# INTERNATIONAL PORTFOLIO CHOICE AND OPTIMAL CURRENCY HEDGE

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## 摘　要

由於國際間資產報酬，彼此的不完全相關，使得國際投資帶來資產風險分散的好處。唯由於匯率變化之不可預測，國際投資亦帶來了新的（外匯）風險。過去國際財管對於資產組合與最適外匯避險策略採取分開處理的方式。本研究採用Merton（1973）連續性（Continuous time）跨時（intertemporal）的動態最適方法將此二課題併於同一個理論架構之上。

由於國際投資可能組合（investment opportunity set）之不恆定性（nonstationary），對於外匯避險工具之需求，除了傳統的投機性需求（speculative demand）及純避險需求（pure hedging demand）之外，亦將包含新的動態避險需求（dynamic hedging demand）。本研究亦將尋求〝m＋2〞個最有效率的共同基金，其中包含了一個國際的資產市場組合，一個無風險的資產及m個最有效率的外匯避險資產組合。此m＋2個基金將可完全替代國際財務市場上所有的投資工具，從而大大地減輕了投資者篩選國際資產時的昂貴交易及資訊成本，本研究亦將探討CAPM運用於國際投資時可能遭遇的問題，並且做必需的修正。

## 1. Introduction

Exchange rate variations can affect an investor's portfolio choice if his portfolio includes foreign assets as well. In the absence of non-zero expected exchange rate movement, exchange risk is undesirable and should be hedged to a certain degree. Once the extent of an investor's vulnerability to foreign currency risk is determined, it may readily be modified using financial hedging instruments such as forward contracts, swap and a variety of insurance schemes. The optimal hedging ratio is usually determined by the covariance $\sigma_{pf}$ between unhedged position and the hedging instrument, e.g., forward exchange contract, divided by the variance of the hedging instrument $(\sigma_f^2)$.

If the exchange rate variations are not zero-sum game and an investor attempts to maximize expected utility rather than simply minimize risk, the optimal hedging strategy depends on a ''speculative demand'' in addition to the aforementioned

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"pure hedging demand," i.e. $\frac{\sigma_{vf}}{\sigma_f^2}$. In the traditional mean-variance approach,[1] the speculative demand is positively related to the expected return of the hedging instrument and negatively related to the variance of the hedging instrument and the degree of an investor's risk tolerance.

From the theoretical point of view, the previous approaches on currency hedging strategy are not optimal in the following several aspects. Firstly, the mean-variance objective functions conforms with the general expected utility maximization only when the utility functions are elliptically distributed. Secondly, portfolio choice and optimal currency hedge ratio are separatedly treated. Virtually, an individual will decide his optimal choice of current consumption and asset investment (including foreign assets) simultaneously. The demand for a hedging instrument is nothing but a demand for an asset that happens to be closely correlated with foreign exchange movement. Finally, the derivation of currency hedging demand should be based on an intertemporal framework. Since any decision about investment policy is intertemporal, one period maximization is not necessarily the optimal result when an investor attempts to maximize his overall utility level within his life horizon or even including the utility of his endless offspring.

In the following I will develop a model that simultaneously resolves the problem of international portfolio choice and optimal currency hedging stratgegy. Most of the recent literature has been devoted to models where nations are defined as zones of a common purchasing power unit. National groups of investors are delineated by deviations from Purchasing Power Parity (PPP) which cause them to evaluate differently the returns from the same security. These works assume that asset prices follow stationary Brownian motions.[2] However, Lucas (1978) shows that the stationary Brownian motion for asset prices is not consistent with the positive risk aversion of investors. Under risk aversion, asset prices multiplied by discounted marginal utilities must follow martingales and so discounted asset prices by themselves do not. This flaw can be corrected by introducing nonstationary Ito process when the parameters $\mu$ and $\sigma$ in the Ito process for asset prices would be functions of a vector of state variables. The resulting portfolio choices would contain one more (hedge) fund per state variable in addition to the two funds: riskless bond fund and market portfolio fund.

This paper will adopt the continuous-time methodology of Merton (1973) that treats $\mu$ and $\sigma$ in the Ito process as nonstationary. The factor resulting in the deviation of PPP for foreign asset prices (or the deviation of Interest Rate Parity for the returns of foreign assets) are treated as state variables that will influence $\mu$ and $\sigma$. Under

[1] See Anderson and Danthine, 1981.

[2] Solnik (1974), and Adler and Dumas (1983).

this structure we will discuss extensively how international portfolio is chosen, how the optimal currency hedge ratio is determined, and what modification should be made for the international CAPM.

## 2. The Model

In a world of free trade, the law of one price should hold theoretically absent the barriers to trade. Exchange rates are used to convert the prices of foreign goods (or assets) to domestic ones according to Purchasing Power Parity (PPP). To discuss the expected returns on financial assets, we focus on the relative version of PPP, that is, the relative price changes between countries should equate the expected change in the foreign exchange rates. However, this relative version of PPP is true only when the law of one price holds not only at present but also at any moment in the future. Otherwise, real interset rates will not necessarily be the same between countries. Allowing for differential real interest rates between countries we can derive the Interest Rate Parity (IRP) condition which asserts the equalization of differential nominal interest rates (which might result from differential expected inflation rates or from differential real interest rates between countries) and expected exchange rate changes between countries.

An investor who considers an international portfolio will compare foreign assets with domestic ones based on IRP. In other words, the foreign assets are added to the investor's investment portfolio by simply adding expected foreign exchange rate's change to the foreign assets' returns. Our model below follows the intertemporal approach introduced by Merton (1973). The choice of consumption and investment level will be made simultaneously. In the international dimension, the content of consumption and investment includes not only domestic goods or assets but also foreign ones.

$$Max \qquad E\{\int_0^t U[c(t),\ t)]dt + B[W(T),\ T]\} \tag{1}$$

subject to

$$dW = [\sum_1^n \omega_i[\alpha_i - r] + r]Wdt + \sum_1^n \omega_i W\sigma_i dz_i + (y - c)dt \tag{2}$$

where $W(t)$ is the wealth at time $t$, which will be used for consumption $c(t)$ or investment in the riskless asset with riskless interest rate $r$ or in the $n$ risky assets whose prices follow Ito processes (Brownian motions) as

$$\frac{dP_i}{P_i} = \alpha_i dt + \sigma_i dz_i , \quad i = 1, 2, \ldots, n. \tag{3}$$

$\omega_i$ is the proportion of $W$ invested in the i-th risky asset. Consequently, $1 - \Sigma_1^n \omega_i$ is the proportion invested in the riskless asset. $y(t)$ is the income other than the revenue from the investment. $B[W(T), T]$ is the bequest at the terminal date.

Let the riskless asset be the (n+1)th asset. We can rewrite the wealth change constraint as

$$dW = \sum_1^{n+1} \omega_i W \frac{dP_i}{P_i} + (y - c)dt. \tag{4}$$

We consider a world of $m$ countries and $n+1$ investment instruments. Each country is characterized by its unique factor that causes the behavior of the country's asset returns to deviate from IRP. These factors denoted by the state variable $S_i$ ($i=1, 2, \ldots, m$) will change according to

$$dS_i = f_i dt + g_i dq_i, \quad i = 1, 2, \ldots, m, \tag{5}$$

or

$$dS = F(S)dt + G(S)dQ$$

where $S = [S_1, S_2, \ldots, S_m]$ , $F=[f_1, f_2, \ldots, f_m]$, $G$ is a diagonal matrix with diagonal elements $[g_1, g_2, \ldots, g_m]$, and $dQ$ is a vector Wiener process $[dq_1, \ldots, dq_m]$.

