國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

一個具擴散性的 SIR 模型之行進波解

Traveling Wave Solutions for a Diffusive SIR Model

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中文摘要

本篇論文討論的是 SIR 模型的反應擴散方程

$$\begin{aligned} s_t &= d_1 s_{xx} - \beta si/(s+i), \\ i_t &= d_2 i_{xx} + \beta si/(s+i) - \gamma i, \\ r_t &= d_3 r_{xx} + \gamma i, \end{aligned}$$

之行進波的存在性，其中模型描述的是在一個封閉區域裡流行疾病爆發的狀態。這裡的 β 是傳播係數，γ 是治癒或移除 (即死亡) 速率，s 是未被傳染個體數，i 是傳染源個體數，d_1、d_2 及 d_3 分別為其擴散之係數。

我們將使用 Schauder 不動點定理 (Schauder fixed point theorem)、Arzela-Ascoli 定理和最大值原理 (maximum principle) 來證明：該系統存在最小速度為 $c = c^* := 2\sqrt{d_2(\beta-\gamma)}$ 之行進波解。我們的結果回答了 [11] 裡所提出的開放式問題。

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Abstract

In this thesis, we study the existence of traveling waves of a reaction-diffusion equation for a diffusive epidemic SIR model

$$s_t = d_1 s_{xx} - \beta si/(s+i),$$
$$i_t = d_2 i_{xx} + \beta si/(s+i) - \gamma i,$$
$$r_t = d_3 r_{xx} + \gamma i,$$

which describes an infectious disease outbreak in a closed population. Here β is the transmission coefficient, γ is the recovery or remove rate, and s, i, and r represent numbers of susceptible individuals, infected individuals, and removed individuals, respectively, and d_1, d_2, and d_3 are their diffusion rates. We use the Schauder fixed point theorem, the Arzela-Ascoli theorem, and the maximum principle to show that this system has a traveling wave solution with minimum speed $c = c^* := 2\sqrt{d_2(\beta-\gamma)}$. Our result answers an open problem proposed in [11].

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Chapter 1

Introduction.

Traveling wave solutions for the epidemic systems have received a lot of attention by several researchers because the existence of traveling waves implies spread of the disease. For the disease control and prevention, it is important to investigate that whether traveling waves exist and what the propagation speed c is.

One important epidemic model is the following SIR model

$$s_t = d_1 s_{xx} - f(s,i), \tag{1.1a}$$

$$i_t = d_2 i_{xx} + f(s,i) - \gamma i, \tag{1.1b}$$

$$r_t = d_3 r_{xx} + \gamma i, \tag{1.1c}$$

where $s(x,t)$ represents the number of individuals not yet infected with the disease at position x and time t, or those susceptible to the disease; infected $i(x,t)$ denotes the number of individuals who have been infected with the disease and are capable of spreading the disease to those in the susceptible category; and $r(x,t)$ represents individuals who have been infected and then removed from the disease, either due to immunization or due to death. Here γ denotes the recovery rate; d_1, d_2, and d_3 are the diffusion rates of the susceptible, infective, and removed individuals, respectively. There are various types of the incidence term $f(s,i)$. The common types include bilinear incidence (or mass action incidence) βsi and standard incidence $\frac{\beta si}{\Lambda}$ or $\frac{\beta si}{s+i}$, where β and Λ are positive constants.

Since r does not occur in the first two equations (1.1a) and (1.1b), we can only consider

$$s_t = d_1 s_{xx} - f(s,i), \tag{1.2a}$$

$$i_t = d_2 i_{xx} + f(s,i) - \gamma i. \tag{1.2b}$$

Such a system with a kinetic planar vector field also provides a simple example of the general diffusive predator-prey system.

The existence and non-existence of traveling wave solutions of system (1.2) for the bilinear incidence have been investigated by Källén [10] for the case $d_1 = 0$, Hosono and Ilyas [7] for the case $d_2 = 0$, Hosono and Ilyas [8] and Dunbar [2, 3] for the case $d_1 \neq 0$ and $d_2 \neq 0$.

Now, we consider system (1.2) for the standard incidence. i.e.,

$$s_t = d_1 s_{xx} - \frac{\beta s i}{(s+i)}, \tag{1.3a}$$

$$i_t = d_2 i_{xx} + \frac{\beta s i}{(s+i)} - \gamma i. \tag{1.3b}$$

In [11], Wang, Wang, and Wu studied the existence and non-existence of traveling wave solutions of system (1.3) by using Laplace transform, shooting method, and a similar method in [1, 9]. They proved that if $R_0 := \beta/\gamma > 1$, then there exists a traveling wave solution with speed $c > c^* := 2\sqrt{d_2(\beta - \gamma)}$ and there exists no traveling wave solution with speed $c < c^*$; Howerver, if $R_0 \leq 1$, then there exists no traveling wave solutions. Nevertheless, when $R_0 > 1$, the existence and non-existence of traveling wave solutions with speed $c = c^*$ of system (1.3) is still unknown. Therefore, in this thesis, we would like to tackle this gap. By a traveling wave solution of system (1.3), we mean a nonnegative solution of system (1.3) of the form $(s,i)(x,t) = (S(x-ct), I(x-ct))$ with the boundary conditions

$$S(\pm\infty) = s_{\pm\infty} \text{ and } I(\pm\infty) = 0, \tag{1.4}$$

for some constants s_∞ and $s_{-\infty}$ with $s_\infty > s_{-\infty}$. Here c denotes speed of the wave.

