# AN URBAN LAND USE MODEL WITH PRICE AND NON-PRICE INTERACTIONS UNDER VARIABLE DENSITY DISTRIBUTION

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## 摘　要

自從Alonso於1964年提出競價地租後，許多學者致力於都市土地使用理論之研究，然而這些研究均有著許多嚴格的假設，其中最爲人批評的爲，「單一都心」，的假設，即都市有一個事先存在的市中心，全市的居民均在此中心就業。這個假設的提出是爲了數學解的方便，但在理論及事實上均有著重大的爭議。爲了修正此一假設，有人提出，「一般均衡的都市土地使用理論」，即市中心的位置、數目等均爲模型之內生變數；此一般均衡的模型主要分爲兩大類別，一爲價格交易（即商業行爲）所導出之理論〔如 Kanemoto（1985）、Papageoriou and Thisse（1985）、Fujita（1986）、Liu and Fujia（1991）〕。另一類別爲由非價格交易（即空間外部性）所導出之理論Ogawa and Fujita（1980）、Imai（1982）、Liu（1988）〕。但一都市的形成應是由此二種機能所組成，但是從未有人將此二種機能同時研究其對都市土地使用之影響，因此本篇報告是同時考慮價格，及非價格交易所建立的都市土地使用模型。

## 1. OBJECTIVE

The study of urban land use, commonly being called as "New Urban Economics"(NUE), was originated from Alonso's bid rent (1964). Since then, a great deal of works regarding various aspects of urban land use have been done[1]. Although

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[1] For example, Beckmann, Mills, Muth, Solow and Vickrey, Anas and Dendorinos, Goldstein and Moses. Mills and Mackinnon, and Richardson.

these studies have contributed a great deal to our understanding concerning the spatial structure of cities. However, most of them still rely on some strong and restrictive assumptions. Among those assumptions[2], the monocentricity is the most restrictive one and is particularly crucial to the formulation of urban land use models. Under this assumption, the city in question has a pre-specified center, the Central Business District(CBD), which employs the entire urban labor force. The introduction of this assumption is essential for mathematical tractability. Without it, solutions of the urban land use model become more difficult to obtain. But it has major drawbacks, both empirical and theoretical, of adapting this assumption[3]. In attempting to relax the monocentric assumption, two approcaches are conceivable in the existing models. The first is a multicentric approach. It assumes in advance that there exist prespecified multiple centers in the city under studying [Papageorgou and Casetti(1971), Romanas(1977), and White(1976)]. This approach has the same flaw as it in those monocentric models since the locations and the number of centers are prespecified and not allowed to emerge endogenously. The second approach is to develop a general equilibrium model of urban land use which does not require the a *priori* locations of either employment or residence of households. According to the *Spatial Impossibility Theorem* of Starrett, three prototypes of general equilibrium model are conceivable: (i) the port city model, (ii) the competitive equilibrium model with spatial externalities and (iii) the noncompetitive equilibrium model with scale economies[4]. Among these three prototypes, the spatial externalities and the noncompetitive equilibrium model with scale economies can explain the urban structure much more meaningful then the port city model (Fujita, 1986). But all the papers talk about spatial externalities assume fixed density distribution in both households and firms sector [Ogawa and Fujita (1978, 1980), Imai (1980), Fujita and Ogawa (1982)]. And no papers talk about how these two factors influence the urban structure simultaneously.

The purpose of this paper is to examine the spatial structure of urban configuration due to the non-price interaction (spatial externalities[5]) within sector, price interaction (imperfect competition[6]) between sectors, and the variable density of both sectors. This may be called a general equilibrium model with spatial interaction and imperfect competition since the distribution of firms and households as well as floor rent are simultaneously determined within the model. In this model we analyze

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2 For these basic assumptions see Richardson (1977).

3 For detail see Ogawa and Fujita (1980).

4 For detail see Fujita (1985, 1986).

5 See Fujita (1985).

6 See Fujita (1985).

the structure of urban configuration without assuming any a *priori* locations of economic agents, and with the within and/or between-sector interactions. Although the spatial externalities and imperfect competition are two main reasons to cause agglomeration economies and have great impact on the urban configuration, they have not been fully investigated in the literature. Most existing studies of urban land use focus either on the effect of spatial externalities[7] or on that of price interaction[8] on the spatial structure of cities. And, to the best of our knowledge, no one has examined how these tow externalities simultaneously affect urban configuration.

In Section 2, we first state the behavior assumptions of firms, households, and constructor then present a model. We also summarize the equilibrium conditions. From Section 3 to Section 5, our attention is focused upon monocentric, completely mixed, and incompletely mixed urban configurations. Important findings and conclusions are discussed in the last section.

## 2. ASSUMPTIONS AND FORMULATION OF THE MODEL

### 2.1 City

A city is developed on a long strip of agricultural land of width 1 (unit distance). Assuming that the width of the land is sufficiently small comparing to its length, the city may be treated as a linear city. Each location in the city is represented by a point, x, on a line coordinate. The center of the city denoted as origin may be arbitrarily chosen.

Economic activity in the city is assumed to be generated by three types of agents: households, industry and constructors. The industry consists of M firms, $i \in [1,M]$. Each firm produces one product which is slightly different from other firms, and all of the products are sold to households in the city. Firms interact with each other. Households consume import goods at a given price(*numerire*) as well as differentiated goods produced by firms in the city. Households supply labor to firms, and conversely, firms pay wages to households. In addition, activity units in both sectors compete for floor space. They are assumed to be perfectly competitive everywhere in the city. Absentee constructors supply the floor space to firms and households.

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[7] In this category, Beckman(1976), O'Hara(1977), Ogawa and Fujita(1980), Fujita and Ogawa(1982), Imai(1982), Tauchen and Witte(1983, 1984), Tabuchi(1984), and Fujita(1985) have presented seminal works.

[8] There are very few studies so far; Kanemoto(1985), Papageoriou and Thisse(1985), and Fujita(1986).

## 2.2 Household and Firm

It is assumed that there are N identical households in the city and that they all have the same preference. We assume that each household occupies the same amount of floor space, $S_h$. The utility function of a household at x is specified as

$$u(x) = \int_{-f}^{f} B[Z(x,y)]f(y)dy + Zo \qquad (2.1)$$

where u(x): utility level,

B(.): the commodity benefit function of local goods,

Z(x,y): the amount of the good purchased by the household at x from a firm at y,

f(y): firm density at y (which equals the number of goods supplied at y),

Zo: the amount of import goods.

