A CHANCE-CONSTRAINED APPROACH TO INSURANCE AND INVESTMENT RISKS MANAGEMENT

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摘　　要

保險公司的業務上常常牽涉着某些不可預知的風險性，因此，如何平衡這些保險投資作業上的風險，則使公司的利潤最大，損失最少實在是一個重要的問題。一九七四年，Thompson, Matthew and Li 三人在美國著名的學術性雜誌 Operations Research 上發表一篇論文，他們利用機會限制規劃方法（Chance-constrained Approach）去處理此類性質的問題，惟他們僅考慮混合投資的情形，却未涉及保費混合的實情。此處，吾人試着利用機會限制規劃方法去處理保費混合暨投資混合的問題，建立一個數學模式，接着應用一些數學性質導出它的相當定常模式（Deterministic Equivalent Form），然後再以三種不同的參數值代入此模式用非線性規劃的電腦程式（SUMΓ）算出各種不同的數值表來，加以詳細的比較、分析並做成結論。

1. INTRODUCTION

An insurance company collects premiums and accumulates amounts of reserve funds for unknown future claims. Then, the company invest other industries and expects to earn returns on the investment. Since there are risks involved with both insurance and investment operations, hence how to balance these two risks will be a major problem. Indeed, underwriting operations and investments are closely related. And, there also imply a particular trade-off function between risk and return.

Recently a number of attempts have been made to relate premium mix to investment portfolio problems, such as statistical correlation modeling, stochastic processes approach, decision theoretic approach, and other statistical analytical techniques, etc. In 1969, Hofflander and Drandell developed a LP model which considers the relationship between total premium volume and investment portfolio. However, their model lacks the specific consideration of risk factors.

In late 1969, Agnew etc. formulated an investment model in CCP, but it treats the insurance side as deterministic [1]. In 1974, Thompson, Matthew and Li considered the effects of risks on both the insurance and investment sides [8]. In particular, they formulate the premium volume-investment mix problems using a CCP approach. However, they treat only the investment-mix problem using pseudo-empirical data. They left the insurance-mix problem and real empirical data for some subsequent paper. But, up to now, no such paper has appeared.

Here, we intend to formulate both a premium-mix and investment-mix problem as a CCP model in which expected return (profit) is maximized subject to chance constraints involving the risk or random elements of returns and demands. And, we apply this model to three experimental runs and illustrate with them the inferences one can make from their solution.

2. MODEL FORMULATION

The model we concerned is to select the appropriate combination of policies to determine an appropriate relation between expected returns (or profits) and the variance of these returns in a single period. In other words, we want to maximize expected returns subject to chance constraints on a minimum level of these returns.

First, let's define the following variables:

X_j denote the dollar volume of premiums of type j. $(j = 1,2,3)$ written in a single period. They denote three types of insurances; life insurance, title insurance, and casualty insurance, respectively.

Y_i denote dollars invested of type $i (i = 1,2,3,4)$ in a single period. They denote four types of investments: bonds, normal stocks, growth stocks, and cash investment, respectively.

φ_j denote the ratio of expenses to premiums of type $j (j = 1,2,3)$ written in a single period; the random variables, and $0 \leqslant \varphi_j \leqslant 1$.

Ψ_j denote the ratio of losses to premiums of type $j (j = 1,2,3)$ written in a single period; the random variables, and $0 \leqslant \Psi_j \leqslant 1$.

λ_i denote a percentage of the realized return of type i investment, where $i = 1,2,3,$ 4; also random variables.

δ_i denote a percentage of the unrealized return of type i investment, where i = 1,2, 3,4; also random variables.

Π denotes the special tax rate for casualty insurance, $0<\Pi\leqslant 1$.

ρ denotes the tax rate for ordinary return (income) except casualty insurance, $0<\rho\leqslant 1$.

θ denotes the tax rate for capital gains, $0<\theta\leqslant 1$.

M denotes the initial surplus.

Hence, the objective function is the maximization of the expected value of the closing surplus, including what would be the capital gains after tax. And it can be formulated as follows:

$$\text{Max } M+(1-\rho)\sum_{j=1}^{2}[E(1-\varphi_j-\Psi_j)X_j]+(1-\Pi)E(1-\varphi_3-\Psi_3)X_3+\sum_{i=1}^{4}(1-\rho)E(\lambda_i)$$
$$+(1-\theta)E(\delta_i)]Y_i$$

where 'E' means the mathematical expectation operator.