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Hence to show the existence of traveling wave solutions of system (1.3) is equivalent to show the existence of nonnegative solutions of the following system on $\mathbb{R}$:

$$d_1S'' + cS' - \frac{\beta SI}{(S+I)} = 0\,, \tag{1.5a}$$

$$d_2I'' + cI' + \frac{\beta SI}{(S+I)} - \gamma I = 0, \tag{1.5b}$$

$$S(\pm\infty) = s_{\pm\infty}, I(\pm\infty) = 0. \tag{1.5c}$$

Specifically, we use the Schauder fixed point theorem, the Arzela-Ascoli theorem, and the maximum principle to show the following:

Theorem 1.1. *Suppose that $R_0 > 1$. For a given $s_\infty > 0$, there exists $s_{-\infty} \in [0,\ s_\infty)$ such that the system (1.3) has a traveling wave solution with speed $c = c^*$.*

Our method in this thesis is mainly based on Fu [4, 5]. We make a compendium of our approach. First, we consider system (1.5a)-(1.5b) in a finite interval $[-l, l]$ with appropriate boundary conditions. Using the Schauder fixed point theorem, we show that this boundary value problem has a solution, denoted by (S_l, I_l), sandwiched by the super- and sub-solutions. Then, letting $l \to \infty$ and using Ascoli-Arzela theorem and a diagonal process, we can get a nonnegative solutions of system (1.5). Note that system (1.5) has no maximum principle. Here we just apply the maximum principle for a single equation to compare the super- and sub-solutions with the solutions.

This paper is organized as follows. In Chapter 2, we first construct the super- and sub-solutions, and consider the system (1.5a)-(1.5b) in a finite interval $[-l, l]$. Then, by passing to the limit $l \to \infty$, we obtain a candidate of a nonnegative solution of (1.5). Finally, in Chapter 3, we use this candidate to show the existence of nonnegative solutions of system (1.5). This completes the proof of our main theorem.

Chapter 2

Preliminary.

2.1 Construction of super- and sub-solutions.

In this section, we construct super- and sub-solutions which will be used in Section 2.2. For a given $s_\infty > 0$, we set $S^+ \equiv s_\infty$. To built I^+, we introduce the polynomial

$$p(\lambda) = d_2\lambda^2 - c\lambda + (\beta - \gamma). \tag{2.1.1}$$

Note that when $c = c^*$, the equation $p(\lambda) = 0$ just has a repeated root, saying $\lambda_1 = c^*/2d_2$, and $p(\lambda) \geq 0$ for all λ in $\mathbb{R}$. Let $b_1 := 1/\lambda_1$ and $\eta := e\lambda_1 s_\infty(\beta - \gamma)/\gamma$ be positive constants.

Lemma 2.1.1. *The function*

$$I^+(z) := \begin{cases} s_\infty(\beta - \gamma)/\gamma, & \text{if } z < b_1, \\ \eta z e^{-\lambda_1 z}, & \text{if } z \geq b_1, \end{cases}$$

satisfies the following inequality

$$d_2(I^+(z))'' + c^*(I^+(z))' + \frac{\beta s_\infty I^+(z)}{s_\infty + I^+(z)} - \gamma I^+(z) \leq 0, \tag{2.1.2}$$

for all $z \neq b_1$.

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Proof. For $z < b_1$, it is clear that inequality (2.1.2) holds. For $z > b_1$, since $I^+(z) = \eta z e^{-\lambda_1 z}$, $c^* = 2d_2\lambda_1$, and $p(\lambda_1) = 0$, it follows that

$$\begin{aligned} & d_2(I^+)'' + c^*(I^+)' + \frac{\beta S^+ I^+}{S^+ + I^+} - \gamma I^+ \\ \leq\ & d_2(I^+)'' + c^*(I^+)' + \beta I^+ - \gamma I^+ = \eta e^{-\lambda_1 z}(c^* - 2d_2\lambda_1) + p(\lambda_1) I^+ = 0. \end{aligned}$$

Thus the inequality (2.1.2) holds.

Select $0 < \varepsilon_3 < \min\{c^*/d_1,\ \lambda_1\}$. Then $c^* - d_1\varepsilon_3 > 0$ and $\lambda_1 - \varepsilon_3 > 0$. Since

$$\eta z e^{-(\lambda_1 - \varepsilon_3)z} \to 0 \text{ as } z \to \infty,$$

there exists $b_2 > 0$ such that

$$\eta z e^{-(\lambda_1 - \varepsilon_3)z} \leq \frac{s_\infty + \eta z e^{-\lambda_1 z}}{\beta}\big[\varepsilon_3(c^* - d_1\varepsilon_3)\big],\ \forall z \geq b_2.$$

Let $z_0 := \max\{b_1, b_2\}$. Then, for $z \geq z_0$, we have

$$I^+(z) = \eta z e^{-\lambda_1 z} \text{ and } \eta z e^{-(\lambda_1 - \varepsilon_3)z} \leq \frac{s_\infty + \eta z e^{-\lambda_1 z}}{\beta}\big[\varepsilon_3(c^* - d_1\varepsilon_3)\big],$$

which implies that

$$e^{-\varepsilon_3 z}\big[\varepsilon_3(c^* - d_1\varepsilon_3)\big] \geq \frac{\beta I^+}{s_\infty + I^+}. \tag{2.1.3}$$

Set $M_3 := e^{\varepsilon_3 z_0} > 1$.

Lemma 2.1.2. *The function $S^-(z) := \max\{0, s_\infty(1 - M_3 e^{-\varepsilon_3 z})\}$ satisfies the inequality*

$$d_1(S^-(z))'' + c^*(S^-(z))' - \frac{\beta S^-(z) I^+(z)}{S^-(z) + I^+(z)} \geq 0, \tag{2.1.4}$$

for all $z \neq z_0$.

