the backward children can't keep up the mathematics lessons!

... the common children are not interested in the mathematics lessons!

While there are vociferous please for reduction of teaching materials from the core curriculum for the salvation of the future citizens, in the classrooms volumes of supplementary materials are being fed into the children for higher mark in the future entrance examinations. Among the most controversial material, there sit the 'elements' of the 'NEW MATH'.

. . . . . . with its fallacy, the NEW MATH stays

Is any kind of reform possible? Or, one ought to turn around?

IT IS TIME TO SEEK THE UNDERSTANDING OF CHILDREN'S LEARNING MATHEMATICS! .

I. GENERAL MENTALITY

*** What has to be defended is
the construction of human normality*
Maria Montessouri

This is an area used to be neglected, or taken for granted but no due amount of attention paid to, in connexion with mathematics learning, for there is such an opinion that mathematicians be of a peculiar specimen(Sawyer, 1945) and 'therefore', mathematics learning requires special ability and is only possible for the special mind - mathematical mind.

Whether there is such a thing as a mathematical mind that is super is not what we want to argue about, we'd rather contend that for the school children, mathematical lessons may be devised as a process of interactions with things and events in their environment, which give rise to experiences of discovery, proof of truth and worthiness, sense of style and beauty, . . . all indispensible with the full growth of personality.

To get aware of the general mentality of the child, we shall attend to the absorbent mind(1969) and the pedagogical anthropology of Montessouri(1913).

Along with the philosophical tradition of Rousseau(1712-1778), Pestalozzi (1746-1827) and Froebel(1782-1852), Montessoui emphasized the innate potential of the child and its ability to develop in environmental conditions. In her attempt to build up a school of scientific pedagogy, Montessouri has made further studies centered at the child and even confronted with the problems of mathematics learning in the elementary level, (1912, 1917).

Through 'observations in the places where the minds of the children can actually develop without their faculties being limited'(1913), Montessouri discovered the Hidden Nature of the child—the absorbent mind—has the following characteristics:

1) The child loves to work and works with streneous concentration. He also desires to complete, and only starts new task after completing old one. It is through this activity of concentration and completion that the child reaches his state of psychic integration, the normal state.

*** The discovery that the child has a
mind able to absorb on its own account
produces a revolution in education*

2) The child needs to act independently. In child, there is an inner guide, the spiritual embryo, for his action and development. He needs independence to listen his own inner guide and make up his choice freely, by which he drives towards greater independence.

3) Once the child chooses a task by his own decision, he attends with intensity and interest, as well as prolonged concentration. He loves to repeat, and through the repetition of the cycle of activity, he achieves a sense of power and independence.

4) The will of the child develops through continuous activities in relationship with the environment, through adaptation to the limits of his chosen tasks. By exercising the will, the child procures the strength to control his own action by choice, and begins spontaneously to choose self-discipline as a way of life, which in turn gives rise to the power of obedience to natural laws and the forces of life.

5) The child develops his intelligence by summing up his reflex and reproductive activities, which enable his mind to construct itself and put it into relation with the environment orderly. The child loves order, beauty, truth, and meanings.

6) The child has inborn powers for imagination and creativity, which only develop through his interaction with the environment, necessarily beautiful, harmonious and based on reality, and which only develop after he has developed realistic and ordered perceptions of the life about him.

7) The child's development occurs in stages of chronological age. Certain character develops in certain stage naturally.

Her observations made on the effects of obstacles to the natural development are even more alarming for us to ponder over:

1) If the child's capacity for work and construction is repressed, his destructive behavior appears

> '... for without work his personality can not organize itself and deviates from the normal line of its construction ... ...'

2) If the unfolding function of the spiritual embryo is usurped, the child is prevented from developing either his will or his concentration, and much of his potential is never realized.

3) If the opportunities for choice and action is limited, not only the opportunity for using the will is denied, but also obstructs and inhibits the expression of the will to make progress and develop powers. His intelligent obedience to cooperate with the forces of life and nature becomes confused, his spontaneity and autonomy degenerated, the child appears idle and dependent.