Each household has one worker who supplies his/or her labor to a firm. The wage earned by the worker is the only income of a household. The travel of each household consists of both the journey to work and the journey to shopping. The budget constraint of a household locating at x working at $x_w$ is given by

$$Z_0 + \int_{-f}^{f} [p(y)+g(x,y)]Z(x,y)f(y)dy+OR(x)S_h+td(x,x_w) = W(x_w) \qquad (2.2)$$

where p(y): f.o.b. price of good produced by firms at y[9],

g(x,y): unit shopping transport cost from x to y[10],

w(xw): the wage paid by the firm locating at $x_w$,

OR(x): floor rent per unit floor space at x,

t: commuting cost per unit of distance for journey to work,

$d(x,x_w) = |x-x_w|$: distance between residence and job site.

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9 Because of identical production technology and transportation costs for all goods, all goods at location y must have the same (f.o.b) price p(y) in equilibrium.

10 This is assumed to be the same in purchasing any good.

Then specifying the benefit function B(Z) as

$$B(Z) = \begin{cases} \dfrac{Z}{A} - \dfrac{Z}{A}\log\dfrac{Z}{A} & \text{if } Z < A \\ 1 & \text{if } Z \geq A \end{cases} \tag{2.3}$$

where A is a positive constant. The optimal demand distribution function is given as

$$Z(x,y) = A\exp\{-A[p(y)+g(x,y)]\}. \tag{2.4}$$

Hence from (2,1), (2,2) and after the choice of an optimal demand distribution, the utility function of a household at location x becomes

$$u(x) = \int_{-f}^{f} \exp\{-A[p(y)+g(x,y)]\}f(y)dy - OR(x)S_h + W(x_w) - td(x,x_w). \tag{2.5}$$

Accordingly, the locational behavior of each household is to choose a residential location x and a job site $x_w$ so as to maximize its utility given by (2,5), given the distribution p (.), f(.), and OR(.).

It is assumed that there are M firms in the city. Each firm produces one product which is slightly differentiated from the other. Hence, there are M types of goods in the market. All firms are assumed to have identical production technology. Moreover, all firms occupy the same constant amount of floor space $S_b$ and use the same amount of labor $L_b$. Then, by assuming full employment, the following must hold at equilibrium;

$$M = N/L_b. \tag{2.6}$$

Firms require to have transactions (e.g., communications or information exchange) with other firms in the city in order to produce goods. Each firm transacts with every other firms with equal frequency. The unit cost of transaction between any two firms is proportional to the distance between them. The profit, $\pi$, of a firm at x, is thus given by

$$\pi(x) = [p(x)-c]\int Z(y,x)h(y)dy - OR(x)S_b - W(x)L_b - \tau T(x) - K \tag{2.7}$$

where p(x): price of product sold at x,

c: marginal production cost,

h(y): the household density at y,

$\tau$: unit transaction cost,

T(x): $|x-y|f(y)dy$: total transaction distance for a firm at x.

K: capital cost[11].

After the choice of the optimal demand distribution given by (2.4), (2.7) becomes

$$\pi(x) = A[p(x)-c]\int_{-f}^{f} \exp\{-A[p(x)+g(x,y)]\}h(y)dy$$

$$- OR(x)S_b - W(x)L_b - \tau T(x). \tag{2.8}$$

where $\pi(x) = \pi(x) + K$.

According to the first order conditions of profit maximization with respect to p, we have

$$p(x) = (1+Ac)/A \equiv p. \tag{2.9}$$

This is the maximum profit price for firms at x and is independent of location.

Finally, from (2.4), (2.7), and (2.9), we have

$$u(x) = \exp(-Ap)\int_{-f}^{f} \exp[-Ag(x,y)]f(y)dy - OR(x)S_h$$
$$+ W(x_w) - t|x-x_w|, \tag{2.10}$$

$$\pi(x) = \exp(-Ap)\int_{-f}^{f} \exp[-Ag(x,y)]h(y)dy$$
$$- OR(x)S_b - W(x)L_b - \tau T(x). \tag{2.11}$$

[11] We assume each firm has the same capital cost.

## 2.3. Constructor

It is assumed that constructors supply floor space to households and firms in a market which is perfectly competitive. Hence, in equilibrium, the profit of each constructor is zero. The cost and profit functions are

$$K(x) = \alpha H(x)^{\beta} \tag{2.12}$$

where K(x): total construction cost at x,

H(x): floor space density at x,

$\alpha$, $\beta$ parameters, $\alpha > 0$, $\beta > 1$.

$$\pi_c(x) = OR(x)H(x) - \alpha H(x)^{\beta} - R(x)g(x) = 0, \tag{2.13}$$

where $\pi_c$: profit of a constructor at x,

R(x): land rent at x.

g(x): the amount of land used by construction at x.

## 2.4 Equilibrium Conditions

In this section, we will first specify the necessary and sufficient conditions for the equilibrium city. Then, our task is to analyze the following set of unknown functions and parameters: (1) fringe and boundary of each zone distance from the center, f, e, (2) floor density function, H(x), (3) household density function, h(x), (4) firm density function, f(x), (5) commuting pattern $p(x,x_w)$, the density distribution of commuting destination xw by the household at x, (6) floor rent profile, OR(x), (7) land rent profile, R(x), (8) wage profile, W(x), (9) utility level, u, and (10) profit level $\pi$ in the equilibrium solutions.

Before stating the equilibrium conditions, we first define the following functions:

(1) $\Psi(x) \equiv \Psi|x;f(x),W(.),u^*]$

$$= \underset{x_w}{\mathrm{Max}}\{\frac{1}{Sh}[\exp(-Ap)\int \exp[-Ag(x,y)]f(y)dy + W(x_w) - t|x-x_w| - u^*]\}, \tag{2.14}$$

(2) $$\Phi(x) \equiv \Phi(x;h(x),\pi^*) = \{\frac{1}{S_b}[\exp(-Ap)\int \exp[-Ag(x,y)]h(y)dy - W(x)L_b - \pi^* - \tau T(x)]\}, \quad (2.15)$$

(3) $$\Gamma(x) \equiv \Gamma(x;g(x),H(x)) = \frac{1}{g(x)}\alpha(\beta-1)H(x)^{\beta}. \quad (2.16)$$

Function $\Psi(x)$ gives the maximum floor rent which would be paid by a household at location x while attaining the equilibrium utility level, $u^*$, given the firm density f(x). $\Psi(x)$ denotes the *bid floor rent function of households.* Function $\Phi(x)$ gives the maximum floor rent which would be paid by a firm at location x while attaining the equilibrium profit level, $\pi^*$, given the household density h(x). $\Phi(x)$ denotes the *bid floor rent function of firms.* Function $\Gamma(x)$ gives the maximum land rent which would be paid by a constructor at x while attaining the zero profit, given the amount of land used by construction g(x) and floor density H(x). $\Gamma(x)$ denotes the *bid land rent of constructors.*