During any single period, the insurance company collects premium, pays expenses of operation, suffer losses, and holds some risks in the form of capital and surplus. In addition, it invests surplus and premium reserves in earning assets, and obtains returns. We assume that premiums are written and collected at the beginning of the period, and investments are made immediately following the receipt of the premiums but before expenses are incurred. Also, we assume that claims are paid at the end of the period after earnings on investments are realized. Two available constraints follow:

(1) The Budget Constraint

Here, we assume that the funds available for investment are the initial surplus plus the premiums after expenses. Since the expenses are incurred throughout the year, the funds that will be available for investment are unknown. Hence, investment must be made under uncertainty, too. By stating the desired probability of the funds available after expenses exceeding the investment amount, we may write the budget constraint as follows:

$$P[M+\sum_{j=1}^{3}(1-\varphi_j)X_j-\sum_{i=1}^{4}Y_i\geqslant 0]\geqslant\alpha \quad \text{where } 0<\alpha\leqslant 1 \qquad \ldots(1)$$

It states that the probability that the funds available after expenses exceed the investment amount is greater than α.

(2) The Liquidity and The Profitability Constraints

Since the funds available at the end of the period exceeding claims can be written as

$$[M+\sum_{j=1}^{3}(1-\varphi_j)x_j-\sum_{i=1}^{4}Y_i]-[\rho\sum_{j=1}^{2}(1-\varphi_j-\Psi_j)X_j+\Pi(1-\varphi_3-\Psi_3)X_3]$$
$$+\sum_{i=1}^{4}[1+(1-\rho)\lambda_i+(1-\theta)\delta_i]Y_i\geqslant\sum_{j=1}^{3}\Psi_j X_j$$

where the first-term represents the original investment in cash, the second-term represents the taxes on insurance income, the third-term represents the original investments plus the after-tax investment returns.

Hence, the probability that the initial surplus will not be completely eroded is greater than some stated probability. Thus, the second constraint may be written as

$$P\{M+\sum_j^3(1-\varphi_j)X_j-\sum_i^4 Y_i-\rho\sum_j^2(1-\varphi_j-\Psi_j)X_j-\Pi(1-\varphi_3-\Psi_3)X_3$$
$$+\sum_{i=1}^{4}[1+(1-\rho)\lambda_i+(1-\theta)\delta_i]Y_i\geqslant\sum_j^3\Psi_j X_j\}\geqslant\beta\qquad\text{where } 0<\beta\leqslant 1$$

$$\Rightarrow\quad P\{M+\sum_j^3(1-\varphi_j)X_j-\sum_i^4 Y_i-\rho\sum_j^2(1-\varphi_j-\Psi_j)X_j-\Pi(1-\varphi_3-\Psi_3)X_3$$
$$+\sum_i^4 Y_i+\sum_i^4[(1-\rho)\lambda_i+(1-\theta)\delta_i]Y_i-\sum_j^3\Psi_j X_j\geqslant 0\}\geqslant\beta$$

$$\Rightarrow\quad P\{M+\sum_j^2(1-\rho)(1-\varphi_j-\Psi_j)X_j+(1-\Pi)(1-\varphi_3-\Psi_3)X_3+\sum_i^4[(1-\rho)\lambda_i$$
$$+(1-\theta)\delta_i]Y_i\geqslant 0\}\geqslant\beta \qquad \ldots(2)$$

Therefore, we have our model as follows:

$$\text{Max}\{M+(1-\rho)\sum_j^2[E(1-\varphi_j-\Psi_j)X_j]+(1-\Pi)E(1-\varphi_3-\Psi_3)X_3$$
$$+\sum_i[(1-\rho)E(\lambda_i)+(1-\theta)E(\delta_i)]Y_i\}$$

Subject to

(1) & (2)

The constraints (1) & (2) may be converted into deterministic equivalent form.