The $n$ risky assets consist of all the investment opportunities in the world, including foreign exchange forward contracts which can be used to hedge the exchange risk associated with $dq_i$. Note that $S$ will influence the individual's choice, i.e. maximizing (1) subject to (2), through its impact on the $\alpha_i$'s and $\sigma_i$'s in the Ito process for the $n$ risky assets.

Let $\eta_{ij}$ be the instantaneous correlation coefficient between $dq_i$ and $dz_i$, $\sigma_{ij}$ be the instantaneous covariance between the returns on the i-th and j-th assets $(=\sigma_i\sigma_j\rho_{ij})$, $\nu_{ij}$ be the instantaneous correlation coefficient between $dq_i$ & $dq_j$.

The Ballman Principle tells us that the necessary optimality condition for the individual who acts to maximize (1) subject to the (2) constraint are that

$$0 = Max_{c,\omega}(\mathrm{U}(c,t) + I_t + I_w \{[\sum_1^n \omega_i (\omega_i - r)+r]W - c\} \tag{6}$$
$$+ \Sigma_1^m I_i f_i + \frac{1}{2} I_{ww} \Sigma_1^n \Sigma_1^n \omega_i \omega_j \sigma_{ij} W^2$$
$$+ \Sigma_1^m \Sigma_1^n I_{iw} \omega_j W g_i \sigma_j \eta_{ij} + \frac{1}{2} \Sigma_1^m \Sigma_1^m I_{ij} g_i g_j \nu_{ij})$$

subject to

$$I(W,S,T) = B(W,T).$$

Note that $y(t)$ is ingored here for simplicity. $I(W, S, T)$ is the indirect utility by solving (1) and (2). And

$I_i$ is the derivative of $I$ with respect to $S_i$.
$I_W$ is the derivative of $I$ with respect to $W$,
$I_{iW}$ is the derivative of $I_i$ with respect to $W$,
$I_{ij}$ is the derivative of $I_i$ with respect to $S_j$,
$I_{WW}$ is the derivative of $I_W$ with respect to $W$.

The $n+1$ first order conditions are

$$0 = U_c(c,t) - I_W(W,S,T), \tag{7}$$

$$0 = I_W(\alpha_i - r) + I_{WW} \sum_{j=1}^{n} \omega_j W \sigma_{ij} + \sum_{j=1}^{m} I_{jw} g_j \sigma_i \eta_{ji}, \quad i=1, \ldots, n. \tag{8}$$

(8) can be rewritten as

$$\omega_i W = A \Sigma_{j=1}^n \nu_{ij} (\alpha_j - r) + \Sigma_{k=1}^m \Sigma_{j=1}^n H_k \sigma_j g_k \eta_{kj} \nu_{ij}, \quad i=1, \ldots, n, \tag{8'}$$

where the $\nu_{ij}$ are the elements of the inverse of the variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$. And

$$A = -\frac{I_W}{I_{WW}}, \text{ and } H_k = -\frac{I_{kW}}{I_{WW}}.$$

From Equation (7), we can interpret $A$ and $H_k$ as

$$A = -\frac{U_C}{U_{CC}(\frac{\partial C}{\partial W})} > 0. \tag{9}$$

$$H_k = -\frac{\frac{\partial C}{\partial S_k}}{\frac{\partial C}{\partial W}} \gtrless 0. \tag{10}$$

(or $H_k = -\frac{\frac{\partial C}{\partial S_k}}{\frac{\partial C}{\partial W}} - \frac{U_{CS_K}}{U_{CC}(\frac{\partial C}{\partial W})}$ , if $U$ is state dependent.)

A can be interpreted as the degree of absolute risk tolerance, or $A/W$ as the degree of relative risk tolerance of the investor. $H_k$ is the compensation variation in wealth required following a change in the $k$-th state variable to maintain the investor's lifetime expected utility.

## 3. Perfectly Correlated Hedging

In the beginning we assume that there is only one foreign country which is characterized by a state variable, $e$. The investment opportunity set is now a function of this state variable $e$. According to Equation (8)' in the last section, we have

$$\omega_i W = A \sum_{j=1}^{n} v_{ij}(\alpha_j - r) + H \sum_{j=1}^{n} \sigma_{je} , \tag{11}$$

where $H \equiv -\frac{\frac{\partial C}{\partial S_k}}{\frac{\partial C}{\partial W}}$ , *and* $\sigma_{je} \equiv \rho_{je}\sigma_{jg}$ .

By transforming all the foreign goods' prices to the domestic currency unit, domestic state variable $S$ can be eliminated because of its role as numeraire. Therefore, one foreign state variable is sufficient to describe the change in the investment opportunity set.

In this section, it is further assumed that there exists an asset, the $n$-th one, whose return is perfectly correlated with changes in the state variable $e$. In other words, we assume that $\rho_{ne}=1$. The foreign exchange forward contract can be used as a proxy for this $n$-th asset. The demand for this asset tells us how optimal currency strategy should be constructed.

Since

$$\rho_{je} = \rho_{jn} ,$$

$$\sigma_{je} = \rho_{je}\sigma_j g = \frac{g(\rho_{in}\sigma_j)}{\sigma_n} ,$$

and

$$\Sigma^n_{j=1} v_{ij}\sigma_{je} = \frac{g(\Sigma^n_{j=1} v_{ij} \ \sigma_{jn})}{\sigma_n} = 0, \ if \ i \neq n,$$
$$= \frac{g}{\sigma_n}, \quad if \ i = n.$$

Therefore, Equation (11) can be rewritten as

$$\omega_i W = A\Sigma^n_{i=1} v_{ij} \ (\alpha_j - r), \ i = 1, \ldots, n-1,$$
$$= A\Sigma^n_{i=1} v_{ij} \ (\alpha_j - r) + \frac{gH}{\sigma_n} , \ i=n.$$

The optimal currency hedge can be seen from Equation (11)'. The demand for the $n$-th asset (or forward exchange contract) can be decomposed as (i)$A\Sigma^n_{i=1} v_{ij}(\alpha_j - r)$ and (ii) $\frac{gH}{\sigma_n}$ .

Illustration of (i):

In a single asset case, (i) becomes $A\ \frac{\alpha_n - r}{\sigma_n^2}$. $\frac{\alpha_n - r}{\sigma_n^2}$ is the excess return per risk (ERPR). The higher ERPR and the greater the risk tolerance $A$, the greater the demand for this hedging asset.

If there are more than one risky assets, say 2 assets, there will be another component of demand for this asset. For the two-asset case, (i) consists of $A\ \frac{-\sigma_{12}(\alpha_1 - r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$ and $A\ \frac{\sigma_1^2(\alpha_2 - r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$.

By defining $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$, we can rephrase the second term as

$$A\ \frac{\alpha_2 - r}{\sigma_2^2(1-\rho^2)} = A \cdot \widetilde{ERPR}(2). \qquad (12)$$

where $\widetilde{ERPR}(2) = \frac{\alpha_2 - r}{\sigma_2^2(1-\rho^2)}$ is the revised $ERPR(2)$. (12)

In other words, only the uncorrelated or nondiversifiable portion of the risk, i.e. $\sigma_2^2\ (1 - \rho^2)$, should be taken into account in computing ERPR for asset 2. We call this component of the demand for asset 2 as the "speculative demand" which corresponds to the entire demand in a single asset case.