Proof. For $z < z_0$, the inequality (2.1.4) holds immediately since $S^- \equiv 0$ in $(-\infty, z_0)$. For $z > z_0$, $S^-(z) = S_\infty(1 - M_3 e^{-\varepsilon_3 z})$. It is easy to see that $0 < S^- < S_\infty$. So we have

$$\frac{S_\infty I^+}{S_\infty + I^+} \geq \frac{S^- I^+}{S^- + I^+}. \tag{2.1.5}$$

Together with (2.1.3) and (2.1.5) and the fact that $M_3 > 1$, we have

$$d_1(S^-)'' + c^*(S^-)' = S_\infty M_3 e^{-\varepsilon_3 z}\big[\varepsilon_3(c^* - d_1\varepsilon_3)\big] \geq \frac{\beta S_\infty I^+}{S_\infty + I^+} \geq \frac{\beta S^- I^+}{S^- + I^+}.$$

Hence the inequality (2.1.4) holds.

Select $M_4 > 0$ sufficiently large such that $M_4 > \max\{(7/(2\lambda_1))^{1/2}, {z_0}^{1/2}\}$ and

$$\beta\eta {M_4}^6 e^{-\lambda_1 {M_4}^2} \leq \frac{d_2}{4}\big[s_\infty(1 - M_3 e^{-\varepsilon_3 {M_4}^2})\big]. \tag{2.1.6}$$

Let $z_1 := {M_4}^2$, then $z_1 > z_0 > 0$.

Lemma 2.1.3. *The function* $I^-(z) := \max\big\{0, \big(\eta z - \eta M_4 z^{1/2}\big)e^{-\lambda_1 z}\big\}$ *satisfies the inequality*

$$d_2(I^-(z))'' + c^*(I^-(z))' + \frac{\beta S^-(z) I^-(z)}{S^-(z) + I^-(z)} - \gamma I^-(z) \geq 0, \tag{2.1.7}$$

for all $z > z_1$.

Proof. For $z > z_1$, $I^-(z) = \big(\eta z - \eta M_4 z^{1/2}\big)e^{-\lambda_1 z}$. Then it is easy to deduce that

$$d_2(I^-)'' + c(I^-)' + (\beta - \gamma)I^- = \frac{d_2}{4}\eta M_4 z^{-3/2} e^{-\lambda_1 z}. \tag{2.1.8}$$

Multiplying both sides of (2.1.6) by ηM_4 and using the fact that $z_1 = (M_4)^2$, we deduce that

$$\beta\eta^2 {z_1}^{7/2} e^{-\lambda_1 z_1} \leq \frac{d_2}{4}\eta M_4\big[s_\infty(1 - M_3 e^{-\varepsilon_3 z_1})\big].$$

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For all $z > z_1$, we get

$$\frac{d_2}{4}\eta M_4\big[s_\infty(1-M_3e^{-\varepsilon_3 z_1})\big] \le \frac{d_2}{4}\eta M_4\big[s_\infty(1-M_3e^{-\varepsilon_3 z})\big] = \frac{d_2}{4}\eta M_4 S^- \le \frac{d_2}{4}\eta M_4(S^-+I^-)$$

and

$$\beta\eta^2 {z_1}^{7/2}e^{-\lambda_1 z_1} \ge \beta\eta^2 z^{7/2}e^{-\lambda_1 z} \ge \beta z^{3/2}e^{\lambda_1 z}\big[e^{-\lambda_1 z}(\eta z - \eta M_4 z^{1/2})\big]^2 = \beta z^{3/2}e^{\lambda_1 z}(I^-)^2.$$

Combining the above three inequalities, we have

$$\frac{-\beta(I^-)^2}{S^-+I^-} \ge -\frac{d_2}{4}\eta M_4 z^{-3/2}e^{-\lambda_1 z}. \tag{2.1.9}$$

Then, by summing up (2.1.8) and (2.1.9), we finally obtain (2.1.7).

2.2 System in a finite interval [-l, l].

In this section, we consider the system

$$d_1 S'' + cS' - \frac{\beta SI}{(S+I)} = 0 \qquad \text{in } (-l,\ l), \tag{2.2.1a}$$

$$d_2 I'' + cI' + \frac{\beta SI}{(S+I)} - \gamma I = 0 \qquad \text{in } (-l,\ l), \tag{2.2.1b}$$

together with the boundary conditions

$$\big(S,I\big)(-l) = \big(S^-,I^-\big)(-l) \ \text{ and } \ \big(S,I\big)(l) = \big(S^-,I^-\big)(l). \tag{2.2.2}$$

We will apply the Schauder fixed point theorem to show the existence of solutions of problem (2.2.1)-(2.2.2). Let $l > z_1$. For convenience, we set $R_l := [-l,\ l]$, $X := C(R_l)\times C(R_l)$, and

$$\mathscr{A} := \big\{(S,I)\in X \ \mid\ 0 \le S^- \le S \le S^+ \equiv s_\infty \text{ and } 0 \le I^- \le I \le I^+ \text{ in } R_l\big\}.$$

To make it more comprehensible, we recall the theorem in the following:

Theorem (Schauder fixed point theorem). *Let $\mathscr{A}$ be a closed convex set in a Banach space and let $T : \mathscr{A} \to \mathscr{A}$ be a continuous mapping such that $T(\mathscr{A})$ is precompact, then T has a fixed point.*

It is easy to verify that $\mathscr{A}$ is a closed convex set in the Banach space X equipped with the norm $\| (F_1, F_2) \|_X = \| F_1 \|_{C(R_l)} + \| F_2 \|_{C(R_l)}$. Because S^- and I^- are non-negative, it follows that $S \geq 0$ and $I \geq 0$ for any $(S,\ I) \in \mathscr{A}$.