4) If the child's cycle of repetition of the tasks of his own choice is per-

sistently interrupted, his self-confidence, cc urage, and ability to persevere in a task are severely jeopardized, and his intellectual growth is impeded. He then cannot make creative use of his abilities and therefore fails to accept the responsibility of his own actions.

Yes, if the child develops resistance towards mathematics learning, if the child could not concentration mathematics learning, if the child appears idle, dependent, and irresponsible in the mathematics learning, if . . . , all in all, it is because his need of environmental conditions is overlooked.

> '. . . the child is endowed with great creative energies which are of their nature so fragile as to need a loving and intelligent care . . . .

With Montessouri's emphasis on the importance of a prepared environment with care and protection, one comes to realize how restraints in action and delay or rush in learning have caused damages in the child's potential for intellectual development.

Since, according to Montessouri, the goal of self-development of the child is rather for the service to mankind as well as individual happiness, the misunderstanding and consequently the mistreating of the children would continue to induce great loss in human community.

While it is unwise to encast the child in an academic strait jacket, it is not wise either to leave the child 'in an unmapped, unmarked intellectual wilderness with no guidelines, rules, or self-discipline'. To further promote the growth of intelligence, which is not a constant as insighted by Montessouri, she devised a method of teaching, which consists of a system of didactic apparatus and programmed exercises for child to work at his own pace, to test for himself the understanding, and to correct the errors by himself with minimum instruction and limited number of 'ground rules'. In her method, mathematics learning goes with the acquiring of perceptual skill, body coordination, and language competence, which are totally for the mastery of one's self and the environment.

Encountering with present-day disconcerting phenomen on of children's failure and low-attainment in mathematics learning, what one could help is not to blame the children or to lower the standards of intellectual development, which would only lead to an inferior education and society. Montessouri's teaching and her proposed scientific padagogy certainly stand a good example for us to follow,

> '. . . Alone a scientific enquiry into human personality can lead us to salvation.

Let us all come along and start 'combatting the mathophobia'(Resek, 1980).

II. COGNITIVE DEVELOPMENT

*** It's probably possible to accelerate,
but maximal acceleration is not desirable.*
Jean Piaget

To further delve into the need of the child relating to the learning of mathematics, one is attracted to the Piaget's structure theory of cognitive development and the science of equilibrium.

Begining as a biologist with an intrinsic interest in philosophy and having then inbued with the idea that biology could profitably brought to bear upon the epistemological problem, Piaget turned to the child psychology as a mediator to tie the two together and addressed himself the simple and clear question:

'HOW DOES A CHILD LEARN?'

Having studied closely the child's behavior through a wide variety of area of cognitive functioning, Piaget came out with a picture of the child as a cognizing organism and draw up a description of cognitive structure and its development:

1) The mind is a self-organizing and self regulating system, responsive to the subject's interaction with the environment, and intelligence is an interiolized action in progressive equilibration. In other words, the child is a cognizing organism, starting from birth he constructs and reconstructs his own model of the reality of the world about him, and adapts himself to environmental conditions through processes of assimilation and accomodation to achieve mental equilibrium. It is through these kind of processes that cognitive development occurs.

2) Every act of intelligence presumes some kind of intellectual structure, some sort of organization within which it operates. It is the different mechanism of the cognitive action and the distinct mode of utilizing experience which give rise to the structural characteristic of intelligence behavior, and accordingly partition the cognitive development into discrete periods and even smaller stages.

3) The development periods:

the period of sensor motor intelligence
the period of preparation for concrete operation
the period of organization of concrete operation
the period of formal operation

form an ordinal whole in an unchanging and constant order. That is, although the chronological age of duration may be affected by all manner of variables a child must pass through each stage of the periods in the invariant order in order to enter into the final stage of the development, which not all individual needs to achieve.

There is virtually no 'operational' structure in the first two periods.