Then, the necessary and sufficient conditions for a city to be in equilibrium are:

(a) Housing market equilibrium conditions:

$$OR(x) = Max\{\Psi(x), \Phi(x), 0\} \quad (2.17)$$

$$OR(x) = \Psi(x) \quad \text{if } h(x) > 0 \quad (2.18)$$

$$OR(x) = \Phi(x) \quad \text{if } f(x) > 0 \quad (2.19)$$

$$OR(x) = \alpha\beta H^*(x)^{\beta-1} \quad (2.20)$$

$$S_h h(x) + S_b f(x) = H(x) \quad (2.21)$$

(b) Land market equilibrium conditions:

$$R(x) = max\{\Gamma(x), R_A\} \quad (2.22)$$

$$R(x) = \Gamma(x) = \alpha H(x)^{\beta} \quad \text{if } H(x) > 0 \quad (2.23)$$

$$g(x) + (\text{land for agriculture use}) = 1 \quad (2.24)$$

$$R(-f) = R(f) = R_A \tag{2.25}$$

where $R_A$ is agriculture land rent which is exogenously given.

(c) Labor Market equilibrium conditions:

$$f(x)L_b = \int h(y)p(y,x)dy \tag{2.26}$$

$$p(x,y) > 0 \text{ only if } (1/Sh)\,\{e^{-AP}\int e^{-Ag}f(y)dy + W(y) - t(x,y) - u^*\} \tag{2.27}$$

(d) Total activity units number constraints:

$$\int_{-f}^{f} h(x)dx = N \tag{2.28}$$

$$\int_{-f}^{f} f(x)dx = N/Lb = M \tag{2.29}$$

(e) Nonnegativity constraints:

$$h(x),\ f(x),\ R(x),\ OR(x),\ H(x),\ W(x),\ p(x,y),\ g(x) \geq 0. \tag{2.30}$$

In the above, condition (2.17) claims that each unit of floor space must be occupied by a household or a firm which bids the highest positive floor rent at that location. Conditions (2.18) and (2.19) require that if households or firms locate at x, they must have succeeded in bidding for floor space at that location. Condition (2.20) says that the equilibrium floor rent is equal to the marginal cost of providing floor space. The physical constraint on the amount of floor space is given by the (2.21). Condition (2.22) says that each unit of land must be used by a constructor or a farm which bid the highest land rent. Condition (2.23) says that if constructors use land at x, they must have succeeded in bidding for land at that location. The physical land constraint and boundary constraint are stated by (2.24) and (2.25). Condition (2.26) assures that the demand for labor must be equal to the supply of labor at all locations in the city. Condition (2.27) states that p(x,y) can be positive only when y is a commuting destination optimally chosen by a household at x.

## 2.5 Further Specifications

In order to obtain an explicit solution to the equilibrium problem described above, we will introduce several additional specifications.

From (2.10) and (2.11), it is clear that the character of the equilbrium configurations is governed by the nature of the optimal demand distribution function. The term $e^{-Ag(x,y)}$ can be expressed in various forms in different transport cost function g(x,y). In this chapter, we still assume a linear trip distribution:

$$e^{-Ag(x,y)} = a - b|x-y|. \tag{2.31}$$

According to the properties of commuting pattern and wage profile[12], there are only three cases to be examined as equilibrium urban configurations. In the subsequent sections, it will be shown that monocentric, completely mixed, and incompletely mixed urban configurations will be equilibrium solutions under specific conditions on parameters.

For the convenience in the subsequent analysis, we introduce some terminologies:

(i) Residential Area: $RA = \{x|h(x)>0, f(x)=0\}$,

(ii) Firm District: $FD = \{x|h(x)=0, f(x)>0\}$,

(iii) Mixed District: $MD = \{x|h(x)>0, f(x)>0\}$.

In order to obtain the explicit analytical solutions, we always assume $\beta=2$.

## 3. Monocentric Urban Configuration

In this section, we will discuss the monocentric urban configuration as depicted in Figure 1. The density functions of households and of firms are given, respectively, by

$$h(x) = 0, \qquad f(x) = \frac{H(x)}{S_b} \qquad \text{for } x\epsilon[-e,e],\ x\epsilon[e,f] \tag{3.1}$$

$$h(x) = \frac{H(x)}{S_h}, \quad f(x) = 0 \quad \text{for } x\epsilon[-f,-e],\ x\epsilon[e,f] \tag{3.2}$$

From Property 2 in Ogawa and Fujita (1978) , the equilibrium wage profile is

$$w(x) = W(0) - t|x| \tag{3.3}$$

where W(0) is the wage paid by firms at the origin, and

[12] All households commute inward from residence toward the firm district. For detail discusion, see the properties 1 and 2 of Ogawa and Fujita(1978).