In addition to the speculative demand described above, we have another demand associated with the first term, $A\ \frac{-\sigma_{12}(\alpha_1 - r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$. This term can be expressed in the following way:

$$-A\ \frac{\sigma_{12}\ (\alpha_2 - r)}{\sigma_2^2\ \sigma_1^2(1-\rho^2)} = -A\ \frac{\sigma_{12}}{\sigma_2^2}\ \widetilde{ERPR}(1). \qquad (13)$$

Adding one unit of asset two will create $\sigma_2^2$ units of variance in which $\sigma_2^2\rho^2$ can be diversified away owing to its correlation with asset one. The speculative demand (12) above is derived from the excess return $(\alpha_2 - r)$ per unit of the nondiversifiable portion of the variance, i.e. $\sigma^2\ (1-\rho^2)$. Similarly, adding one unit of asset one will generate $\sigma_1^2$ unit of variance of which $\sigma_1^2\rho^2$ needs to be hedged by asset two. Let $h$ be the units of asset two required to hedge the diversifiable risk, i.e. $\sigma_1^2\rho^2$, from the addition of one unit of asset one. The optimal $h$ should satisfy the condition that the increase of variance from $h$ units of asset two, i.e.

$h^2\sigma_2^2$, must be equal to the decrease of the diversifiable risk in one unit of asset one, i.e. $\sigma_1^2\rho^2$. In other words,

$$h^2\sigma_2^2 = \sigma_1^2\rho^2$$

implies

$$h = \frac{\sigma_1\rho}{\sigma_2} = \frac{\sigma_{12}}{\sigma_2^2}.$$

Since the speculative demand for asset one is equal to $A\cdot\frac{ERPR(1)}{1-\rho^2}$ or $A\cdot\widetilde{ERPR}(1)$, the hedging demand for asset two in order to diversify $A\cdot\widetilde{ERPR}(1)$ units of asset one equals

$$-hA\cdot ERPR(1) = -A\,\frac{\sigma_{12}}{\sigma_2^2}ERPR(1).$$

The $h$ thus derived is exactly Eqution (13) above.

Illustration of (ii):

The (i) portion of the demand for the perfectly correlated hedging instrument (asset two) corresponds to the speculative demand and the pure hedge demand that are addressed in the Anderson and Danthine (1981). These two demands are derived based on the assumption that the investment opportunity set is constant or the expected return, variance and covariance with other assets are all stationary for all the assets in the world. However, we know that the expected return (or variance, covariance) of foreign assets denominated in the domestic currency unit will not remain unchanged once the exchange rate fluctuates not in line with *ex post* IRP. This added risk stemming from the variation of exchange rate in violation of *ex post* IRP will create another hedge demand for asset two.

This added hedging demand is called "dynamic hedging demand", which is distinguished from the previous "static hedging demand", as described in Equation (13). The dynamic hedging demand, $\frac{gH}{\sigma_n}$, can be illustrated as follows: Let $x$ be the amount of dynamic hedging demand. This demand will result in an increase of $x^2\sigma_n^2$ variance. Since the impact of one unit of variance on utility is equal to $I_{WW}$, this demand will contribute to a change of utility by the amount of $I_{WW}x^2\sigma_n^2$. However, this utility change has to be balanced by the change of utility from the indirect effect of the resulting change in state variable, which amounts to $-I_{nW}[x\sigma_n\rho_{ne}g] = I_{nW}[x\sigma_n g]$ since $\rho_{ne}=1$ by assumption.

Hence,

$$I_{WW}x^2\sigma_n^2 = I_{nW}\ x\sigma_n g$$

implies

$$x=-\frac{I_{nW}}{I_{WW}}\frac{g}{\sigma_n} = \frac{Hg}{\sigma_n}\ . \tag{14}$$

In fact, we can interpret $H$ as the implicit price per unit of $\frac{g}{\sigma_n}$, similar to $A$ as the implicit price per unit of $\widetilde{ERPR}$.

From the mutual fund (separation) theorem, we know that the asset demand in Equation (11)' can be represented by three funds. Let the first fund hold the proportion $\delta_k$ in asset $k$. $\delta_k$ is determined as follows:

$$\delta_k = \frac{\Sigma_{j=1}^n v_{ij}\ (\alpha_j - r)}{\Sigma_{i=1}^n \Sigma_{j=1}^n v_{ij}(\alpha_j - r)'} \qquad k=1,\ \ldots,\ n. \tag{15}$$

Let the second fund hold only the $n$-th asset and the third fund only the riskless asset. Let $\lambda_i$ be the fraction of the investor's wealth invested in the $i$-th fund $(i=1,2,3)$, so that $\lambda_1+\lambda_2+\lambda_3=1$. Assume

$$\lambda_1\delta_i = \frac{A}{W}\Sigma_{j=1}^n v_{ij}\ (\alpha_j - r), \qquad i=1,\ \ldots,\ n-1,$$

$$\lambda_1\delta_n + \lambda_2 = \frac{A}{W}\Sigma_{j=1}^n v_{ij}\ (\alpha_j - r) + \frac{gH}{\sigma_n W}\ ,$$

which exactly replicate the demand functions (11)'. Therefore

$$\lambda_1 = \frac{A}{W}\sum_{j=1}^{n}\sum_{j=1}^{n} v_{ij}\ (\alpha_i - r), \tag{16}$$

$$\lambda_2 = \frac{gH}{\sigma_n W},$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2.$$

$\lambda_1$ represents the proportion of wealth held in the risky market portfolio which is constructed according to Equation (15). The speculative demand (Equation (12)) and static hedging demand (Equation (13)) for asset $n$ have been subsumed in this market portfolio. $\lambda_2$ represents the proportion of wealth held in the second fund which reflects the "dynamic hedging demand" for asset $n$. In other words, due to the nonstationary nature of the investment opportunity set caused by the change in state variable, the market portfolio can not sufficiently represent the whole asset demand. We need this second fund to capture the dynamic characteristic of opportunity set. The third fund addresses the minimum asset demand in a riskless world.

## 4. Imperfectly Correlated Hedging

In the last section we assume that asset $n$ is perfectly correlated with the state variable. However, the perfectly correlated asset usually does not exist. Empirical evidence shows that the regression of interest rate differential on the exchange rate changes turns out poorly.[3] The factors that cause the deviation from IRP are "news" that may not be expected at present according to the assertion of rational expectation theorists. Therefore, the forward exchange contract as discussed in the last section would not be perfectly correlated with the state variable $S_i$. The demand for the $n$-th asset (the forward exchange contract) will be Equation (11) instead of Eqution (11)'. The speculative and the static hedging demands remain the same. However, the dynamic hedging demand will become more complicated. It requires the knowledge of all the covariances among traded assets (i.e. $\sigma_{ij}$) and the correlation between traded assets and the state vriable (i.e. $\sigma_{je}$). More importantly, the forward exchange contract will not be the most efficient hedging instrument for the unfavorable change in the state variable. We need to find a substitute that has the largest correlation with the state variable $S_i(t)$.

Let this highest-correlated hedging instrument be represented by a fund $V_i$. Let $X_j(t)$ denote the fraction of the fund's portfolio allocated to asset $j$ at time $t$, $j=1, \ldots, n$.[4] Then

[3] e.g., R. Roll and B. Solnik (1977); E. Fama (1984); R.E. Cumby and M. Obstfeld (1981); and R.J. Hodrick and S. Srivastava (1984).