4) Cognition, like digestion, is an organized affair. Cognitive functioning, like biological functioning, over the whole developmental span, has two fundamental, dual, characteristics:

organization, in its internal, static aspect
adaptation, in its external, dynamic aspect

While these two characteristics occur as invariances through the stages, in the organizational aspect there are new structural units created by the continuous operation of the two interaction functions of adaptation:

assimilation and accomodation

which together cause the ontogenetic changes in the cognitive structure. 'It is by adapting to things that cognition organizes itself, and it is by organizing itself that it structures things.'

Using his genesis-with-structure conception to deal with questions about knowledge and about knowing, Piaget stressed the 'ROLE' of 'ACTIVE CHILD' as a significant agent in the learning process and the nature of his SPONTANEOUS interactions with the ENVIRONMENT. Dialectically, he made further converse arguments:

1) Cognitive development is not totally innate along with natural maturation, nor determined solely by the outer stimulations. There is no such thing as a faculty of force which is to bear upon the reality as development proceeds, nor an biological entlechy which directs intellectual growth. 'A particular social environment remains indispensable', and 'the right to education' should mean 'the right to be placed in a scholastic environment during one's formation until the basic tools of adaptation are completed'(1974).

2) Cognition is not of the cold 'pure reason' variety, cognitive and affective reactions are essentially two sides of the same coin, (1951)

> 'Affective life, like intellectual life, is a continual adaptation, and the two are not only parallel but interdependent, since feelings express the interest and value given to actions of which intelligence provides the structure.'
>
> 'Interests, pleasures, and difficulties, joy at success and disappointment at failure, . . . intervene as regulations of action constructed by intelligence.'
>
> 'Affectivity regulates the energetics of the action while intelligence provides the technique.'
>
> 'Personal schemes, like all others, are both intellectual and affective. We do not love without seeking to understand, and we do not even hate without a subtle use of judgement.'

therefore, problems of emotionality, values, personality development, and the like should be discussed and studied in a cognitive setting, and similarly, the realm of the affective-personalsocial viewed in its cognitive context.

3) The need to cognize is not fundamentally an extrinsic motive, separate from intellectual activity and pushed from behind, rather, the need to cognize is contained in and almost synonymous with intellectual activity itself. Therefore, it is important to stress the curiosity, exploratory drive, activity and sensory needs in opposition to an exclusive preoccupation with primary drive reinforcement, and to place all educational emphasis on the spontaneous aspects of the child's activity.

4) The child does grope in his contacts with reality, but the act of groping cannot induce success unless it gets established with corrections and the after-the-fact selection. Successful groping never occurs in complete independence from the milieu, it has some reality-oriented aim. Therefore, teacher as an organizer in the milieu is indispensable in promoting learning, not only to present useful problems to the child but also to provide counterexamples that compel reflection and reconsideration of solutions resulted by groping.

5) '*No* structure is ever radically new, each one is simply a generation of action drawn from the preceding structure!

All present cognitive behavior is constructed on a based of past accommodatory experiences with the outer world. New and complex forms of intellectual organization are activated by abstracting from earlier, simpler organization. The organism can assimilate only those things which past assimilation has prepared it to assimilate, can incorporate only those components of the reality which its ongoing structure can.

Therefore, the child cannot understand a general principle unless he has worked with that piece of principle in a more concrete and action-oriented context suitable to his stage or development. The Child's incapability in a particular subject is owing to a neglection of, or a too-rapid passage from the lower, non-operational structure to the higher, operational structure.

Concerning specifically about the problems of mathematics, education, Piaget made out some serious comments:

1) Every normal child is capable of good mathematical reasoning if attention is directed to activities of his interest, and if by this method the emotional inhibitions that too often give him a feeling of inferiority in lessons of mathematics are removed. Mathematics questions are solved by the students with their general intelligence and not by special individual aptitudes.

2) One of the basic causes of passivity in children in mathematics learning is due to the insufficient dissociation between questions of logic and numerical questions.