Lemma 2.2.1. *For a given $(S_0, I_0) \in \mathscr{A}$, there exists a unique solution to the boundary value problem*

$$d_1 S'' + cS' - \phi(S, z)S = 0 \qquad \text{in } (-l,\ l), \tag{2.2.3a}$$

$$d_2 I'' + cI' + \phi(S_0, z)S_0 - \gamma I = 0 \qquad \text{in } (-l,\ l), \tag{2.2.3b}$$

$$(S, I)(-l) = (S^-, I^-)(-l),\ (S, I)(l) = (S^-, I^-)(l), \tag{2.2.3c}$$

where

$$\phi(\xi,\ z) = \begin{cases} \dfrac{I_0(z)\beta}{\xi + I_0(z)}, & \text{if } I_0(z) \neq 0, \\ 0, & \text{if } I_0(z) = 0. \end{cases}$$

Moreover, this solution $(S,\ I)$ satisfies $S > 0$, $I > 0$, and $S' > 0$ in $(-l,\ l)$.

Proof. Note that system (2.2.3) is not a coupled system, so that we can consider the existence and uniqueness of S and I, respectively. Because $l > z_1 > z_0 > 0 > -l$, the definition of S^- and I^- implies that $S^-(-l) = I^-(-l) = 0$, $S^-(l) > 0$, and $I^-(l) > 0$.

Since the equation for I is a non-homogeneous linear equation, we can use [6, Theorem 3.1 of Chapter 12] to obtain the existence and uniqueness of I. Moreover, since $d_2 I'' + cI' - \gamma I = -\phi(S_0, z)S_0 \leq 0$ in $(-l,\ l)$ and $I(\pm l) \geq 0$, it implies that $I > 0$ in $(-l,\ l)$ by the maximum principle.

Now we check the existence and uniqueness of S. First, we consider the initial value problem

$$d_1 S'' + cS' - \phi(S,z)S = 0, \tag{2.2.4a}$$

$$S(-l) = \left(S^-\right)(-l),\ S'(-l) = m, \tag{2.2.4b}$$

where m is constant. Using the existence and uniqueness theorem, we can prove that, for each m, the initial value problem (2.2.4) has a unique local solution $S(z,\ m)$ and this solution can be continued as long as $S + I_0 \neq 0$. When $m = 0$, $S(z,0) \equiv 0$ due to the uniqueness. For any fixed $m < 0$, since $S(-l,m) = \left(S^-\right)(-l) = 0$ and $S'(-l,m) = m < 0$, implies that there exists $\delta > 0$ such that $S(z,m) < 0$ for all $z \in (-l,\ -l+\delta]$. On the other hand, integrating (2.2.4a), we have

$$S'(z,m) = me^{-c(z+l)/d_1} + e^{-cz/d_1}\int_{-l}^{z} \frac{\beta}{d_1}\frac{S(\tau,m)I_0(\tau)}{S(\tau,m)+I_0(\tau)}e^{c\tau/d_1}d\tau, \tag{2.2.5}$$

which implies that $S(z,m) < 0$ and $S'(z,m) < 0$ as long as $S(z,m)$ exists for $z > 0$. For each fixed $m > 0$, we can use a similar method as the case $m < 0$ to find that $S'(z,\ m) > 0$ and $S(z,\ m) > 0$ as long as S exists for $z > 0$. So that the solution can be extended to the interval R_l. Note that $S^-(l) > 0$ because that $l > z_1 > z_0$ and definition of S^-. From above reasoning, we see that $S(l,\ m) = S^-(l)$ unless $m > 0$. Now, we will use the shooting method to show that there exists $m^* > 0$ such that $S(l,m^*) = S^-(l)$. First, discussing $m > 0$. Recall that $S(z,\ m) > 0$ and $I_0(z) \geq 0$ for $z \in (-l,\ l]$. Together with equation (2.2.5), we can infer that

$$S'(z,m) \geq me^{-c(z+l)/d_1}.$$

Then, integrating both sides of the above inequality from $-l$ to l yields

$$S(l,m) \geq \frac{md_1}{c}\left(1 - e^{-2cl/d_1}\right) > S^-(l)$$

if m is large enough. Note that $S(l,0) < S^-(l)$ since $S(z,0) \equiv 0$ and $S^-(l) > 0$.

Since $S(z,\ m)$ is a continuous function with respect to $m \geq 0$, there exists $m^* > 0$ such that $S(l, m^*) = S^-(l)$. At last, set $S(z) := S(z, m^*)$. Then S is a solution of equation (2.2.3a) with $S(-l) = \left(S^-\right)(-l)$ and $S(l) = \left(S^-\right)(l)$. This implies the existence of S. Moreover, we infer that $S > 0$ and $S' > 0$ in $(-l,\ l)$ from the above discussion. Using the maximum principle, we can easily get the uniqueness of S. Hence the proof of this lemma is complete.

Now we define the mapping $T : \mathscr{A} \to X$ by

$$T(S_0, I_0) = (S, I), \quad \forall \left(S_0, I_0\right) \in \mathscr{A},$$

where $\left(S, I\right)$ is the unique solution of the boundary value problem (2.2.3). Clearly, any fixed point of T must be a solution of the problem (2.2.1)-(2.2.2).

Lemma 2.2.2. $T\left(\mathscr{A}\right) \subseteq \mathscr{A}$.