[4] $1 - \Sigma_1^n x_j(t)$ allocated to the riskless asset.

$$\frac{dV_i}{V_i} = [\Sigma_1^n x_j(\alpha_j - r) + r]dt + \Sigma_1^n x_j\sigma_j dz_j$$

Define $$Y_i\ (t) \equiv \frac{S_i}{V_i(t)}\ .$$

The probability distribution for $\frac{Y_i(t)}{Y_i(0)}$ provides an indicator against changes in state variable $S_i$. By Ito Lemma,

$$d(logY_i) = d(logS_i) - d(logV_i) = \theta_i(t)dt + \phi_i(t)d\epsilon_i\ , \tag{17}$$

where

$$\theta_i\ (t) \equiv f_i - \frac{1}{2}\ g_i^2 + \frac{1}{2}\ \Sigma_{j=1}^n\ \Sigma_{k=1}^n\ x_jx_k\sigma_{jk} - r - \Sigma_{j=1}^n\ x_j(\alpha_j - r),$$

$$\phi_i^2(t) \equiv g_i^2 + \Sigma_{j=1}^n\ \Sigma_{k=1}^n\ x_jx_k\sigma_{jk} - 2\Sigma_{j=1}^n\ x_j\sigma_j\eta_{ij}g_i,$$

$$d\epsilon_i \equiv 0, \quad \text{if } \phi_i(t) = 0; \text{ otherwise,}$$

$$d\epsilon_i \equiv \frac{g_i dq_i - \Sigma_{j=1}^n\ x_j\sigma_j dz_j}{\phi_i(t)}\ ,$$

where $d\epsilon_i(t)$ is a standard Wiener process. Note that we have redefined the dynamics for $S_i(t)$ as

$$dS_i = f_iS_idt + g_iS_idq_i$$

to facilitate our discussion in the dynamics for $Y_i(t)$.

We attempt to find a fund that has a minimum $\phi_i^2(t)$. The first order conditions for a minimum $\phi_i^2(t)$ are

$$\frac{\partial\phi_i^2}{\partial x_j} = 2(\Sigma_{k=1}^n\ x_k\sigma_{jk} - \sigma_i\eta_{ij}g_i) = 0,\ j = 1,\ \ldots,\ n.$$

Solving the linear system above gives

$$x_j = g_i \sum_{k=1}^{n} v_{kj}\sigma_k\eta_{jk} = g_i\delta_{ij}, \; j = 1, \ldots, n, \tag{18}$$

where

$$\delta_{ij} \equiv \Sigma_{k=1}^{n} v_{kj}\sigma_k\eta_{ik}.$$

Define the maximal-feasible correlation coefficient $\rho_i^*$ as

$$\rho_i^* \equiv (\Sigma_{j=1}^{n} \delta_{ij}\eta_{ij}\sigma_j)^{\frac{1}{2}}$$
$$= (\Sigma_{j=1}^{n} \Sigma_{k=1}^{n} v_{kj}\sigma_k\sigma_j\eta_{ik}\eta_{ij})^{\frac{1}{2}}$$

$\rho_i^* = 1$ is the case of perfectly correlated hedging. By substitution of $x_j = g_i\delta_{ij}$ into $dV_i/V_i$ we get

$$\frac{dV_i}{V_i} = \alpha_i^* + \sigma_i^* dz_i^*, \tag{19}$$

$$\alpha_i^* \equiv r + g_i\Sigma_i^n \delta_{ij}(\alpha_j - r),$$

$$\sigma_i^* \equiv g_i\rho_i^*,$$

$$dz_i^* \equiv \frac{\Sigma_{i=1}^{n} \delta_{ij}\sigma_j dz_j}{\rho_i^*},$$

where $dz_i^*$ is a standard Wiener process.

## Property of Fund $V_i$

For each state variable $S_i$, we can create a fund $V_i$, $i=1, \ldots, m$. In the following we investigate some properties of $V_i$.

(a) Fund $V_i$ has the maximal feasible correlation with $S_i$, i.e., the unanticipated change in country $i$'s exchange rate movement that violates IRP.[5]

(b) $\frac{dP_k}{P_k} d\epsilon_i = 0, \quad for\ i = 1, \ldots, m, \ k=1, \ldots, n.$

[5] By definition of $V_i$.

From Equation (17) we can interpret $d\epsilon_i$ to be the remaining risk of $S_i$ after taking into account the impact of $V_i$ which has the maximal feasible correlation with $S_i$. Therefore, property (b) means that this remaining risk is a white noise for the return on each traded asset. The dynamics of $S_i$ after extracting $V_i$ can be rewritten in a concise way. By substitution of the portfolio weight $x_j = g_i\delta_{ij}$ into Equation (17), we have

$$\theta_i = f_i - \alpha_i^* - \frac{\phi_i^2}{2}$$

$$\phi_i^2 = [1 - (\rho_i^*)^2]g_i^2$$

$$d\epsilon_i = (dq_i - \rho_i^* dz_i^*) / [1 - (\rho_i^*)^2]^{\frac{1}{2}}$$

Moreover,

$$\frac{dP_k}{P_k} d\epsilon_i = \frac{dP_k}{P_k} \cdot \frac{g_i dq_i - \Sigma_1^n x_j\sigma_j dz_j}{\phi_i(t)}$$

$$= \frac{(g_i\eta_{ik}\sigma_k - \Sigma_1^n x_i\sigma_{jk})dt}{g_i (1-\rho_i^{*2})^{\frac{1}{2}}}$$

$$= 0,$$ [6]

From Equation (17) and (19), we get

$$d(logS_i) = d(logV_i) + d(logY_i)$$

$$= (\alpha_i^* + \theta_i)dt + \sigma_i^* dz_i^* + \phi_i d\epsilon_i.$$

Therefore,

$$\frac{dS_i}{S_i}\frac{dP_k}{P_k} = g_i\sigma_k\eta_{ik}dt$$

$$= \sigma_i^* \sigma_k (dz_i^* dz_k)$$ [7]

$$= \frac{dV_i}{V_i}\frac{dP_k}{P_k},$$

[6] Since $g_i\eta_{ik}\sigma_k - \Sigma_i^n x_i\sigma_{jk}=0$, by the definition of $x_j$.

[7] Since $\frac{dP_k}{P_k} d\epsilon_i = 0$.

which is rewritten as property (c) below:

(c) $$\frac{dS_i}{S_i}\frac{dP_k}{P_k} = \frac{dV_i}{V_i}\frac{dP_k}{P_k}, \quad i=1, \ldots, m, \; k=1, \ldots, n.$$

As a matter of fact, an investor's international portfolio choice can be generated by $m+2$ funds: $m$ funds decribed above to hedge unfavorable exchange rate changes (i.e. $S_i$, $i = 1, \ldots, m$); a mean-variance efficient market portfolio of risky assets and a riskless asset.

Let $V_{m+1}(t)$ denote the price per share on the $(m+1)$th fund and $V_{m+2}(t)$ the price per share on a fund that holds riskless assets only. Then

$$\frac{dV_{m+2}}{V_{m+2}} = rdt \text{ which implies } \frac{V_{m+2}(T)}{V_{m+2}(t)} = exp[\textstyle\int_t^T r(\tau)d\tau]$$

As for fund $m+1$, the portfolio weight $\chi_k = [\Sigma_1^n v_{kj} (\alpha_j - r)]/\lambda$, $k = 1, \ldots, m$, where $\lambda = \Sigma_1^n \Sigma_1^n v_{ij} (\alpha_j - r)$, will satisfy the mean-variance efficiency condition when the opportunity set generated by all traded assets is stationary or independent of any state variable. From property (b) above, we know that any asset return becomes independent of the remaining risk of $S_i$ in the presence of the $m$ funds, i.e., $V_i$, $i=1, \ldots, m$.