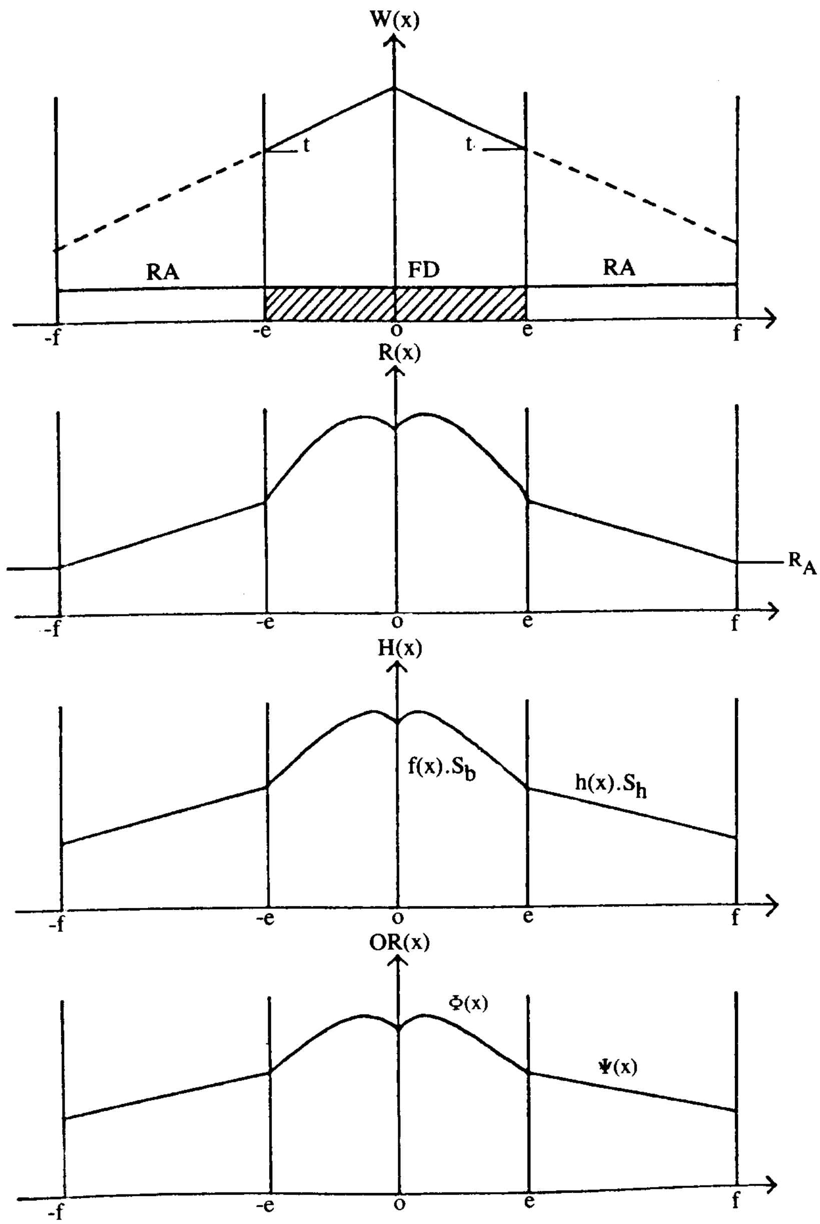


Figure 1. Monocentric Urban Configuration

W(x) is the wage paid at x if $x \epsilon FD$ and the disposable income for households at x if $x \epsilon RA$.

According to the above density functions and wage profile, the equilibrium conditions in the floor and land markets are restated as follows:

$$OR(x) = \Phi(x) \geq \Psi(x) \quad \text{for } x\epsilon[-e,e] \tag{3.4}$$

$$OR(x) = \Phi(x) = \Psi(x) \quad \text{at } x=e,\ x=-e \tag{3.5}$$

$$OR(x) = \Psi(x) \geq \Phi(x) \quad x\epsilon[-f,-e],\ x\epsilon[e,f] \tag{3.6}$$

$$\Phi(x) = \frac{1}{S_b}[\exp(-Ap)\int_{-f}^{f}(a\text{-}b|x\text{-}y|)h(y)dy - LbW(x) - \tau T(x) - \pi^*] \tag{3.7}$$

$$\Psi(x) = \frac{1}{S_h}[\exp(-Ap)\int_{-e}^{e}(a-b|x\text{-}y|)f(y)dy + W(x) - U^*)] \tag{3.8}$$

$$OR(x) = 2\alpha H(x) \quad \text{for } x\epsilon[-f,f] \tag{3.9}$$

$$R(x) = \Gamma(x) = \alpha H(x)^2 \quad \text{for } x\epsilon[-f,f] \tag{3.10}$$

$$R(x) = R_A \quad \text{at } x = -f,f \tag{3.11}$$

First, we calculate the density function of firms in the FD. From (3.1), (3.4), (3.7), and (3.9), we get[13]

$$f(x) = \begin{cases} \dfrac{V\text{-}D\sin(Ke)}{\cos(Ke)}\cos(Kx) + D\sin(Kx) & \text{for } x\epsilon[0,e] \\ \dfrac{V\text{-}D\sin(Ke)}{\cos(Ke)}\cos(Kx) - D\sin(Kx) & \text{for } x\epsilon[-e,0] \end{cases} \tag{3.12}$$

where $V = \dfrac{\sqrt{\alpha R(e)}}{\alpha Sb}$

$$D = \frac{(L_b)t}{2S_b\sqrt{\alpha\tau}}$$

$$K = \frac{\sqrt{\alpha t}}{\alpha S_b}$$

Next, we calculate the density functions of households in the RA.

13 Appendices which contain the solutions are available on request from the author.

Because the city is assumed to be symmetric, it suffices only to examine the right half of the city. From (3.2), (3.6), (3.8), and (3.9), we get

$$h(x) = -\frac{be^{-AP}M+t}{2\alpha S_h^2}x + \frac{2S_h\sqrt{\alpha R_A} + (be^{-AP}M+t)f}{2\alpha S_h^2}. \tag{3.13}$$

However, the density function of firms and households given by (3.12) and (3.13) are still undefined, since it depends on two unknowns: the CBD boundary distance e[14], and urban fringe f. We need two equations to solve these two unknowns. From (3.1), (3.2), (3.5), (3.9), (3.14), and (3.17), we can obtain f and V as

$$f = e + \frac{\alpha S_h N}{\alpha S_b V + \sqrt{\alpha R_A}}, \tag{3.14}$$

$$V = \left[\frac{2\alpha R_A + (be^{-AP}M+t)N}{2S_b^2\alpha^2}\right]^{1/2}. \tag{3.15}$$

Hence, both V and f are dependent on the value of e.

The CBD boundary, e, can be obtained from the total activity unit number constraint (2.26) under the density function of firms given by (3.12) as

$$e = \frac{1}{K}\sin^{-1}\frac{VD+[V^2D^2-(V^2+D^2-B+E)(B-E)]}{V^2+D^2-B+E}, \tag{3.16}$$

where $B = \dfrac{(L_b)tM}{2\alpha S_b^2}$,

$$E = \frac{M^2K^2}{4}.$$

Putting V into (3.16), we can get the CBD boundary, e. Then, the urban fringe, f, can be obtained from (3.14).

From (3.1), and (3.2), we know that the floor properties as firm density and household density functions. Therefore, from (3.12) we can conclude that the firm density

14 The density function of firms, f(x), depends on the value of V which is a function of the land rent at e, R(e). Hence, if we can obtain the CBD boundary distance, e, f(x) can be determined uniquely.

function f(x), and floor density function H(x) are concave in the FD. However, their maxima are not at the center. Instead, they locate at some distance from the center. From (3.13), we know that the household density and floor density functions are linearly decreasing. These density functions are shown in Figure 1.

Having obtained the density functions of households and firms, we can determine the floor rent function from the following manipulation. Differentiating (3.7) and (3.8) with respect to x, we observe that the bid floor rent of firm is concave in the region where $x \geq 0$ and its maximum is not at the center, but at some distance from the center. The bid floor rent of household is linearly decreasing in the RA and concave in the FD. From these observations, $g(x)=1$ for $x \epsilon [0,f]$.