Proof. For $\left(S_0,\ I_0\right) \in \mathscr{A}$, let

$$\left(S,\ I\right) := T(S_0,\ I_0).$$

We are going to claim that $I^- \leq I \leq I^+$ on R_l. Note that $0 \leq S^- \leq S_0 \leq S^+ \equiv s_\infty$ and $0 \leq I^- \leq I_0 \leq I^+$, then we get that

$$\phi(S_0, z) S_0 \leq \frac{\beta s_\infty I^+}{s_\infty + I^+},$$

which yields

$$d_2 I'' + cI' + \frac{\beta s_\infty I^+}{s_\infty + I^+} - \gamma I \geq 0. \tag{2.2.6}$$

Let $I^* := s_\infty(\beta - \gamma)/\gamma$. Note that for all $z \in \mathbb{R}$, $I^+(z) \leq I^*$. Then, we consider the system on $(-l, l)$:

$$d_2(I^*)'' + c(I^*)' + \frac{\beta s_\infty I^*}{s_\infty + I^*} - \gamma I^* = 0,$$
$$d_2 I'' + cI' + \frac{\beta s_\infty I^*}{s_\infty + I^*} - \gamma I \geq 0.$$

It is easy to see that $(I^* - I)(z)$ satisfies $(I^* - I)(-l) = I^* > 0$, $(I^* - I)(l) = I^* - I(l) \geq I^+(l) - I(l) \geq 0$, and $d_2(I^* - I)'' + c(I^* - I)' - \gamma(I^* - I) \leq 0$. Then by using the maximum principle theorem, we get that $I^* - I \geq 0$ on $(-l,\ l)$, and so $I \leq I^*$ on $(-l,\ l)$, i.e.

$$I \leq I^+ \ \text{ in } (-l,\ b_1). \tag{2.2.7}$$

Together with (2.1.2) and (2.2.6), we find that the function $\omega_1 := I^+ - I$ satisfies

$$d_2 \omega_1{}'' + c\omega_1{}' - \gamma \omega_1 \leq 0 \qquad \text{in } (b_1,\ l)$$

and

$$\omega_1(b_1) = I^+(b_1) - I(b_1) = I^* - I(b_1) \geq 0,$$
$$\omega_1(l) = I^+(l) - I(l) = I^+(l) - I^-(l) \geq 0.$$

By the maximum principle theorem, we get $I \leq I^+$ in $[b_1,\ l)$. Together with (2.2.7), we have $I \leq I^+$ in $(-l,\ l)$.

Next, let $\omega_2 := I - I^-$. Since $I^- = 0$ and $I \geq 0$ in $[-l,\ z_1]$, it follows that

$$\omega_2 \geq 0, \ \text{ in } [-l,\ z_1]. \tag{2.2.8}$$

Since $S^- \leq S_0$, it follows that

$$\phi(S_0, z) S_0 \geq \frac{\beta S^- I^-}{S^- + I^-}$$

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and therefore,

$$d_2 I'' + cI' + \frac{\beta S^- I^-}{S^- + I^-} - \gamma I \leq 0, \tag{2.2.9}$$

for all $z \in (z_1,\ l)$. Moreover, note that (2.1.7) and (2.2.9) imply that $d_2\omega_2'' + c\omega_2' - \gamma\omega_2 \leq 0$ in $(z_1,\ l)$, and $\omega_2(\pm l) = 0$ from (2.2.3c). So that we have $\omega_2 \geq 0$ in $[z_1,\ l]$ by the maximum principle. At last, together with (2.2.8), we obtain that $I \geq I^-$ in R_l.

Now we prove that $S^- \leq S$ in R_l. Since $S^- \equiv 0$ in $[-l,\ z_0]$ and $S \geq 0$ in $[-l,\ z_0]$, it follows that

$$S \geq S^- \text{ in } [-l,\ z_0]. \tag{2.2.10}$$

So it remains to show that $S \geq S^-$ in $(z_0,\ l]$. Due to $I_0 \leq I^+$, we get that

$$\frac{\beta S I_0}{S + I_0} \leq \frac{\beta S I^+}{S + I^+},$$

and thus we have the inequality

$$d_1 S'' + cS' - \frac{\beta S I^+}{S + I^+} \leq 0 \text{ in } (z_0,\ l). \tag{2.2.11}$$

Together with (2.1.4) and (2.2.11), we find that the function $\nu_1 := S - S^-$ satisfies

$$d_1 \nu_1'' + c\nu_1' - q_1(z)\nu \leq 0 \text{ in } (z_0,\ l),$$

where

$$q_1(z) = \begin{cases} \dfrac{\beta (I^+)^2}{(S + I^+)(S^- + I^+)}, & \text{if } S \neq S^-, \\ 0, & \text{if } S = S^-. \end{cases}$$

It is easy to verify that $q_1(z) \geq 0$. Moreover, from (2.2.10) and (2.2.3c), we get that $\nu_1(z_0) \geq 0$ and $\nu_1(l) = 0$. Then, by using maximum principle, we have $\nu_1 \geq 0$ in $[z_0,\ l]$. Hence $S^- \leq S$

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in $[-l,\ l]$.

Finally, we show that $S \le S^+$ in R_l. Since $S^+ \equiv s_\infty$ and $I_0 \ge 0$, we see that S^+ satisfies

$$d_1(S^+)'' + c(S^+)' - \frac{\beta S^+ I_0}{S^+ + I_0} \le 0 \quad in\ (-l,\ l),$$

and $S(\pm l) = s_\infty \ge S^-(\pm l) = S(\pm l)$. By a similar argument as the proof for the case $S \ge S^-$ in $[z_0,\ l]$, we get that $S \le S^+$ in R_l. This completes the proof of this lemma.