The dynamics for the return on the $(m+1)$th fund can be written as $\dfrac{dV_{m+1}}{V_{m+1}} = \alpha^*_{m+1}dt + \sigma^*_{m+1}dz^*_{m+1}$,

where

$$\alpha^*_{m+1} \equiv r + \Sigma_1^n\Sigma_1^n v_{kj}(\alpha_j - r)(\alpha_k - r)/\lambda,$$

$$(\sigma^*_{m+1})^2 \equiv \omega'\Omega\omega \text{ [8]}$$

$$= \omega'\Sigma_1^n \Sigma_1^n v_{kj}(\alpha_j - r)\sigma_{ij}/\lambda,$$

$$= \frac{1}{\lambda^2}\Sigma_1^n \Sigma_k^n (\alpha_j - r)(\alpha_k - r)v_{kj}$$

$$= (\alpha^*_{m+1} - r)/\lambda.$$

and

$$dz^*_{m+1} = \frac{1}{\lambda} \Sigma^n_k \Sigma^n_j v_{kj} (\alpha_j - r)\sigma_k dz_k / \sigma^*_{m+1}$$

Morever, we can show that CAPM will hold for any feasible portfolio that is constructed from the $n$ risky assets, as long as the market portfolio is constructed according to the following portfolio weight:

$$x_k = \frac{\Sigma^n_{i=1} v_{kj}(\alpha_j - r)}{\Sigma^n_{i=1} \Sigma^n_{j=1} v_{ij}(\alpha_j - r)}, \qquad k=1, \ldots, n.$$

In the presence of nonstationary opportunity set, the market portfolio above is not necessarily mean-variance efficient, and the $\beta$ thus generated is not "stationary" any more unless some revision is made. This revised CAPM will be discussed in Section V. Here we focus on the conventional CAPM. This property is stated as follow:

Let $dV/V$ be the return on any feasible portfolio constructed from the $n$th risky assets. If $\frac{dV_{m+1}}{V_{m+1}}$ is the return on a mean-variance efficient portfolio, then $\alpha - r = \beta(\alpha^*_{m+1} - r)$, where $\alpha$ is the expected rate of return on the portfolio $V$ and

$$\beta = \frac{(dV/V)\ (dV_{m+1}/V_{m+1})}{dV_{m+1}/V_{m+1}{}^2}.$$

Proof:

$$\frac{dV}{V}\frac{dV_{m+1}}{V_{m+1}} = (\Sigma^n_1 x_j\sigma_j dz_j)\ (\sigma^*_{m+1} dz^*_{m+1}) \quad = \sigma^*_{m+1}\ \Sigma^n_1 x_j\sigma_j dz_j\, dz^*_{m+1}$$

$$\beta \equiv (\frac{dV}{V}\frac{dV_{m+1}}{V_{m+1}}) / \frac{dV^2_{m+1}}{V_{m+1}} = \frac{\Sigma^n_1 x_j\sigma_j dz_j dz^*_{m+1}}{\sigma^*_{m+1}\ dt}$$

But

[8] $\omega' = (x_1, x_2, \ldots, x_n)$, $\Omega = [\sigma_{ij}]$.

$$\sigma_j dz_j dz^*_{m+1} = \sigma_j dz_j \; [\Sigma^n_i \Sigma^n_k v_{ki}(\alpha_i - r)\sigma_k dz_k] / \; (\lambda\sigma^*_{m+1})$$

$$= [\Sigma^n_i \; \Sigma^n_k v_{ki}(\alpha_k - r)\sigma_{ki} \; / \; (\lambda\sigma^*_{m+1})]dt$$

$$\beta = \Sigma^n_j \Sigma^n_j x_j(\alpha_i - r)(\Sigma^n_k v_{ki}\sigma_{kj})/[\lambda(\sigma^*_{m+1})^2].$$

Therefore,

$$\beta = \frac{\Sigma^n_i x_j(\alpha_j - r)}{\lambda(\sigma^*_{m+1})^2} \; {}^{9}$$

$$= \frac{\Sigma^n_j x_j(\alpha_j - r)}{\alpha^*_{m+1} - r} \; {}^{10}$$

Hence,

$$\alpha - r = \Sigma^n_j x_j(\alpha_j - r) = \beta \; (\alpha^*_{m+1} - \mathrm{r}). \qquad Q.E.D.$$

Now we are ready to show that the investor who selects his portfolio from the m+2 funds ends up holding the identical portfolio that he would have chosen from among the n individually traded risky assets and the riskless assets. Let $\sigma^*_{ij}$ denote the covariance between $dV_i/V_i$ and $dV_j/V_j$, $i, j = 1, 2, \ldots, m+1$, and $v^*_{ij}$ denote the ijth element of the inverse of the variance-covariance matrix $[\sigma^*_{ij}]$. Consider an investor who selects his portfolio by combining only the m+2 funds. This investor's optimal demand for fund i is

$$d^*_i = A \; \Sigma^{m+1}_{j=1} v^*_{ij} \; (\alpha^*_i - r) + \Sigma^m_{k=1} \Sigma^{m+1}_{j=1} H_k S_k \sigma^*_j g_k \eta^*_{kj} v^*_{ij}, \; i = 1,2, \ldots, m+1.$$

Since $\quad \sigma^*_k \sigma^*_j \eta^*_{kj} \equiv (dq_k dz^*_j)/dt = \sigma^*_k \; (dz^*_i dz^*_k)/g_k dt$, and[11]

$$\eta^*_{kj} = \sigma^*_{kj} \; / g_k \sigma^*_j \; ,^{12}$$

[9] because $\Sigma^n_k v_{ki} \; \sigma_{kj} = 0$, if $i \neq j$; $= 1$, if $i = j$.

[10] because $\alpha^*_{m+1} - r = \lambda(\alpha^*_{m+1})^2$

[11] since $g_k dq_k = \sigma^*_k \; dz^*_k + \phi_k d\epsilon_k$

$$\Sigma_{k=1}^{m} \Sigma_{j=1}^{m+1} H_k S_k \sigma_j^* g_k \eta_{kj}^* \, v_{ij}^* = \Sigma_{k=1}^{m} H_k S_k \, (\Sigma_{j=1}^{m+1} \sigma_{kj}^* \, v_{ij}^*)$$

$$= \begin{cases} H_i S_i, & \text{if } i = 1, 2, \ldots, m; \\ 0, & \text{if } i = m+1, \end{cases}$$

Since $\alpha_j^* - r = \beta \, (\alpha_{m+1}^* - \mathrm{r}) = \dfrac{\sigma_{j,m+1}^*}{(\sigma_{m+1}^*)^2} (\alpha_{m+1}^* - r) = \lambda \sigma_{j,m+1}^*$

$$\Sigma_1^{m+1} v_{ij}^* \, (\alpha_j^* - r) = \lambda \, \Sigma_1^{m+1} v_{ij}^* \sigma_{j,\, m+1}^*$$

$$= \begin{cases} 0, \text{ if } i = 1, \ldots, m; \\ \lambda, \text{ if } i = m+1. \end{cases}$$

Therefore,

$$d_i^* = H_i S_i \; , \quad i = 1,2, \ldots, m,$$

$d_{m+1}^* = \mathrm{A}\lambda$, and

$d_{m+2}^* = W - \Sigma_1^{m+1} d_i^*$.