Since $\Psi(x)$ is linearly decreasing in the RA and concave the in FD and $\Phi(x)$ is concave in both the RA and the FD, the equilibrium conditions, (3.4), (3.5), and (3.6) are equivalent to the following two conditions:

$$OR(0) = \Phi(0) \geq \Psi(0), \tag{3.17}$$

$$OR(e) = \Phi(0) = \Psi(e). \tag{3.18}$$

From these two conditions, we obtain the following inequality:

$$\frac{(S_b+S_hL_b)e}{ShSb} t \leq \frac{\tau}{Sb}[eM-2\int_o^e yf(y)dy] - \frac{e^{-AP}}{Sh}\{beM-2b\int_o^e yf(y)dy\}. \tag{3.19}$$

Accordingly, the monocentric urban configuration is an equilibrium if and only if condition (3.11) and (3.23) are satisfied. The associated equilibrium floor rent profile OR(x). wage profile W(x), land rent profile R(x), household density function h(x), and firm density function f(x) are summarized in Figure 1.

Because e depends on t, it is very difficult to solve (3.23) analytically. Hence, we apply the Bisection method to solve it numerically. For this purpose, we must have set of prespecified values of parameters which govern the character of the solutions. We arbitrarily choose the values of N, M, $\tau$, $\alpha$, $R_A$, $S_h$, $S_b$, b, A, c as follows: {1000, 100, 0.05, 2.0, 10.0, 0.01, 0.2, 0.005, 20.0, 0.05}.

The numerical analysis indicates that the monocentric urban configuration is an equilibrium only when $t < 0.065$. We also observe that the size of the FD and the RA increase as increases, and t decreases. The size of the FD increases also as increases (see Figure 4). Namely, as t increases, households tend to locate closely to the FD in order to save commuting cost. When increases, firms and households will disperse in order to save construction costs. When increases, firms cluster in order to save transaction costs.

However, the equilibrium utility level $U^*$, profit level $\pi^*$, and wage profile (particularly W(0)) cannot be endogenously determined within the model. If one of these three variables is exogenously specified, then the other two will be determined.[15]

In order to assure each household has nonnegative budget (see condition (2.2)). We assume $\pi^*=0$. From condition (2.7), and (2.2), by using numerical analysis (all values of parameters are the same as before), we observe $Z_0$ is greater than zero. Therefore, we can conclude that the assumption of budget is reasonable under the zero profit.

## 4. Completely MIxed Urban Configuration:

In this section, we will examine the completly mixed urban configuration as depicted in Figure 2. From the property of no-commuting in the FD, the density functions of households and of firms are given by

$$h(x) = \frac{L_b H(x)}{S_h L_b + S_b},$$

$$f(x) = \frac{H(x)}{S_h L_b + S_b}, \qquad \text{for } x \epsilon [-f, f]. \tag{4.1}$$

The equilibrium conditions in the floor market and in the land market are summarized as follows: for $x \epsilon [-f, f]$

$$OR(x) = \Phi(x) = \Psi(x) \tag{4.2}$$

$$\Psi(x) = (1/Sh)[e^{-AP} \int_{-f}^{f} (a - b|x-y|) f(y) dy + W(x) - u^*] \tag{4.3}$$

$$\Phi(x) = (1/sb)[e^{-AP} \int_{-f}^{f} (a - b|x-y|) h(y) dy - LbW(x) - \pi^* - \tau T(x)] \tag{4.4}$$

$$OR(x) = 2\alpha H(x) \tag{4.5}$$

$$R(x) = \Gamma(x) = \alpha H(x)^2 \tag{4.6}$$

15 This is also true for the case of completely and incompletely mixed urban configurations as shown in the following. This indeterminancy is due to the fixed population os households and firms and to the fixed coefficient utility and production functions.

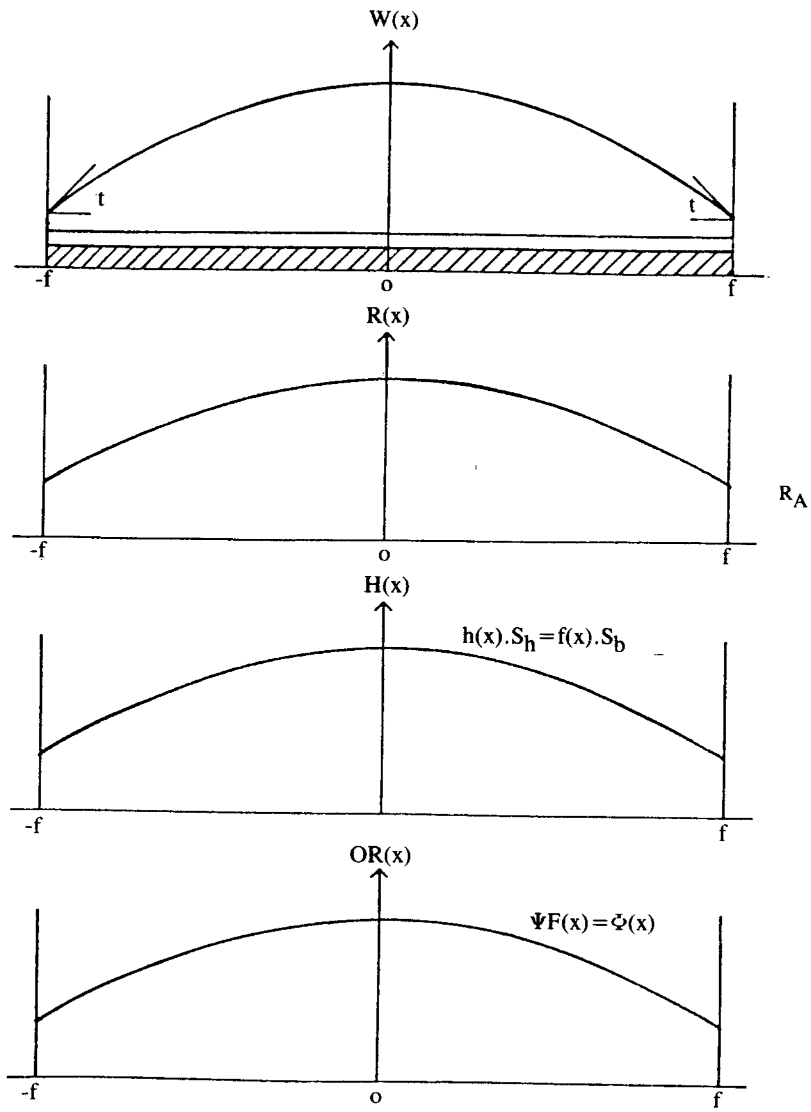


Figure 2 Completely Mixed Urban Configuration

$R(x) = R_A$ at $x = -f, f.$ (4.7)