'Logic is never innate to the child!'
'Logic is built up step by step through his activities!'

Only when the two types of factors are dissociated, the child can advance more surely and attain the true goal of mathematics learning.

3) The true cause of failures in mathematics education is that: mathematics is taught as if it were only a question of truths that are accessible exclusively through an abstract language, and even of that special language which consists of working symbols. The child will not understand such lessons unless they appear to him as a continuation of actions with concrete embodiment.

III. THE CONDITIONS OF LEARNING

Here one attends to the varieties of learning and their conditions enunciated by Gagne(1965) to study the content aspect of learning.

Being unsatisfied with the opinion that 'there is virtually only one kind of learning'(Thorndike, 1931), Gagne attempted 'to consider the sets of circumstances that obtain when learning occurs', 'when certain observable changes in human behavior take place that justify the inference of learning', He came up with eight varieties of learning.

His question is exactly the following:

'How can one determine what learning is?'

and his approach is based on controlled experimentations as well as common life observations, in which an information processing viewpoint is employed and an hierarchy is set up for the PRODUCT of learning. By taking into consideration the three elements: learner, stimulus situation, and response of the learning, and the nature of new capabilities established by learning, he brought about the conditions as follows:

TYPE	CONDITION
Signal Learning	The individual learns to make a general, diffuse response to a signal or a symbol.
Stimilus-Response learning	The learner acquires a precise response to a discriminated stimulus.
Chaining	The learner acquires a connected chain with two or more

	stimulus-response connections.
Verbal Association	When the learning chains are verbalized in terms of the symbols in one's repertoire of language, the learner is said to acquire the capability.
Discrimination learning	The individual learns to make several different identifications to as many different stimuli, which may resemble each other in physical appearance to a greater or lesser degree.
Concept learning	The capability of making common response to a class of stimuli with different outlooks. The content of the common response is the 'CONCEPT'.
Rule learning	Rule is formed of a chain of two or more concepts. The individual learns to control the verbalized rule of the form: 'If A, then B', where A and B are previously learned concepts.
Problem solving	The capability of combining two or more previously learned rules to produce a new rule to attain a solution of the problem.

The learnings listed above differ from each other in the sense that each one begins with some different initial capabilities, 'prerequisites', and end with some new capabilities, different in content and performance. Further, these learnings are in an hierarchical order, that the prerequisites of the learning listed later are those acquired in the previous 'lower' type of learning. It is to this ordering, that a hierarchy of learning of a specific task can be formed accordingly, which reveals the 'hidden steps' in the learning process.

By this kind of analysis, the process of acquisition of knowledge is no longer kept in the black box, and every piece of new capability is seen explicitely to build on a foundation established by previously attained capability.

Besides the learning hierarchy, Gagne insighted the significance of the 'time dimension' in the event of learning, and subscribed to all types of learning a common 'time sequence' as consisting of four major phases: apprehending, acquisition, storage, and retrieval. In diagram,

PHASE	BEHAVIOR	Shorthand Terms
Apprehending	attending perceiving coding	LEARNING
Acquisition	acquiring	
Storage	retention memory storage	REMEMBERING
Retrieval	recognition recall of verbal information reinstatement and transfer of intellectual skills	

Both the hierarchical ordering and the time dimension play eminent roles in children's learning, for mathematics learning it is more critical:

1) Any individual is ready to learn something new if the prerequisites are prepared, but often symbolical manipulation started without meaning.
2) The newly acquired capability is to be retented for transfer if the subject is allowed enough time to store the knowledge, but only rush!

Therefore, according to Gagne, to say the pupil is not 'mature' enough to learn any particular content is likely to become a convenient excuse for not planning well with prerequisites and time.

. . . which we can afford to spare, a few minutes or a life time?

IV. MATHEMATICS SYLLABUS AND HIERARCHY

*** . . . deceptively simple material . . .
children are called upon to learn - - - *
Jean Piaget

Mathematics syllabus sits eminently outside the child in his environment of mathematics learning, which contents and methods are essentially the things and events (or 'stimulations') for the child to interact and to cope with (or 'respond').