Also, from the definition of $V_i$, $i = 1,2, \ldots, m+1$, the demand for asset j is equal to

$$\Sigma_{i=1}^{m} d_i^* g_j \, (\Sigma_{k=1}^{n} v_{kj} \sigma_k \eta_{ik}) + d_{m+1}^* \, \Sigma_{k=1}^{n} v_{jk} \, (\alpha_k - r)/\lambda$$

$$= \Sigma_{i=1}^{m} \Sigma_{k=1}^{m} \mathrm{H}_i \mathrm{S}_i \mathrm{g}_i v_{kj} \sigma_k \eta_{ik} + \mathrm{A} \Sigma_{k=1}^{n} v_{jk} (\alpha_k - r)$$

$$= d_i \, , \; j = 1, 2, \ldots, n.$$

That is, the investor who selects his portfolio from the m+2 funds ends up holding an identical portfolio that he would have chosen from the n traded risky assets and the risky asset.

## 5. Revised CAPM

As mentioned in the last section, nonstationarity in the opportunity set will cause some problems about the international portfolio $V_{m+1}$. The portfolio $\mathrm{V}_{m+1}$ may not

[12] since $dz_j^* \, dz_k^* = \sigma_{kj}^* \; dt$

be efficient unless it is accompained by other portfolios, $V_i$, $i = 1, \ldots, m$, that are constructed so as to hedge unfavorable changes in exchange rates, $S_i$, $i = 1, \ldots, m$. In other words, we need $m+2$ funds, instead of two funds in the case of stationary opportunity set, for completely separating the construction of these funds from the impact of investor's preference.

The other problem with the international portfolio $V_{m+1}$ is that the $\beta$ value in CAPM is not stationary. Any change in the state variables will generate changes in $\mu_i$'s & $\sigma_i$'s and thus in the estimate of $\beta$. To remedy this drawback we need revise CAPM in either the estimates of excess returns for risky assets or the estimates of $\beta$ for each fund.

Revision of Estimate of Excess Return

From equation (8)' above, we know that the demand for each traded risky asset is

$$d_j = A \ \Sigma^{n}_{k=1} v_{jk}(\alpha_k - r) \ + \ \Sigma^{m}_{i=1} \Sigma^{n}_{k=1} H_i S_i g_i v_{kj} \sigma_k \eta_{ik}, \ j=1, \ \ldots, \ n.$$

Note that $S_i$ is included in $d_j$ because we assume the dynamics for $S_i$ is $dS_i = f_i S_i dt + g_i S_i dq_i$ here instead of $dS_i = f_i dt + g_i dq_i$ in equation (8)'. $d_i$ can be further simplified as

$$d_j \ = \ A\Sigma^{n}_{k} v_{jk}[\alpha_k - r \ + \sigma_k \Sigma^{m}_{i} \frac{H_i}{A} \ S_i g_i \eta_{ik}]$$

$$= \ - \ \frac{I_{WW}}{I_W} \ v_{jk} \ [\alpha_k - r \ + \ \sigma_k \ \Sigma^{m}_{i} \frac{I_{Wi}}{I_W} \ S_i g_i \eta_{ik}]$$

Define

$$\alpha'_k \ = \ \alpha_k \ + \ \sigma_k \ \sum^{m}_{i} \frac{I_{wi}}{I_W} S_i g_i \eta_{ik} \ , \ k \ = \ 1,2, \ \ldots, \ n. \qquad (20)$$

By using $\alpha'_k$ instead of $\alpha_k$ we can obtain the results of optimal portfolio choice similar to the one obtained in the absence of state variables. Consequently, we can convert the problem of nonstationary investment opportunity set into the stationary one by simply revising the expected return of each traded asset according to equation (20).

We can revise the conventional CAPM as

$$\alpha' - r = \beta(\alpha'^{*}_{m+1} - r)$$

where $\alpha'$ is the expected return of any portfolio after adjusting the risk premium associated with the changes in $S_i$, that is,

$$\alpha' = \Sigma_j^n x_j[\alpha_j + \sigma_j \Sigma_i^m \frac{I_{Wi}}{I_W} S_i g_i \eta_{ij}].$$

And

$$\beta = \frac{\frac{dV'}{V'} \cdot \frac{dV'_{m+1}}{V'_{m+1}}}{(\frac{dV'_{m+1}}{V'_{m+1}})^2} = \frac{(\Sigma_i^n x_j \sigma_j dz_j)\ (\sigma'^{*}_{m+1}\ dz'^{*}_{m+1})}{(\sigma'^{*}_{m+1})^2 dt}$$

where

$$\frac{dV'}{V'} = \Sigma_j^n x_j\ (\alpha'_j - r) + \Sigma_j^n x_j \sigma_j dz_j$$

$$\frac{dV'_{m+1}}{V'_{m+1}} = \alpha'^{*}_{m+1} dt + \sigma'^{*}_{m+1} dz'^{*}_{m+1}$$

$$\alpha'^{*}_{m+1} = r + \Sigma_k^n \Sigma_j^n v_{kj}\ (\alpha'_j - r)\ (\alpha'_k - r)/\lambda$$

$$(\lambda = \Sigma_k^n \Sigma_j^n v_{kj}(\alpha'_j - r))$$

$$(\sigma'^{*}_{m+1})^2 = \frac{1}{\lambda^2} \Sigma_k^n \Sigma_j^n v_{kj}\ (\alpha'_j - r)\ (\alpha'_k - r) = \frac{\alpha'^{*}_m - r}{\lambda}$$

and

$$dz'^{*}_{m+1} = \frac{1}{\lambda^2} [\Sigma_k^n \Sigma_j^n v_{kj}\ (\alpha'_j - r)\ \sigma_k dz_k]/\sigma'^{*}_{m+1}$$

Revision of Estimate of ß:

There is one problem in the revision of expected returns for traded assets. It requires the information of $H_i$ which is related to an investor's preference function. It is difficult to estimate these $H_i$'s. Moreover, it is not consistent with the separation theorem. Alternatively, we revise ß and retain the original estimates of $\alpha_i$'s. We will show that the separation theorem still holds under this revision of ß.

From equation (8), we know that

$$0 = I_W(\alpha_i - r) + I_{WW}\Sigma_j^n \omega_j W \sigma_{ij} + \Sigma_j^m I_{jW} S_j g_j \sigma_i \eta_{ij}, \quad i = 1,2, \ldots, n.$$

$$\rightarrow \alpha_i - r = \frac{W}{A} \Sigma_j^n \omega_j \sigma_{ij} + \Sigma_j^m \frac{H_j}{A} S_j g_j \frac{\hat{\sigma}_{ji}}{g_j},$$

where $\hat{\sigma}_{ji}$ is the covariance between asset i and $S_J$, i.e., $\hat{\sigma}_{ji} = \eta_{ji}\sigma_i g_j$.

Define $\alpha_{m+1} \equiv \Sigma_j^n \omega_j(\alpha_j - r) + r$,

$$\sigma_{im+1} \equiv \Sigma_j^n \omega_j \sigma_{ij},$$

and $\sigma_{m+1}^2 \equiv \Sigma_i^n \omega_i \sigma_{im+1}$.