Lemma 2.2.3. *T is a continuous mapping.*

Proof. For (S_0, I_0) and $(\tilde{S_0}, \tilde{I_0})$ in $\mathscr{A}$, let

$$(S,\ I) = T(S_0,\ I_0) \ \text{ and } \ (\tilde{S},\ \tilde{I}) = T(\tilde{S_0},\ \tilde{I_0}). \tag{2.2.12}$$

Clearly, $\omega_1 := S - \tilde{S}$ satisfies $\omega_1(\pm l) = 0$ and

$$\omega_1'' + \frac{c}{d_1}\omega_1' + g_1(z)\omega_1 = h_1(z),$$

where

$$g_1(z) = -\frac{\beta I_0 \tilde{I_0}}{d_1(S + I_0)(\tilde{S} + \tilde{I_0})} \text{ and } h_1(z) = \frac{\beta S \tilde{S}}{d_1(S + I_0)(\tilde{S} + \tilde{I_0})} \cdot \left(I_0 - \tilde{I_0}\right).$$

It is easy to see that

$$-g_1(z) = \frac{\beta I_0 \tilde{I_0}}{d_1(S + I_0)(\tilde{S} + \tilde{I_0})} \le \frac{\beta}{d_1},$$

and

$$|\ h_1\ | \le \frac{\beta}{d_1} \cdot \|\ I_0 - \tilde{I_0}\ \|_{C(R_l)}\ ,$$

so we find that $-K_1 \le g_1 \le 0$ with $K_1 = \beta/d_1$.

Then, by [4, Lemma 3.2], there exists a positive constant C_1, depending only on d_1, c, K_1, β, and l, such that

$$\| \omega_1 \|_{C(R_l)} \leq \frac{\beta C_1}{d_1} \cdot \| I_0 - \tilde{I_0} \|_{C(R_l)} ,$$

i.e.

$$\| S - \tilde{S} \|_{C(R_l)} \leq \frac{\beta C_1}{d_1} \cdot \| I_0 - \tilde{I_0} \|_{C(R_l)} . \tag{2.2.13}$$

Now we set $\omega_2 := I - \tilde{I}$. Then ω_2 satisfies $\omega_2(\pm l) = 0$ and

$$\omega_2'' + \frac{c}{d_2}\omega_2' - \frac{\gamma}{d_2}\omega_2 = h_2(z),$$

where

$$h_2 = \frac{1}{d_2} \cdot \left[\tilde{\phi}(\tilde{S_0}, z)\tilde{S_0} - \phi(S_0, z)S_0 \right].$$

Clearly,

$$| h_2 | \leq \frac{\beta}{d_2} \cdot \| I_0 - \tilde{I_0} \|_{C(R_l)} + \frac{\beta}{d_2} \cdot \| S_0 - \tilde{S_0} \|_{C(R_l)} .$$

Again, by [4, Lemma 3.2], there exists a positive constant C_2, depending only on d_2, c, β, and l, such that

$$\| \omega_2 \|_{C(R_l)} \leq \frac{\beta C_2}{d_2} \cdot \| I_0 - \tilde{I_0} \|_{C(R_l)} + \frac{\beta C_2}{d_2} \cdot \| S_0 - \tilde{S_0} \|_{C(R_l)} ,$$

i.e.

$$\| I - \tilde{I} \|_{C(R_l)} \leq \frac{\beta C_2}{d_2} \cdot \| I_0 - \tilde{I_0} \|_{C(R_l)} + \frac{\beta C_2}{d_2} \cdot \| S_0 - \tilde{S_0} \|_{C(R_l)} . \tag{2.2.14}$$

Together with inequality (2.2.12), (2.2.13), (2.2.14), and the definition of the norm $\| \cdot \|_X$, we obtain

$$
\begin{aligned}
& \| T(S_0, I_0) - T(\tilde{S_0}, \tilde{I_0}) \|_X , \\
= & \| (S, I) - (\tilde{S}, \tilde{I}) \|_X , \\
\leq & \| S - \tilde{S} \|_{C(R_l)} + \| I - \tilde{I} \|_{C(R_l)} , \\
\leq & \frac{\beta C_1}{d_1} \| I_0 - \tilde{I_0} \|_{C(R_l)} + \frac{\beta C_2}{d_2} \| I_0 - \tilde{I_0} \|_{C(R_l)} + \frac{\beta C_2}{d_2} \| S_0 - \tilde{S_0} \|_{C(R_l)} , \\
= & \frac{\beta C_2}{d_2} \| S_0 - \tilde{S_0} \|_{C(R_l)} + \left(\frac{\beta C_1}{d_1} + \frac{\beta C_2}{d_2} \right) \| I_0 - \tilde{I_0} \|_{C(R_l)} , \\
\leq & C_3 (\| I_0 - \tilde{I_0} \|_{C(R_l)} + \| S_0 - \tilde{S_0} \|_{C(R_l)}), \\
= & C_3 \| (S_0, I_0) - (\tilde{S_0}, \tilde{I_0}) \|_X ,
\end{aligned}
$$

where

$$C_3 = \frac{\beta C_1}{d_1} + \frac{\beta C_2}{d_2}.$$

For a given $\varepsilon > 0$, we choose $0 < \delta < \varepsilon / C_3$. Then, if $\| (S_0, I_0) - (\tilde{S_0}, \tilde{I_0}) \|_X < \delta$, then

$$\| T(S_0, I_0) - T(\tilde{S_0}, \tilde{I_0}) \|_X < \varepsilon,$$

for any (S_0, I_0) and $(\tilde{S_0}, \tilde{I_0})$ in $\mathscr{A}$. This shows that T is a continuous mapping. Hence the proof of this lemma is done.