The successfulness of child's learning of mathematics therefore depends heavily on the selection of syllabus and the design of curricular, activities, which, no matter in what form, are imposed on the child externally.

Encountering the child's failure in mathematics learning, one ought to ask:

> 'Is the topic well selected for the mathematics lesson?'
> 'Is the selected topic well sequenced for the child?'
> 'Is the learning method suitably devised for the child?'
> 'Is the child ready to use the learning method to learn the topic?'

To answer the questions, that is, to straighten out the critical issues in the dependence of child's learning of mathematics on the syllabus, there has emerged two lines of study: one, following Piaget's development theory, is trying to measure the level of cognitive demands of mathematics topics in the syllabus; the other, using Gagne's notion of the conditions of learning, is making task analysis of the learning material.

Let us start with the Piagetian approach:

Although Piaget's theory of cognitive development has brought forth an analogy between the structure of child's intelligence and the logiomathematical structure of the knowledge, the hierarchy of child's formation of mathematical concepts is not identical to the logical construction of mathematical concepts,

> 'In fact, the operational structures of the intelligence, although they are of a logio-mathematical nature, are not present in children's minds as conscious structures, ...' (Piaget, 1965)

Practically speaking, a logically valid inference in the mathematics texts is not necessarily psychologically acceptable to the child if he has not yet achieved the corresponding cognitive functioning.

It is the lack of explicit connection between the psychological processes of acquisition of the concepts and the logical construction of the concepts, pointed out by Suppes(1966), that inspired researchers in mathematics education for exploration.

A quite systematic one has been carried out, to a certain extent, by the Concepts in Secondary Mathematics and Science Program (CSMS):

By using problems in secondary school mathematics to investigate children's understanding of mathematical concepts, CSMS group has worked out a criteria to form hierarchy of concepts in 11 topic areas:

> measurement, graph, ratio and proportion, fraction(12-13 yr), fraction(14-15), positive and negative numbers, vectors, matrices, place value and decimals, algebra, number operation, reflection and rotation;

and came on with a description of the understanding level of each hierarchy (Hart, 1981), for example:

"Levels of the Fraction Problems Hierarchy"
(12-13 year olds)

Level	Description of the group of Items
0	The criterion of 2/3 of level 1 items correct, not satisfied.
1	The meaning of a fraction using pieces, 1/2, 1/5, 2/3.
2	The meaning of a fraction as a subset of a set. Naming a given configuration of pieces of a whole. Equivalent fractions obtained by doubling. Addition of two fractions with the same denominator.
3	Using equivalence to name parts with familiar fractions or when a diagram is provided. Equivalent fractions not obtained by doubling or when the fraction is less familiar.
4	More than one operation is needed, for example equivalence followed by addition or subtration.

As a coordinate system, each hierarchy can be used to indicate the relative position of the inherited mathematical concepts and therefore constitutes a qualitative reference for inducing successful understanding in concept learning, and also for the check-up of child's failure.

Although the author reported 'the difficulty in establishing a reasonable link between the Piagetian tasks', and the contents of items in each developmental stage were only briefly described together with some abstract feathers common to the items in each stage, on the other hand, however, they did draw up some meaningful inferences:

1) Since the majority of children can cope with only the simplest and most concrete of the levels identified, a number of mathematical topics which are generally taught across the ability range in the early years of the secondary school may be unsuitable for the majority of pupils - at least, careful consideration to the level of the concept in the topic areas need to be taken into the instructional approach.

2) On the average, children progressed only one level over two years, and the majority either stayed within the same level or progressed one level. Therefore, only relatively slow rate of growth in mathematical understanding can be expected.

3) In general, the levels attained by children in any one year spanned the whole range in each topic: that is, the range in each year was very wide compared with the average progress. Therefore, it is not wise to base the selection of mathematical material primarily on age.