From (4.2), (4.3), and (4.4) we get

$$W(x) = \frac{(S_h L_b - S_b)e^{-AP}}{(S_h L_b + S_b)^2} \int_{-f}^{f} (a - b|x-y|)H(y)dy + \frac{U^* S_b}{S_b + S_h L_h}$$



30 B

$$- \frac{S_h \pi^*}{S_b + S_h L_b} - \frac{S_h \tau T(x)}{S_h L_b + S_b} . \qquad (4.8)$$

From first and second order condition of W(x), it can be said, W(x) is strictly concave function and has its maximum at x=0. From (4.2), (4.4), (4.5), and (4.8) together we have

$$2a(S_h L_b + S_b) f''(x) + \frac{2be^{-AP} L_b + t}{S_h L_b + S_b} f(x) = 0. \qquad (4.9)$$

Solving the above second-order differential equation, the firm density function is given by

$$f(x) = \frac{\sqrt{\alpha R_A}}{\alpha(S_h L_b + S_b)\mathrm{Cos}(Qf)} \cos(Qx), \qquad (4.10)$$

where $Q = \frac{\sqrt{\alpha 2be^{-AP} L_b + t}}{\alpha(S_h L_b + S_b)}$ .

From (4.5) and (4.10), the floor space rent profile, the floor density function, and household density function are immediately obtained as

$$H(x) = \frac{\sqrt{\alpha R_A}}{\alpha \cos(Qf)} \cos(Qx), \qquad (4.11)$$

$$h(x) = \frac{2\sqrt{\alpha R_A}}{\alpha(S_h L_b + S_b)\cos(Qf)} \cos(Qx), \qquad (4.12)$$

$$OR(x) = \frac{2\sqrt{\alpha R_A}}{\cos(Qf)} \cos(Qx) \qquad (4.13)$$

Obviously, all of them are concave in the MD, and attain their maxima at x=0 (see Figure 2). Because OR(x) is concave everywhere, the g(x) must be equal to 1.

The urban fringe f can be obtained from the total activity unit number constraints as

$$f = \frac{1}{Q} \tan^{-1} \frac{\alpha MQ(S_h L_b + S_b)}{2\sqrt{\alpha R_A}} =$$

From the numerical analysis, we find when construction parameter increases, city size increases. However, when the transaction cost increases, city size will decrease. This result can be interpreted as follows: as increases in transaction cost, firms will cluster in order to save the total transaction cost; on the contrary, when increases in construction cost, firms and households will disperse in order to save construction cost.

Conditions (2.14) and (4.3) imply that $|W'(x)| \geq t$ for all $x \epsilon [-f,f]$. Since $W(x)$ is a strictly concave function, the following condition must hold:

$$W'(f) \geq -t \quad \text{and} \quad W'(f) \leq t.$$

From this, we obtain

$$t \geq \frac{be^{-AP}(S_h L_b - S_b)M + S_h \tau M}{S_h L_b + S_b}. \tag{4.14}$$

Accordingly, the completely mixed urban configuration is an equilibrium pattern if and only if (4.7) and (4.14) are satisfied. Figure 4 illustrates condition (4.14) with the same values of parameters given in Section 3.

## 5. Incompletely Mixed Urban Configuration

Finally, we examine the incompletely mixed urban configuration where a completely mixed zone exists in the center with one residential zone on each side and separated by a firm district as depicted in Figure 3. The density functions are given by

$$h(x) = \begin{cases} \dfrac{L_b H(x)}{S_h L_b + S_b} & x \epsilon [-c,c] \\ 0 & x \epsilon [-e,-c] \text{ and } x \epsilon [c,e] \\ \dfrac{L_b H(x)}{S_h} & x \epsilon [-f,-e] \text{ and } x \epsilon [e,f] \end{cases} \tag{5.1}$$

$$f(x) = \begin{cases} \dfrac{H(x)}{S_h L_b + S_b} & x \epsilon [-c,c] \\ \dfrac{H(x)}{S_b} & x \epsilon [-e,-c] \text{ and } x \epsilon [c,e] \\ 0 & x \epsilon [-f,-e] \text{ and } x \epsilon [e,f]. \end{cases} \tag{5.2}$$