If we assumes that φ_j, Ψ_j, λ_i, & δ_i are mutually independently random variables with known normal distribution. Hence, constraint (1) may be rewritten as

$$P[\sum_j^3 (1-\varphi_j) X_j \geqslant \sum_i^4 Y_i - M] \geqslant \alpha.$$

Since the mean of $\sum_j^3 (1-\varphi_j) X_j$ is $E(\sum_j^3 (1-\varphi_j) X_j) = \sum_j^3 (1-E(\varphi_j)) X_j$ ... (4)

and, the variance of $\sum_j^3 (1-\varphi_j) X_j$ is $V(\sum_j^3 (1-\varphi_j) X_j) = \sum_j^3 X_j^2 V(\varphi_j)$

From (4), we have that

$$P\left\{\frac{\sum_j^3 (1-\varphi_j) X_j - \sum_j^3 (1-E(\varphi_j)) X_j}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}} \geqslant \frac{\sum_i^4 Y_i - M - \sum_j^3 (1-E(\varphi_j)) X_j}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}}\right\} \geqslant \alpha$$

or

$$P\left\{\frac{\sum_j^3 \varphi_j X_j - \sum_j^3 E(\varphi_j) X_j}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}} \leqslant \frac{M - \sum_i^4 Y_i + \sum_j^3 (1-E(\varphi_j)) X_j}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}}\right\} \geqslant \alpha \quad \ldots (5)$$

Now, since $Z = \dfrac{\sum_j^3 \varphi_j X_j - \sum_j^3 X_j E(\varphi_j)}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}}$ is a standard normal deviate, (5) may be re-written as

$$P\left\{Z \leqslant \frac{M - \sum_i^4 Y_i + \sum_j^3 (1-E(\varphi_j)) X_j}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}}\right\} \geqslant \alpha$$

$$\Rightarrow \quad \frac{M - \sum_i^4 Y_i + \sum_j^3 (1-E(\varphi_j)) X_j}{[\sum_j^3 X_j^2 V(\varphi_j)]^{1/2}} \geqslant F^{-1}(\alpha)$$

or

$$\sum_i^4 Y_i - \sum_j^3 X_j (1-E(\varphi_j)) + F^{-1}(\alpha) [\sum_j^3 X_j^2 V(\varphi_j)]^{1/2} \leqslant M \quad \ldots (6)$$

where F denotes the cumulative normal distribution function.

Similarly, the constraint (2) can be converted to deterministic equivalent form, too. Analogous to (6) we have,

$$-F^{-1}(\beta) \geqslant \frac{-\{M+\sum_j^2(1-\rho)E(1-\varphi_j-\psi_j)X_j+(1-\pi)E(1-\varphi_3-\psi_3)X_3+\sum_i^4[(1-\rho)E(\lambda_i)+(1-\theta)E(\delta_i)]Y_i\}}{\{(1-\rho)^2[\sum_j^2 X_j^2(V(\varphi_j)+V(\psi_j)]+(1-\pi)^2X_3^2[V(\varphi_3)+V(\psi_3)]+\sum_i^4[(1-\rho)^2V(\lambda_i)+(1-\theta)^2V(\delta_i)]Y_i^2\}^{1/2}} \quad \ldots (7)$$

Letting

$$\mu_j' = (1-\rho)E(1-\varphi_j-\psi_j) \qquad j=1,2$$

$$\mu_3' = (1-\pi)E(1-\varphi_3-\psi_3)$$

$$\mu_i = (1-\rho)E(\lambda_i)+(1-\theta)E(\delta_i)$$

$$\sigma_j'^2 = (1-\rho)^2[V(\varphi_j)+V(\psi_j)]$$

$$\sigma_3^2 = (1-\pi)^2[V(\varphi_3)+V(\psi_3)]$$

$$\sigma_i^2 = (1-\rho)^2V(\lambda_i)+(1-\theta)^2V(\delta_i) \qquad i=1,2,3,4$$

(7) may be rewritten as

$$-F^{-1}(\beta) \geqslant -\frac{M+\sum_j^2\mu_j'X_j+\mu_3'X_3+\sum_i^4\mu_iY_i}{[\sum_j^2\sigma_j'^2X_j^2+\sigma_3'^2X_3^2+\sum_i^4\sigma_i^2Y_i^2]^{1/2}}$$