Moreover, the research program has resulted in revealing the 'usage' of some of the problem-solving strategies, e.g., addition strategy instead of multiplication schema, not taught officially at school, to get around right answer, as well as some inappropriate strategies committed by large numbers of children independent of type of school, textbooks, or teaching schemes. A new project is already undertaken in these reported strategies and errors by 'The Strategies and Errors in Secondary Mathematics (SESM), in which not only the phenomenon is analysized but also modules for intervention are designed.

Yes, a picture of children's lacking of certain operational functionings has emerged and becomes clearer as study goes on. Concerning about children's setbacks in learning, one should not only make inquiry into their behavior elements in groping or struggling with the situations but also provide different embodiments for them to ponder over what had happened, in order to make them aware of the errors for self-correction, as well as the constraints on the successful strategies for transcend.

For, children, as the significant agent of themselves in the process of learning, have the right to become conscious of what they are doing in the mathematical activities, not to remain inert or totally receptive of the material. The promotion of intelligence through mathematics learning is only possible if the material is adaptable.

V. ATTITUDES AND MOTIVATIONAL FACTORS

Here one is concerned about the negative attitudes children have towards mathematics, and the motivations they are in need of to promote learning, or to promote the positive attitude towards learning.

Negative attitudes towards mathematics have long been 'rumored'. In Cajori's "A HISTORY OF ELEMENTARY MATHEMATICS", it was quoted from a manuscript of the date 1570 or near it: (曹丹文譯 , 1925)

'Multiplication is mie vexation	乘法原非易
And Division is quite as bad.	除法亦不良
The Golden Rule is mie stumbling stule	比例之法更艱澀
And Practice drives me mad.'	習之每令吾發狂

Even Diderot(1713-1771), the 18th century great French philosopher, was frightened away by Euler(1707-1783), who asked him to solve a deceptively algebraic problem in contrast with atheism. (Hogben, 1924)

International Project for the Evaluation of Educational Achievement (Husen,

1967) has conducted a study on the achievement and attitude of math in school children across 12 countries, in which five dimensions of attitude were assessed:

1. Attitudes toward mathematics as process.
2. Attitudes about the difficulties of learning mathematics.
3. Attitudes toward the place of mathematics in society.
4. Attitudes toward school and social learning.
5. Attitudes toward man and his environment.

In connexion with the result of achievement, it was reported that 'in those countries in which achievement is high pupils have a greater tendency to perceive mathematics as a fixed and closed system, as difficult to learn and for an intellectual elite, and as important to the future of human society.'

Generally speaking, attitudes have two components: cognitive and affective components. While affective component as motivational factor has long been known to be the primary variable in the influence of attitudes on learning, it is found (Peak, 1955) that if a person has a positive attitude toward a new idea, he is likely to establish a cognitive component of that attitude possessing clear, stable, and relevant anchoring ideas to incorporate the new idea. Cognitive and affective components are two sides of the coin in the formation of the attitude, interdependent in the attitude structure, (Ausubel, 1969). It is not wise to neglect either of them, or to weigh more on the affective side; for motivation.

To promote positive attitude toward mathematics learning, one may as well ask the following questions: (Hoyles, 1975)

1) How does negative attitude such as dislike, fear, anxiety, phobia originate and come into being? toward mathematics learning.
2) Why mathematics is conceived as 'useful but boring' subject?
3) Why 'an active dislike' toward mathematics even generates among pupils with upper achievement in mathematics?
4) Why there is also such evidence as a positive correlation between attitude to and attainment in mathematics?
5) How could it be possible to bring about improvement in performance by discovering the fundamental factors leading to involvement in mathematics?
6) Is it the nature of mathematics itself that results in the attitude?
7) Are the origins of the attitudes more embedded in teaching methods or classroom organization?