Then,

$$\alpha_i - r = \frac{W}{A} \sigma_{i,m+1} + \sum_j^m \frac{H_j}{A} S_j \hat{\sigma}_{ji}, \quad i=1, \ldots, n. \tag{21}$$

Multiplying (21) by $\omega_j$ and summing it yields

$$\alpha_{m+1} - r = \frac{W}{A} \sigma_{m+1}^2 + \sum_j^m \frac{H_j}{A} S_j \hat{\sigma}_{j,m+1}, \tag{22}$$

where $\hat{\sigma}_{j,m+1} = \Sigma_i^n \sigma_{ji}\omega_i$.

[13] because $\delta_{ik} = \Sigma_h^n \nu_{hk}\sigma_h\eta_{ik}$.

[14] because $\Sigma_k^n \nu_{hk}\sigma_{kj} = 0$, if $j \neq h$; and $= 1$, if $j = h$.

Also, the m funds that are constructed to have maximal feasible correlation with $S_i$, that is, $V_i$, $i = 1, 2, \ldots, m$ which satisfy equation (19), will satisfy the following equation:

$$\alpha_i^* - r = \sum_k^n x_k \ (\alpha_k - r) + \sum_k^n x_k (\sum_j^m \frac{H_j}{A} S_j g_j \sigma_k \eta_{jk}) \qquad (23)$$

$$= \underbrace{g_i \sum_k^n \delta_{ik} \ (\alpha_k - r)}_{A} + \underbrace{g_i \sum_k^n \delta_{ik} (\sum_j^m \frac{H_j}{A} S_j g_j \sigma_k \eta_{jk})}_{B}, \ i = 1, 2, \ldots, m.$$

where $\alpha_i^*$ is the expected return for fund $V_i$. The "A" portion of equation (23) can be simplified as:

$$A = g_i \ \Sigma_k^n \ \delta_{ik} (\frac{W}{A} \ \Sigma_j^n \ \omega_j \sigma_{kj})$$

$$= g_i \frac{W}{A} \ \Sigma_k^n \ [\Sigma_h^n \ v_{hk} \sigma_h \eta_{ih} \ (\Sigma_j^n \ \omega_j \sigma_{kj})] \ ^{13}$$

$$= g_i \frac{W}{A} \ \Sigma_k^n \ \sigma_h \eta_{ih} \ \Sigma_j^n \ \omega_j \ \Sigma_k^n \ v_{hk} \sigma_{kj}$$

$$= \frac{W}{A} \ \Sigma_h^n \ g_i \sigma_h \eta_{ih} \omega_h \ ^{14}$$

$$= \frac{W}{A} \ \Sigma_h^n \ \hat{\sigma}_{ih} \omega_h$$

$$= \frac{W}{A} \ \hat{\sigma}_{i,\ m+1}$$

The "B" portion of equation (23) can be simplified as

$$B = g_i \Sigma_k^n \Sigma_h^n v_{hk}\sigma_h\eta_{ih} \; (\Sigma_j^m \frac{H_i}{A} \; S_j g_j \sigma_k \eta_{jk})$$

$$= g_i \Sigma_j^m \frac{H_i}{A} \; S_j g_j \; (\Sigma_k^n \Sigma_h^n v_{hk}\sigma_h\sigma_k\eta_{jk}\eta_{ih})$$

$$= g_i \Sigma_j^m \frac{H_j}{A} \; S_j g_j (\rho_i^*)^2 v_{ij} \text{ [15]}$$

$$= g_i(\rho_i^*)^2 \; \Sigma_j^m \frac{H_j}{A} \; S_j g_j v_{ij}.$$

Equation (22) and the m equations in (23) can jointly solve $\frac{W}{A}$ and $\frac{H_j}{A} S_j$, $j = 1, 2, \ldots, m$. With these $\frac{W}{A}$ and $\frac{H_j}{A} S_j$ we can rewrite equation (21) as

$$\alpha_i - r = (\alpha_{m+1} - r \;\; \alpha_1^* - r \;\; \alpha_2^* - r \; \ldots \; \alpha_m^* - r) \; (\mathrm{II}^{-1})' \begin{bmatrix} \sigma_{i,m+1} \\ \hat{\sigma}_{1,i} \\ \hat{\sigma}_{2,i} \\ \vdots \\ \hat{\sigma}_{m,i} \end{bmatrix} \qquad (21')$$

$i = 1, 2, \ldots, n.$

where

$$\mathrm{II} = \begin{bmatrix} \sigma_{m+1}^2 & \hat{\sigma}_{1,m+1} & \cdots & \hat{\sigma}_{m,m+1} \\ \hat{\sigma}_{1,m+1} & g_1(\rho_1^*)^2 g_1 v_{1,1} & \cdots & g_1(\rho_1^*)^2 g_m v_{1,m} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\sigma}_{m,m+1} & g_m(\rho_m^*)^2 g_1 v_{m,1} & \cdots & g_m(\rho_m^*)^2 g_m v_{m,m} \end{bmatrix}$$

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15 because $\rho_i^* \equiv (\Sigma_k^n \Sigma_h^n v_{hk}\sigma_k\sigma_h\eta_{jk}\eta_{jh})^{\frac{1}{2}}$, and $\eta_{ih} = \Sigma_j^m v_{ij}\eta_{jh}$, where $v_{ij}$ is the correlation coefficient between $S_i$ and $S_j$.

Therefore, the $\beta$'s in the CAPM can be revised as

$$(\beta_1' \ \beta_2' \ \ldots \ \beta_m') = (\text{II}^{-1})' \begin{bmatrix} \sigma_{i,m+1} \\ \hat{\sigma}_{1,i} \\ \hat{\sigma}_{2,i} \\ \vdots \\ \hat{\sigma}_{m,i} \end{bmatrix}. \qquad (24)$$

Or, specifically,

$$\beta_q' = \frac{\sigma_{i,m+1}}{|II|} \begin{vmatrix} g_1(\rho_1^*)^2 g_1 \nu_{1,1} & \cdots & g_1(\rho_1^*)^2 g_m \nu_{1,m} \\ \vdots & \ddots & \vdots \\ g_m(\rho_m^*)^2 g_1 \nu_{m,1} & \cdots & gp_m(\rho_m^*)^2 g_m \nu_{m,m} \end{vmatrix}$$

$$- \frac{\hat{\sigma}_{1i}}{|\text{II}|} \begin{vmatrix} \hat{\sigma}_{1,m+1} & g_1(\rho_1^*)^2 g_2 \nu_{1,2} & \cdots & g_1(\rho_1^*)^2 g_m \nu_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m,m+1} & g_m(\rho_m^*)^2 g_2 \nu_{m,2} & \cdots & g_m(\rho_m^*)^2 g_m \nu_{m,m} \end{vmatrix}$$

$$+ \frac{\hat{\sigma}_{1i}}{|\text{II}|} \begin{vmatrix} \hat{\sigma}_{1,m+1} & g_1(\rho_1^*)^2 g_1 \nu_{1,1} & \cdots & g_1(\rho_1^*)^2 g_m \nu_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m,m+1} & g_m(\rho_m^*)^2 g_1 \nu_{m,1} & \cdots & g_m(\rho_m^*)^2 g_m \nu_{m,m} \end{vmatrix}$$

$$+ \cdots\cdots\cdots$$

$$+ (-1)^m \frac{\hat{\sigma}_{mi}}{|\text{II}|} \begin{vmatrix} \hat{\sigma}_{1,m+1} & g_1(\rho_1^*)^2 g_1 \nu_{1,1} & \cdots & g_1(\rho_1^*)^2 g_m \nu_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m,m+1} & g_m(\rho_m^*)^2 g_1 \nu_{m,1} & \cdots & g_m(\rho_m^*)^2 g_m \nu_{m,m} \end{vmatrix}$$

Similarly, $\beta_2'$, ..., and $\beta_m'$ can be solved expressively.