Lemma 2.2.4. $T(\mathscr{A})$ *is precompact.*

Proof. For a given sequence $\{ (S_{0,n}, I_{0,n}) \}_{n \in \mathbb{N}}$ in $\mathscr{A}$, let $(S_n, I_n) = T(S_{0,n}, I_{0,n})$. Note that $S^{\pm}$ and $I^{\pm}$ are bounded in R_l. From definition of the set $\mathscr{A}$, the sequences

$$\{S_{0,n}\}, \ \{I_{0,n}\}, \ \{S_n\}, \text{ and } \{I_n\}$$

are uniformly bounded in R_l. In addition, by lemma 2.2.2, $S^- \geq 0$ and $I_0 \neq 0$, the sequences

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$$\left\{ \frac{\beta I_{0,n} S_n}{S_n + I_{0,n}} \right\} \text{ and } \left\{ \frac{\beta I_{0,n} S_{0,n}}{S_{0,n} + I_{0,n}} \right\}$$

are also uniformly bounded in R_l. Therefore, by [4, Lemma 3.3], it follows that the sequences

$$\{S_n{}'\} \text{ and } \{I_n{}'\}$$

are also uniformly bounded in R_l. By using Arzela-Ascoli theorem, we have a subsequence $\{(S_{n_j}, I_{n_j})\}$ of $\{(S_n, I_n)\}$ such that

$$(S_{n_j}, I_{n_j}) \to (S, I)$$

uniformly on R_l as $j \to \infty$, for some pair of functions $(S,\ I) \in \mathscr{A}$. Hence the set $\overline{T(\mathscr{A})}$ is compact in $\mathscr{A}$. So $T(\mathscr{A})$ is precompact.

According to all the above lemmas of this section, we have already proved that the mapping T satisfies all the conditions of Schauder fixed point theorem. So T has a fixed point, which is a non-negative solution of problem (2.2.1)-(2.2.2). Therefore we have the following lemma:

Corollary 2.2.5. *System (2.2.1)-(2.2.2) has a solution* $(S,\ I)$ *on* R_l*. Moreover,*

$$0 \le S^- \le S \le s_\infty \text{ and } 0 \le I^- \le I \le I^+ \tag{2.2.15}$$

on R_l*.*

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Chapter 3

Proof of Theorem 1.1.

Now we are in a position to show Theorem 1.1.

Proof of Theorem 1.1: Let $\{l_n\}_{n\in\mathbb{N}}$ be an increasing sequence in (z_1, ∞) such that $l_n \to \infty$ as $n \to \infty$, and let $\left(S_n,\ I_n\right)_{n\in\mathbb{N}}$ be a solution of problem (2.2.1)-(2.2.2) in R_{l_n}. For any fixed $N \in \mathbb{N}$, the functions I^+ and S^+ are bounded above in $[-l_N,\ l_N]$. Thus, by (2.2.15), the sequences

$$\{S_n\}_{n\geq N} \ \text{ and } \ \{I_n\}_{n\geq N}$$

are uniformly bounded in $[-l_N,\ l_N]$. Moreover we can use [4, Lemma 3.3] to obtain that the sequences

$$\{S_n{}'\}_{n\geq N} \ \text{ and } \ \{I_n{}'\}_{n\geq N}$$

are also uniformly bounded in $[-l_N,\ l_N]$. In addition, it is easy to see that the sequence

$$\{\frac{\beta I_n S_n}{S_n + I_n}\}_{n\geq N}$$

is uniformly bounded in $[-l_N,\ l_N]$. So, by (2.2.1), we get that $\{S_n{}''\}_{n\geq N}$ and $\{I_n{}''\}_{n\geq N}$ are uniformly bounded in $[-l_N,\ l_N]$. Moreover, differentiating (2.2.1) yields that the sequences $\{S_n{}'''\}_{n\geq N}$ and $\{I_n{}'''\}_{n\geq N}$ are also uniformly bounded in $[-l_N,\ l_N]$. With the help of Arzela-

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Ascoli theorem, we can use a diagonal process to get a subsequence $\{(S_{n_j},\ I_{n_j})\}$ of $\{(S_n,\ I_n)\}$ such that

$$S_{n_j} \to S,\ {S_{n_j}}' \to S',\ {S_{n_j}}'' \to S'' \text{ and } I_{n_j} \to I,\ {I_{n_j}}' \to I',\ {I_{n_j}}'' \to I''$$

uniformly in any compact interval of $\mathbb{R}$ as $n \to \infty$, where $(S,\ I)$ is a non-negative solution of system (1.5a)-(1.5b) with $S' \geq 0$ over $\mathbb{R}$ and satisfies (2.2.15). Due to $S^+, S^- \to s_\infty$ and $I^+, I^- \to 0$ as $z \to \infty$, (2.2.15) implies that

$$(S,\ I)(+\infty) = (s_\infty,\ 0). \tag{3.1}$$

Now it remains to show that $S(-\infty) = s_{-\infty}$, for some constant $s_{-\infty}$ with $s_{-\infty} < s_\infty$, and $I(-\infty) = 0$. We divide the proof into several steps:

Step 1: We claim

$$(S',\ I')(+\infty) = (0,\ 0). \tag{3.2}$$

Integrating both sides of (1.5a) from 0 to z , we have

$$d_1[S'(z) - S'(0)] + c[S(z) - S(0)] = \int_0^z \frac{\beta S(\tau) I(\tau)}{S(\tau) + I(\tau)} d\tau. \tag{3.3}$$

Recall that $S(+\infty)$ exists. From equality (3.3), we get that $S'(\infty)$ exists if and only if the improper integral

$$\int_0^\infty \frac{\beta S(\tau) I(\tau)}{S(\tau) + I(\tau)} d\tau \tag{3.4}$$

converges. Note that if (3.4) diverges, then the equation (3.3) gives that $S'(\infty) = \infty$ as $z \to \infty$ and so $S(\infty) = \infty$, which leads to a contradiction to the existence of $S(\infty)$. Hence $S'(\infty)$ exists. Moreover, it is easy to see that $S'(\infty) = 0$ since $S(\infty) = s_\infty$ and $S \leq s_\infty$. Similarly, we show that $I'(\infty) = 0$ by a similar argument.