Employing Herzberg's TWO FACTORS THEORY OF MOTIVATION (1967), Hoyles(1975) made study of children's attitudes along the above lines, and found out qualitatively that, influencing factors of positive attitude seem to be predominantly 'content factors', such as achievement, recognition, responsibility, interest, which are revolved around the actual doing of a job and have the potential to give the subject motivation for growth and self-actualisation; on the other hand, those of negative attitude seem to be 'context factors', such as teacher competence, class organization or teacher friendliness, which meet the need of an individual for avoiding unpleasant situations but will not produce the urge to learn.

Content factors are seen to answer the 'growth need' of an individual and context factors are to the "deficit need'. While the above two needs are very different, and the deficit need seems nonconstructive, neglection could certainly induce 'deficit', as pointed out by Montessouri. External need such as the deficit need is a sort of environmental condition, as also suggested by Piaget, is not to be overlooked, although on the part of his theory, motive from pushing from behind is not essential.

Negative attitudes towards mathematics are really results of bad experiences, to reorient it, both growth and deficit needs, both cognitive and affective components, both content and context factors, are to be taken care.

SUMMARY AND DISCUSSION

... The Problem of Education is an OLD problem, BUT
the teachings of Great Educators of all the ages
are always FRESH ... .

Briefly speaking, the problems arising from children's learning of mathematics fall into two domains: cognitive and affective domains.

In the affective domain, there are factors such as: attitudes, motivations, interest, anxiety, and self-concept. (Wilson, 1971)

In the cognitive domain, one is concerned with children's achievements and the act of achieving, or the knowledge and the process of knowing, or in a more detailed account, the areas such as: mathematics syllabus, their content and method, concepts and their cognitive demand, hierarchical order of the topics, errors and strategies.

Summing up what we have had in the above sections, we anticipate that for the CHILD, a cognising organism endowed with an absorbent mind, any factor of learning, no matter it be cognitive one or affective one, no matter it occur before or during or after an act of learning, no matter it be internal or external to the child, ought to be viewed as an integral part of the whole situation in relation to all other beings. Factors are interlocked, which total could determine what would or would not result when the child is called upon to learn in the mathematics classroom.

From the observations, investigations, and analysis made by past or present researchers, in different settings and with independent approaches, an almost non-astonishing common theory can be deduced which claims that: the failure of children's learning of mathematics, their negative attitudes, if not hundred percent fault of the adults, is at least caused by letting to go worse without adult's making sufficient effort to seek the intelligent understanding of the children and their need of growth and normalization, while stubbornly driving the children to perform according to what we please.

It is intriguing to make a few further remarks on the operational aspects of children's cognitive structure and the genetic epistemological nature of children's cognition to illustrate this.

Since there is virtually no operational structure in the first two stages of development in that the child is incapable of performing the reciprocal thinking as well as sequential thinking, the child fails to relate in a logical way successive impressions and pays no thought to any contradiction involved, by judging every event from his own point of view. It is therefore not fair to assess his functioning of thought with adult s logic and to draw up such conclusion as he is careless.

Children, like adults, do make mistakes. However, according to the findings of Piaget, their mistakes have much to do with their cognitive structures relating to their age. For example, it has been encountered by many primary school teachers that many of their kids, when given the following question: "Da'Ming has twelve sheets of coloured papers, his is two less than his brother's, how much does his brother get?", tend to answer alike:

$$12 - 2 = 14$$

They then ask: "Should we give these kids any credit?"

Certainly, the above answer is not satisfactory in that the number equation is absurd. Nevertheless, it does reflect that these kids fail to make out the reverse statement of "his is two less than his brother's" into "his brother's is two more than his" and embed it back into the original setting, although he is surly aware of the whole situation and has come up to the right answer. With egocentrism, each child tends to identify himself with Da'Ming, the protagonist, and processes whatever the information in relation with Da'Ming. Therefore, when there is such relation as "less than" the child simply writes down a subtraction symbol in his number equation not realising that an absurdity is induced.

In this case, if the item is designed to test whether the kid is able to use the right operation, to perform the number calculation, and to get to the right answer, than one should not hesitate to assume that in the child's logics the addition is chosen virtually to lead to the right answer, and be generous to give him full credit; the notational fault here is minor indeed.