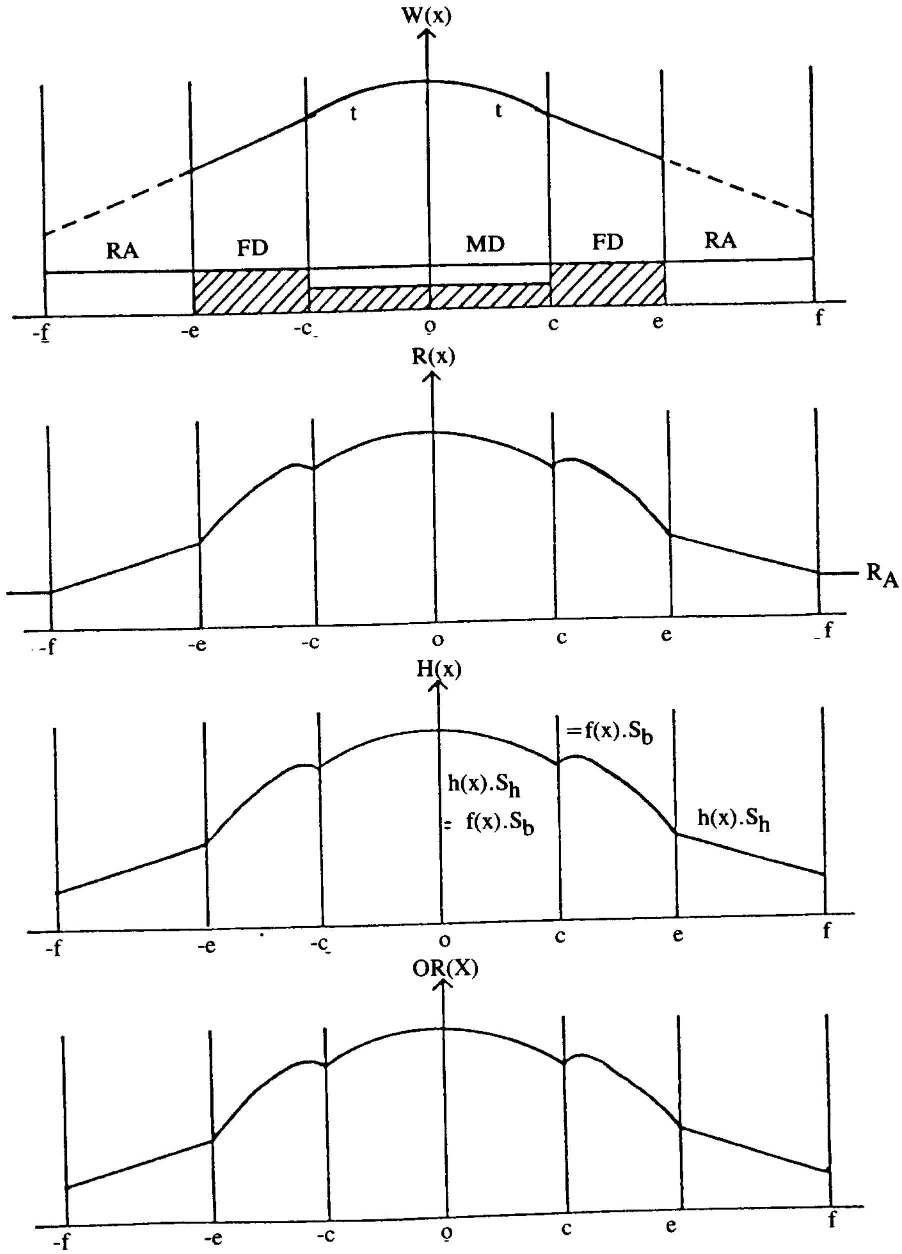


Figure 3 Incompletely Mixed Urban Configuration

The equilibrium conditions in the floor and land markets are restated as follows:

$$OR(x) = \Psi(x) = \Phi(x) \qquad x\epsilon[-c,c] \tag{5.3}$$

$$OR(x) = \Phi(x) \geq \Psi(x) \qquad x\epsilon[c,e] \text{ and } x\epsilon[-e,-c] \tag{5.4}$$

$$OR(x) = \Phi(x) = \Psi(x) \qquad \text{at } x=e,-e \tag{5.5}$$

$$OR(x) = \Psi(x) \geq \Phi(x) \qquad x\epsilon[e,f] \tag{5.6}$$

$$\Psi(x) = (1/S_h)[e^{-AP}\int_{-f}^{f} (a-b|x-y|)f(y)dy + W(x) - U^*] \tag{5.7}$$

$$\Phi(x) = (1/S_b)[e^{-AP}\int_{-f}^{f} (a-b|x-y|)h(y)dy - W(x)Lb - \tau T(x) - \pi^*] \tag{5.8}$$

$$OR(x) = 2\alpha H(x) \tag{5.9}$$

$$R(x) = \Gamma(x) = \alpha H(x)^2 \tag{5.10}$$

$$R(x) = R_A \quad \text{at } x = -f,f \tag{5.11}$$

First, we calculate the wage profile in the MD. Because in the MD, all households must work and live in the same location, the wage profile can be obtained in the similar way as that in the completely mixed pattern as follows:

$$W(x) = \frac{e^{-AP}(S_hL_b - S_b)}{(S_hL_b + S_b)^2}\int_{-c}^{c} (a-b|x-y|)H(y)dy + \frac{1}{S_b + S_hL_b}[S_bU^* - S_h\pi^* - S_h\tau T(x) - S_be^{-AP}an + S_he^{-AP}am] \tag{5.12}$$

where $n = \int_{c}^{f} h(y)dy + \int_{-f}^{-c} h(y)dy$: the number of households in the RA,

$m = \int_{-e}^{-c} f(y)dy + \int_{c}^{e} f(y)dy$: the number of firms in the FD.

From the first and second order condition of (5.12), the wage function in the MD has the same functional form as that in the case of completely mixed configuration, when c is equal to f. So, the properties of wage function are the same as in the last section. It has a maximum point at x=0, and strictly concave in the MD.

All households in the RA commute to firms in the FD. Therefore, the wage profile in the FD and the RA must be a linear function of distance with slope −t:

$$W(x) = W(0) - t|x|. \tag{5.13}$$

Under the wage profile given by (4.9), from (5.2), (5.3), (5.7), (5.9), and together with the boundary condition, we get

$$f(x) = \frac{\sqrt{\alpha R(c)}}{\alpha(S_h L_b + S_b)\cos(Qc)} \cos(Qx). \tag{5.14}$$

Hence, we can get H(x), h(x), and OR(x) in the MD as follows:

$$H(x) = \frac{\sqrt{\alpha R(c)}}{\alpha\cos(Qc)} \cos(Qx), \tag{5.15}$$

$$h(x) = \frac{L_b\sqrt{\alpha R(c)}}{\alpha(S_h L_b + S_b)\cos(Qc)} \cos(Qx), \tag{5.16}$$

$$OR(x) = \frac{2\sqrt{\alpha R(c)}}{\cos(Qc)} \cos(Qx). \tag{5.17}$$

Next, we calculate the firm density in the FD. From (5.2) and (5.8), we get, for $x \epsilon [c,e]$, as follows:

$$f(x) = \begin{cases} \dfrac{V - [D - \dfrac{\tau m}{2K\cos(Ke)}]\sin(Ke)}{\cos[K(e-c)]} \cos(Kx) + [D - \dfrac{\tau m}{2K\cos(Ke)} \sin(Kx) \\ \\ \dfrac{V - [D - \dfrac{\tau m}{2K\cos(Ke)}]\sin(Ke)}{\cos[K(e-c)]} \cos(Kx) - [D - \dfrac{\tau m}{2K\cos(Ke)} \sin(Kx) \end{cases} \tag{5.18}$$

where $V = \dfrac{\sqrt{\alpha R(c)}}{\alpha S_b}$, $D = \dfrac{L_b t}{2S_b\sqrt{\alpha\tau}}$, and $K = \dfrac{\sqrt{\alpha\tau}}{\alpha S_b}$.