Squaring both sides, we have

$$(F^{-1}(\beta))^2 \leqslant \{M+\sum_j^3\mu_j'X_j+\sum_i^4\mu_iY_i\}^2 / \{\sum_j^3\sigma_j'^2X_j^2+\sum_i^4\sigma_i^2Y_i^2\}$$

$$\Rightarrow (F^{-1}(\beta))^2\sum_j^3\sigma_j'^2X_j^2+(F^{-1}(\beta))^2\sum_i^4\sigma_i^2Y_i^2 \leqslant \{M+\sum_j^3\mu_j'X_j+\sum_i^4\mu_iY_i\}^2$$

$$\Rightarrow (F^{-1}(\beta))^2\sum_j^3\sigma_j'^2X_j^2+(F^{-1}(\beta))^2\sum_i^4\sigma_i'^2Y_i^2 \leqslant M^2+(\sum_j^3\mu_j'X_j)^2+(\sum_i^4\mu_iY_i)^2$$
$$+2M(\sum_j^3\mu_j'X_j)+2M(\sum_i^4\mu_iY_i)+2(\sum_j^3\mu_j'X_j)(\sum_i^4\mu_iY_i)$$

$$\Rightarrow (F^{-1}(\beta))^2\sum_j^3\sigma_j'^2X_j^2+(F^{-1}(\beta))^2\sum_i^4\sigma_i^2Y_i^2-\sum_j^3\mu_j'^2X_j^2-2(\mu_1'\mu_2'X_1X_2+\mu_1'\mu_3'X_1X_3$$
$$+\mu_2'\mu_3'X_2X_3)-\sum_i^4\mu_i^2Y_i^2-2(\mu_1\mu_2Y_1Y_2+\mu_1\mu_3Y_1Y_3+\mu_1\mu_4Y_1Y_4+\mu_2\mu_3Y_2Y_3$$
$$+\mu_2\mu_4Y_2Y_4+\mu_3\mu_4Y_3Y_4)-2M(\mu_1'X_1+\mu_2'X_2+\mu_3'X_3)-2M(\mu_1Y_1+\mu_2Y_2$$
$$+\mu_3Y_3+\mu_4Y_4)-2(\mu_1'\mu_1X_1Y_1+\mu_1'\mu_2X_1Y_2+\mu_1'\mu_3X_1Y_3+\mu_1'\mu_4X_1Y_4$$
$$+\mu_2'\mu_1X_2Y_1+\mu_2'\mu_2X_2Y_2+\mu_2'\mu_3X_2Y_3+\mu_2'\mu_4X_2Y_4+\mu_3'\mu_1X_3Y_1$$
$$+\mu_3'\mu_2X_3Y_2+\mu_3'\mu_3X_3Y_3+\mu_3'\mu_4X_3Y_4) \leqslant M^2 \quad \ldots (8)$$

Therefore, we may rewrite our CCP model as the following deterministic form:

$$\text{Max}\ \ M+(1-\rho)\sum_{j}^{2}E(1-\varphi_j-\psi_j)X_j+(1-\pi)E(1-\varphi_3-\psi_3)X_3$$
$$+\sum_{i}^{4}[(1-\rho)E(\lambda_i)+(1-\theta)E(\delta_i)]Y_i$$

s.t.

(6) & (8)

Since (6) & (8) both are non-linear, numerical solution of the problems require sophisticated computing routines to get numerical solutions to specific problems. We may use SUMT [6] & [7] and U.T. dual cyber 170/750 computer to solve them.

3. EXPERIMENTAL COMPUTATION AND SOME COMMENTS

Here, we will make three experimental runs with the model as given in deterministic equivalent form in former section. The parameter for these runs are given in Table I.

In these runs, for simplicity reason, zero covariance is assumed. That is, the loss ratios, expense ratios, and investment rates of return all are normally independently distributed with zero mean and variance one. And, the risk-acceptance level (α or β) be 95%.

The run set (A, B, C) is designed to accesses the effects of changes in insurance risks on premium-mix and investment-mix policy. All the given data are assumed equal except '*' marked items as shown on the Table I.

The relationship between insurance risks and investment risks is frequently stated in terms of the growth of premiums versus the proportion of common stock in the investment portfolio.

About the optimal solutions to the three runs, please see Table II.