It is necessary, though, to call upon these kids to reexamine their answer to find out the flaw themselves. Or, if not successful, pointing out the left-hand-side of the number equation to make them respond and compare the answers. When the child comes to realise the contradiction, he is ready to follow the lesson that an explicit reverse statement ought to be taken into consideration. In other words, the child learns to perform the reciprocal thinking by his try-an-error in such a concrete situation.

If no further learning is expected of the child of age 7 or 8 to perform the reciprocal thinking, the original item should be rephrased as:

> "Da'Ming has twelve sheets of coloured papers, his brother has two more, how much does his brother get?"

to make it straight and simple enough to serve the aim of the test.

What we ought to be cautious against is that the children with right answer "14" should not be confused without being informed why they are discredited. If the children deserve no credit due to their error, they at least deserve an explanation to learn from their failure.

Yes, just as Piaget said, there is nothing more difficult for the adult than to know how to appeal to the spontaneous and real activity of the child. But one may always keep the genetic epistemological view point to observe, watch, and provide ground for the chldren for their self-correction of their behavior, which is especially essential for the learning of mathematics, both cognitively and affectively. For the negative attitudes are mostly the results of unnecessary failures.

When one compares the child, a cognising body, with mathematics, a body of knowledge; the cognitive structure of the child with logical structure of mathematical system; the act of knowing with the evolution of knowledge; the intelligence of the child with the power of mathematics; the absorbent mind with the extendability of the mathematical tree; it is natural to ask:

> "While it has taken thousands of years for mathematics to develop, millions of problems to explore and ponder, trillions of puzzles to resolve and justify, is it not justifying to give due consideration to the need of the child in TIME and SPACE to spare in the learning of mathematics?"

Children's intelligence as an authentic activity has not been rightly brought up, children's love for work, for order, for truth, has been overlooked, children's need of adaptation for equilibrium has been restrained, children's role as constructors for future world has not been looked up to, ... Children have turned away from mathematics before they get chance to know what is mathematics.

Now, while we insist that the core job of mathematics education is to administer the mathematics lessons to the youngsters to broaden their knowledge and to prepare the skills for their future life, we ought to be honest and humble to accept the simple and unfortunate fact that such a job can hardly get started when the resistance or indifference towards mathematics learning among the youngsters is unsurmountable.

Indeed, mathematics education is not really functioning when the children are not really learning, not really been helped to learn, not been expected of learning genuine mathematics.

If we contend that it is for the children that mathematics education is constituted, then we should agree that it is only by the understanding of children's learning of mathematics that the constitution of mathematics education could undergo any meaningful reform. What to do next are then the revision of curricular activities, allocation of time for thorough learning, compilation of textbooks, devise of teaching programs, design of examination items and scales, and most importantly, the novel approach of teachers training research.

To seek the understanding of children's learning of mathematics might be the last measure we could do for mathematics education, in comparison with the effort of exploring the nature of mathematics or any other kind of work of innovations. As matter of facts, the movements of educational reforms associated with great educators of the history were initiated in most cases in a concern for the needs of "backward children" (Lloyd, 1960), in which the attempt was always to revive the capabilities of the children under stress.

It is therefore an adaptation of our own mind to the REALITY to correct the had-been egocentrism of our own conviction of our job, to awaken our own creativity in the field of mathematics education.

Mathematics lessons will not be able to awaken the creativity of the children until the mathematics teacher's creativity is awaken.

"Why can Jennifer not do arithmetic?"

Freudenthal (1981) broached this problem again in presenting the major problems of mathematics education in the recent International congress of Mathematics Education. (ICME IV). Researches of mathematics education were called upon to be responsible for the learning of mathematics and not to exist for one's own sake.

There is no room for mathematics educators to stay in one's own imagery. There is no royal road to mathematics education. To save the failure of mathematics education, we should observe the children's failure and try to UNDERSTAND.

***To understand is to Invent*
Jean Piaget

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