The boundary between the FD and RA, e, can be obtained from the total activity unit number constraint as

$$e = \frac{1}{K} \sin^{-1} \frac{VH + \sqrt{V^2H^2 - [V^2 + H^2 - B + E](B - E)}}{V^2 + H^2 - B + E} \tag{5.19}$$

where $H = \dfrac{L_b t}{4\alpha S_b \cos(Ke)\cos(Kc)}$, $B = \dfrac{L_b tM}{2\alpha S_b^2}$ and $E = \dfrac{M^2K^2}{4}$

The household density in the RA can be obtained from (5.1), (5.7) and (5.9),

$$h(x) = -\frac{be^{-AP}M+t}{2\alpha S_h^2}x + \frac{2S_h\sqrt{aR_A}+(be^{-AP}M+t)f}{2\alpha S_h^2}. \qquad (5.20)$$

The urban fringe, f, can be obtained from the total activity constraint

$$f = e + \frac{\alpha S_h(N-n)}{\alpha S_b V+\sqrt{\alpha R_A}} \qquad (5.21)$$

From (5.20), (5.21), we obtain

$$V = \sqrt{\frac{(N-n)(be^{-AP}M+t)+2\alpha R_A}{2\alpha^2 S_b^2}},$$

and

$$R(c) = \frac{(N-n)(be^{-AP}M+t)+2R_A}{2}.$$

When c=0, e and f are equal to the CBD boundary and urban fringe, respectively, of monocentric urban configuration. When c=f, the urban configuration will be the completely mixed pattern. The density functions of households and firms are depicted in Figure 3.

Having derived the density functions of firms and households, the slope and curvature of bid floor rents can be obtained straightforwardly from (5.7) and (5.8) by differentiating them twice. From these calculations, we conclude that the bid floor rent of firms is strictly concave in the MD, FD, and RA, and the bid floor rent of households is strictly concave in the MD and FD, and monotonically decreasing in the RA. The bid floor rent functions are depicted in Figure 3.

Next, from (5.3) and (5.5), we have the following condition on t

$$\frac{e^{-AP}}{S_b}\left[b(e-c)\int_{-c}^{c} h(y)dy\right] + \frac{\tau}{S_b}\left[eM-c\int_{-c}^{c} f(y)dy - 2\int_{c}^{e} yf(y)dy\right]$$

$$- \frac{e^{-AP}}{S_h} [b(e-c)M - 2b\int_c^e yf(y)dy] = \frac{(S_b+S_hL_b)(e-c)}{S_hS_b} t \tag{5.22}$$

Accordingly, if and only if condition(5.11) and (5.22) are satisfied, the incompletely mixed urban configuration is an equilibrium pattern. According to the numerical analysis (the values of parameters are fixed as before), when c is zero, t is the same as the maximum value of it in the monocentric urban configuration. If the value of c increases as it does in the urban fringe of completely mixed urban configuration, t will be the same as the lower bound of it in the completely mixed urban configuration. Figure 4 illustrates the conditions of t and simultaneously according to condition (5.22)

## 6. Conclusion

In this paper, we have presented a model of urban land use pattern in which the density of both households and firms are not fixed and households and firms have price interactions. The model explicity considers the effect of spatial interactions among activities on the behavior: each firm's locational decision reflects its transactions with all other firms and the locations of households; each household's locational decision reflects its commuting trip to the firm chosen by the household and the locations of firms. In this framework, we obtained three equilibrium urban configurations: monocentric urban configuration, completely mixed urban configuration and incompletely mixed configuration. The type of equilibrium configuration to be realized depends on the values of parameters in the model, especially the commuting rate and the transaction cost. In this model, construction parameter is not an influential factor in determining the existence of which type of urban configuration, but an important factor regarding the city size.

From the findings of this analysis, we can concluded that when commuting rate t is relatively small, monocentric city will prevail. However, when t is considerably high, the completely mixed urban configuration will exist at equilibrium and no dominant CBD. The incompletely mixed urban configuration can be considered as the intermediate equilibrium configuration between them. These results are summarized in Figure 4.

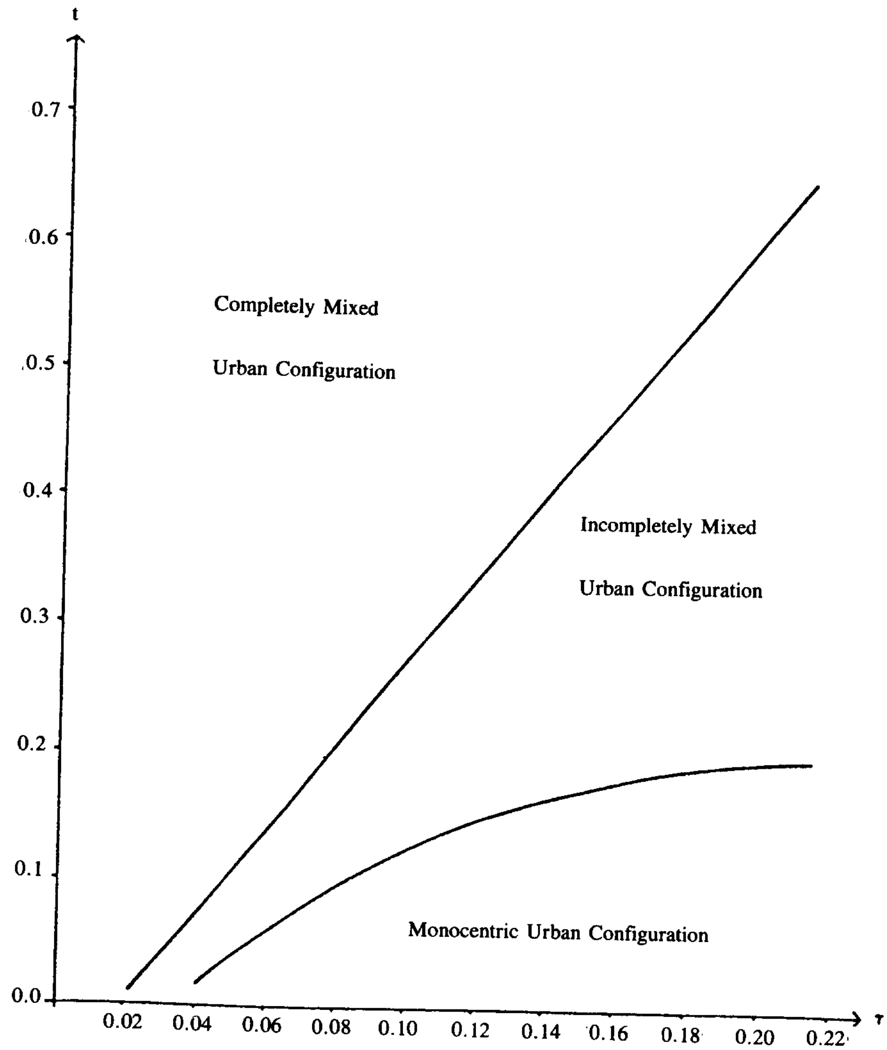


Figure 4 Range of Equilibrium Solution

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