Table III are concerned with the effects of insurance risks on investment mixes, premium-surplus ratios, and proportion of expected profits in insurance and investment.

Table III shows that, as insurance risks increase, the investment side also varies. This also is accompanied by a premium-to-surplus ratio that varies irregularly with the increase in insurance risk and a larger proportion of expected profit in investment income.

We see that insurance mix and investment portfolio are closely related to premium growth. Risks on both sides are compensated for by varying the growth rate.

Indeed, growth and portfolio selection can't be separated.

It also appears that the problem of insurance-mix and investment-mix is a very complicated one. The proper solution is highly dependent on business environments, especially the tax laws. As we know, realized returns and capital gains are taxed differently, so as are different types of securities such as preferred stocks and municipal bonds. The relationships between risks and rates of return become very difficult to assess operationally without rather elaborate mathematical models.

Table I

Parameter Values for Experimental Runs

Surplus \ Run		A	B	C	Remarks
Insurance Type I	$E(\varphi_1)$	.20	.20	.20	
	$\sqrt{V(\varphi_1)}$	.05	.06	.06	*
	$E(\psi_1)$	.65	.65	.65	
	$\sqrt{V(\psi_1)}$	.1	.12	.15	*
Insurance Type II	$E(\varphi_2)$	.25	.25	.25	
	$\sqrt{V(\varphi_2)}$	.07	.08	.08	*
	$E(\psi_2)$	.70	.70	.70	
	$\sqrt{V(\psi_2)}$	.14	.168	.21	*
Insurance Type III	$E(\varphi_3)$	.23	.23	.23	
	$\sqrt{V(\varphi_3)}$	.03	.04	.04	*
	$E(\psi_3)$	.68	.68	.68	
	$\sqrt{V(\psi_3)}$	.12	.144	.18	*
Investment Type I	$E(\lambda_1)$	.07	.07	.07	
	$\sqrt{V(\lambda_1)}$	.00	.00	.00	
	$E(\delta_1)$	.00	.00	.00	
	$\sqrt{V(\delta_1)}$	.02	.02	.02	
Investment Type II	$E(\lambda_2)$	.05	.05	.05	
	$\sqrt{V(\lambda_2)}$	.02	.02	.02	
	$E(\delta_2)$	.06	.06	.06	
	$\sqrt{V(\delta_2)}$	.04	.04	.04	
Investment Type III	$E(\lambda_3)$	.02	.02	.02	
	$\sqrt{V(\lambda_3)}$	.01	.01	.01	
	$E(\delta_3)$	.14	.14	.14	
	$\sqrt{V(\delta_3)}$	.15	.15	.15	
Investment Type IV	$E(\lambda_4)$	.01	.01	.01	NOTE: '*' denotes differences among the runs.
	$\sqrt{V(\lambda_4)}$	.25	.25	.25	
	$E(\delta_4)$	.20	.20	.20	
	$\sqrt{V(\delta_4)}$	.40	.40	.40	
$\pi = .50$, $\rho = .45$, $\theta = .30$ for all the three runs					

Table II
Optimal Solutions to the Runs

RUN		A	B	C
Premiums:	X_1	$ 804,202	$ 898,968	$ 741,070
	X_2	300,961	330,818	280,296
	X_3	461,344	512,839	426,936
Investment:	Y_1	402,441	448,756	372,568
	Y_2	686,053	769,203	632,287
	Y_3	1,043,765	1,164,154	962,975
	Y_4	1,334,441	1,393,957	1,265,859
Total Expected Profit		$ 466,490	$ 506,790	$ 435,493

NOTE: Let the initial surplus M = $250,000. The expected profit is the objective function itself. It may be derived by means of NLP codes [6] and U.T. dual cyber 170/750 computer system within six seconds.

Table III
Insurance Mixes and Premium-Surplus Ratios
as a
Function of Insurance Risks

RUN		A	B	C
Insurance	I	–	+20%	+50%
	II	–	+20%	+50%
	III	–	+20%	+50%
Available for Investment		100%	100%	100%
Investment	I	11.6%	11.9%	11.5%
	II	19.8%	20.4%	19.6%
	III	30.1%	30.8%	29.8%
	IV	38.5%	36.9%	39.1%
Premium/Surplus		6.266	6.97	5.79

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