Step 2: We show that

$$S(-\infty) = s_{-\infty} \text{ and } S'(-\infty) = 0, \tag{3.5}$$

for some $s_{-\infty} \in [0,\ s_\infty)$.

First, since S is increasing and $S > 0$, it follows that $S(-\infty)$ exists, denoted by $s_{-\infty}$. Clearly, $s_{-\infty} \geq 0$. Next, we show that $S'(-\infty) = 0$. Integrating equation (1.5a) from z to ∞ and recalling that $S'(\infty) = 0$, we have

$$-d_1 S'(z) + c(s_\infty - S(z)) = \int_z^\infty \frac{\beta S(\tau) I(\tau)}{S(\tau) + I(\tau)} d\tau. \tag{3.6}$$

Since $S > 0$, $d_1 > 0$ and $S' \geq 0$, equation (3.6) implies that

$$\int_z^\infty \frac{\beta S(\tau) I(\tau)}{S(\tau) + I(\tau)} d\tau \leq c s_\infty.$$

Thus the improper integral

$$\int_{-\infty}^\infty \frac{\beta S(\tau) I(\tau)}{S(\tau) + I(\tau)} d\tau$$

converges. Then we get the fact $S'(-\infty)$ exists by letting $z \to -\infty$ in equation (3.6) and using the fact that $S(-\infty)$ exists. Moreover, since $S' \geq 0$, it implies that $S'(-\infty) \geq 0$. Actually, $S'(-\infty) = 0$ since $S'(-\infty) > 0$ leads to $S(-\infty) = -\infty$, which is a contradiction to the fact that $S(-\infty) = s_{-\infty}$ exists. Finally, letting $z \to \infty$ in equation (3.6) and recalling that $S'(-\infty) = 0$ yields $s_{-\infty} < s_\infty$.

Step 3: We show that $I(-\infty) = 0$.

To this end, we first claim that $B := (d_2 I' + cI)(-\infty)$ exists. Summing up (1.5a) and (1.5b) and then integrating the resulting equation over $\mathbb{R}$ and using the fact (3.1), (3.2) and (3.5), we get

$$c(s_\infty - s_{-\infty}) - (d_2 I(-\infty)' + cI(-\infty)) = \gamma \int_{-\infty}^\infty I(\tau) d\tau. \tag{3.7}$$

Note that the improper integral

$$\int_{-\infty}^{\infty} I(\tau)d\tau \tag{3.8}$$

converges. If not, then $(d_2 I' + cI)(-\infty) = -\infty$. Therefore, together with the boundedness of I, it follows that $I'(-\infty) = -\infty$, which contradicts to the fact that I is bounded over $\mathbb{R}$. Thus, $(d_2 I' + cI)(-\infty)$ exists.

Next, we prove that $I(-\infty) = 0$. Since $I > 0$ on $\mathbb{R}$ and the improper integral (3.8) converges, it follows that $\liminf_{z\to-\infty} I(z) = 0$. Recall that I is bounded. For contradiction, we assume that $\xi := \limsup_{z\to-\infty} I(z) > 0$. Choose two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}} \searrow -\infty$ such that $y_{n+1} < z_n < y_n$, $I(y_n) < \xi/2$, $I(z_n) > \xi/2$ for all $n \in \mathbb{N}$, and

$$\lim_{z\to\infty} I(y_n) = 0 \text{ and } \lim_{z\to\infty} I(z_n) = \xi. \tag{3.9}$$

For each $n \in \mathbb{N}$. Since I is continuous, it follows that there exist $y_n{}^* \in [y_{n+1}, y_n]$ and $z_n{}^* \in [z_{n+1}, z_n]$ such that

$$I(y_n{}^*) = \max_{z\in[y_{n+1}, y_n]} I(z) \text{ and } I(z_n{}^*) = \min_{z\in[z_{n+1}, z_n]} I(z).$$

Since $y_{n+1} \in [z_{n+1}, z_n]$, the minimality of I at $z_n{}^*$ implies that $0 \le I(z_n{}^*) \le I(y_{n+1})$. Together with (3.9), we have

$$\lim_{n\to\infty} I(z_n{}^*) = 0. \tag{3.10}$$

Note that, if $y_n{}^*$ is not a critical point, then it must be an endpoint of $[y_{n+1}, y_n]$. Thus, $I(y_n{}^*) < \xi/2$. Note that $z_n \in [y_{n+1}, y_n]$, and $I(z_n) > \xi/2$. It implies that $I(y_n{}^*) < \xi/2 < I(z_n)$, which contradicts the definition of $I(y_n{}^*)$. From similar arguments, we know that $z_n{}^*$ is also a critical point. Therefore

$$I'(y_n{}^*) = 0 \text{ and } I'(z_n{}^*) = 0. \tag{3.11}$$

Using (3.11) and (3.10), we have the equality

$$B = (d_2 I' + cI)(-\infty) = \lim_{n\to\infty} (d_2 I' + cI)(z_n{}^*) = 0.$$

Therefore, $\lim_{n\to\infty}(d_2I' + cI)(y_n{}^*) = (d_2I' + cI)(-\infty) = 0$. Moreover, by (3.11), we get that

$$\lim_{n\to\infty} I(y_n{}^*) = 0. \tag{3.12}$$

Since $z_n \in [y_{n+1},\ y_n]$, it follows that $0 \le I(z_n) \le I(y_n{}^*)$. By (3.12), we obtain $\lim_{n\to\infty} I(z_n) = 0$, which contradicts (3.9). Hence, we have $\limsup_{z\to-\infty} I(z) = 0$ and so $I(-\infty) = 0$. This completes the proof of Theorem 1.1.

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