In the following we illustrate these revised $\beta$'s by using an example of one state variable, i.e., m = 1. In this example, we need one fund to hedge the unfavorable change in the state variable. Also, $(\rho_1^*)^2 = 1$ and $\nu_{1,1} = 1$.

$$(24) \rightarrow \beta_1' = \sigma_{i,m+1} g_1^2 / \begin{vmatrix} \sigma_{m+1}^2 & \hat{\sigma}_{1,m+1} \\ \hat{\sigma}_{1,m+1} & g_1^2 \end{vmatrix} - \hat{\sigma}_{1i}\hat{\sigma}_{1,m+1} / \begin{vmatrix} \sigma_{m+1}^2 & \hat{\sigma}_{1,m+1} \\ \hat{\sigma}_{1,m+1} & g_1^2 \end{vmatrix}$$

$$= \frac{\sigma_{i,m+1} g_1^2 - \hat{\sigma}_{1i}\hat{\sigma}_{1,m+1}}{\sigma_{m+1}^2 g_1^2 - (\hat{\sigma}_{1,m+1})^2} = \frac{\sigma_i(\rho_{i,m+1} - \hat{\rho}_{1i}\hat{\rho}_{1,m+1})}{\sigma_{m+1}[1 - (\hat{\rho}_{1,m+1})^2]}$$

$$\beta_2' = -\sigma_{i,m+1}\hat{\sigma}_{1,m+1} / \begin{vmatrix} \sigma_{m+1}^2 & \hat{\sigma}_{1,m+1} \\ \hat{\sigma}_{1,m+1} & g_1^2 \end{vmatrix} + \hat{\sigma}_{1i}\sigma_{m+1}^2 / \begin{vmatrix} \sigma_{m+1}^2 & \hat{\sigma}_{1,m+1} \\ \hat{\sigma}_{1,m+1} & g_1^2 \end{vmatrix}$$

$$= \frac{\sigma_i(\hat{\rho}_{1i} - \rho_{i,m+1}\hat{\rho}_{1,m+1})}{g_1[1 - (\hat{\rho}_{1,m+1})^2]}$$

Note that any $\sigma$ with "Λ" indicates the relation with the state variable $S_1$; otherwise, it relates with traded assets.

Therefore, we have the following revised CAPM for the case of one state variable:

$$\alpha_i - r = \frac{\sigma_i(\rho_{i,m+1} - \hat{\rho}_{1i}\hat{\rho}_{1,m+1})}{\sigma_{m+1}[1 - (\hat{\rho}_{1,m+1})^2]}(\alpha_{m+1} - r) + \frac{\sigma_i(\hat{\rho}_{1i} - \rho_{i,m+1}\hat{\rho}_{1,m+1})}{g_1[1 - (\hat{\rho}_{1,m+1})^2]}(\alpha_1^* - r), \tag{25}$$

$i = 1, \ldots, n.$

When $\hat{\rho}_{1i} = \hat{\rho}_{1,m+1} = 0$, equation (25) becomes

$\alpha_i - r = \dfrac{\sigma_i \rho_{i,m+1}}{\sigma_{m+1}}(\alpha_{m+1} - r) = \dfrac{\sigma_{i,m+1}}{(\sigma_{m+1})^2}(\alpha_{m+1} - r)$, which is exactly the conventional CAPM with a stationary investment opportunity set.

Finally I remark on a country's $\beta$. If we stick to the conventional CAPM and define $\beta_i$ as the covariance of asset i with the international market portfolio $V_{m+1}$, $V_{m+1}$ divided by the variance of $V_{m+1}$ we can decompose this $\beta_i$ as follows:

$$\beta_i = \frac{\sigma_{i,m+1}}{\sigma_{m+1}^2} = \frac{\rho_{i,m+1}\sigma_i}{\sigma_{m+1}}$$

$$= \Sigma_j^m \frac{\hat{\rho}_{ji}\hat{\rho}_{j,m+1}\sigma_i}{\sigma_{m+1}}$$ [16]

$$= \Sigma_j^m \frac{\hat{\sigma}_{ji}}{g_j^2}\frac{\hat{\sigma}_{j,m+1}}{\sigma_{m+1}^2}$$

[16] because $\rho_{i,m+1} = \Sigma_j \hat{\rho}_{ji}\hat{\rho}_{j,m+1}$.

$$= \Sigma_j^m \beta_{ji}^{CI} \beta_j^C, \qquad i = 1,2, \ldots, n,$$

where

$$\beta_{ji}^{CI} \equiv \frac{\hat{\sigma}_{ji}}{g_j^2}, \quad i=1,2, \ldots, n, j=1,2,\ldots,m.$$

$$\beta_j^C \equiv \frac{\hat{\sigma}_{j,m+1}}{\sigma_{m+1}^2}, \quad j=1,2,\ldots,m.$$

$\beta_{ji}^{CI}$ measures the $\beta$ of asset i within the jth country. $\beta_j^C$ corresponds to the $\beta$ of country with respect to the international market portfolio $V_{m+1}$. If we restrict asset i to be available only in country j, then $\rho_{j,m+1} = \rho_{ji}\rho_{j,m+1}$. And $\beta_i = \beta_{ji}^{CI} \beta_j^C$. In other words, an asset's $\beta$ becomes the product of a country's $\beta$ and the asset's $\beta$ within this country. However, this conclusion is valid only in the conventional CAPM ignoring the impact of state variables on the investment opportunity set.

## 6. Conclusion

This paper relates the issue of currency hedge to an investor's interntional portfolio choice. Under a general framework these two issues are resolved simultaneously. Country risks are introduced because of a violation of *ex post* IRP. These unexpected exchange rate changes will engender a nonstationary investment opportunity set. As a result, the demand for a hedging instrument such as forward exchange contract will be comprised of three elements: the speculative demand due to nonzero expected exchange rate change, the static hedging demand due to the imperfect correlation of this hedging instrument with other traded assets, and the dynamic hedging demand which is intrigued by its correlation with the unexpected exchange rate movement.

This paper illustrates how the most efficient hedging funds are constructed to cope with the unexpected exchange rate movement when a single perfectly correlated hedging instrument is not available. It is interesting to note that $m+2$ funds are sufficient to replicate all the investment possibility in the market. In other words, an optimal portfolio choice can be generated from these $m+2$ funds instead of the original risky assets plus a riskless asset. A lot of transaction costs may be saved especially when the number of traded assets ($n$) is much greater than the number of countries in the world ($m$). These $m+2$ funds are formed by a riskless asset, an international market portfolio and $m$ funds each of which is created to cope with the unexpected exchange rate movement for each country.

A single $\beta$ in the conventional CAPM is no longer a reliable estimate in the case of a nonstationary opportunity set. One can revise the expected returns for the traded risky assets by adding a premium that reflects the risk associated with the unexpected exchange rate variation. After this adjustment for the risk premium the single $\beta$ estimate in the CAPM can sustain its stability. Alternatively, we add another m $\beta$'s each of which deals with the risk resulting from a country's unexpected exchange rate change. The resulting $\beta$ estimate for the market risk will eradicate the unfavorable elements caused by the exchange rate movement and thereby become more stable. Finally, the conventional treatment of a country's $\beta$ is linked to this general framework as a special case.

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