3. A small restaurant has only one cook. Orders are handled to the only cook according to a Poisson process with rate 5 per hour. The average time required to prepare an order in 8 minutes, the actual preparation time is exponentially distributed. Since the cook is new on the job, she handles only one order at a time. Assume the queueing system involved here is in steady-state.
   (a) What is the probability that at an arbitrary time the cook has no orders to prepare? (5%)
   (b) What is the average number of orders waiting to be completed by the cook (including the order the cook is currently working on)? (5%)
   (c) The restaurant's policy is to ensure that the average time an order takes to be complete (from the time it is submitted to the cook) is no more that 20 minutes. Is that the policy satisfied under the current situation? (5%)
   (d) What is the average time an order waits before the cook starts working on it? (5%)

4. Consider a situation where a particular product is produced and placed in in-process inventory until it is needed in a subsequent production process. The number of units required in each of the next 4 months, and the unit production cost (all in units of thousands of dollars) are as follows:

month t_i	1	2	3	4
demand d_i	5	4	6	2
setup cost K_i	5	5	5	5
holding cost h_i	1	1	1	1
unit cost c_i	2	2	2	2

Assume that the initial inventory (I_0) equals 0 and the inventory at the end of fourth month (I_4) is 0.
   (a) Formulate this problem as a mixed integer linear program. (10%)
   (b) Show that this problem can be formulated as a shortest path problem. (10%)
   (c) Determine the optimal production schedule by using dynamic programming approach. (10%)

考試科目	微積分	所別	應用數學所	考試時間	4月28日 星期日 上午第一節

一. Find the maximum and minimum values of

$$f(x,y,z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$. 10%

二. Compute the volume of the solid enclosed by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. 10%

三. Let $u(x,y)$ be a harmonic function near the closed disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$. Find the line integral

$$\int_C \frac{\partial u}{\partial y}\,dx - \frac{\partial u}{\partial x}\,dy ,$$

where C is the unit circle, $x^2+y^2=1$. 10%

四. Evaluate the following indefinite integrals.

(a) $\int \sec x \, dx$

(b) $\int \ln x \, dx$. 15%

五. Determine whether the following series converge or diverge.

(a) $\sum\limits_{n=1}^{\infty} \left(\sin \frac{1}{n}\right)^{3/2}$

(b) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^{\alpha}}$, $\alpha > 0$. 15%

六. (a) State the Fundamental Theorem of Calculus.

(b) Use (a) to prove the function

$$G(x)=\int_{\sin^2 x}^{x^2+x} e^{t^2+2}\,dt$$

is continuously differentiable everywhere on $\mathbb{R}$ and find $G'(0)$.

20%

七. Let $f:\mathbb{R}^n\to\mathbb{R}$ have continuous partial derivatives, and satisfy $\left|\frac{\partial f}{\partial x_j}(x)\right|\le M$ for all $x=(x_1,\cdots,x_n)$, $j=1,2,\cdots,n$.

Prove that

$$|f(x)-f(y)|\le\sqrt{n}\,M\,\|x-y\|,$$

where $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ and $\|x-y\|=\sqrt{\sum_{j=1}^{n}(x_j-y_j)^2}$.

20%.

考試科目	線性代數	所別	應用數學所	考試時間	月 日 星期 日 上下午第 2 節

(I) Let A and B be two nxn matrices over field F.

(1) Prove that I-AB is invertible if and only if I-BA is invertible.

(2) Show that AB and BA have precisely the same eigenvalues.

(3) If A and B are diagonalizable, show that A and B can be simultaneousl diagonalized if and only if AB=BA.

(II) (1) Show that the least squares solution to an inconsistent linear system $Ax = b$ satisfies the normal equations $A^TAx = A^Tb$, where A is an mxn matrix and A^T is the transpose of A.

(2) Is the system of the normal equations consistent? Why?

(3) Show that if A has linearly independent columns, then A^TA is invertible.

(4) If columns of A are linearly independent, what kind of role does the matrix $A(A^TA)^{-1}A$ play in (1)?

(III) Let C be the space of all real-valued continuous functions defined on $[-\pi, \pi]$. For f and g in C, define

$$(f, g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

(1) Show that (,) is an inner product on C.

(2) Show that $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$ are mutually orthogonal.

(3) Find a function in the span of $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$ that is closest to the function $g(x) = |x|$ on $[-\pi, \pi]$.

(4) Sketch the graphs of g and the function that you found in (3) on the same coordinate system.

(IV) Prove that det(A), the determinant of $A_{n\times n}$, equals the volumn of the parallelepiped(平行六面體) with rows of A as edges.

(V) Prove the Fitting's lemma, that is, show that if T is a linear transformation on V, then there exist subspaces W_1 and W_2 of V such that

(1) $V = W_1 \oplus W_2$, (2) W_1 and W_2 are T-invariant,

(3) The restriction $T|W_1$ is nilpotent and $T|W_2$ is invertible.

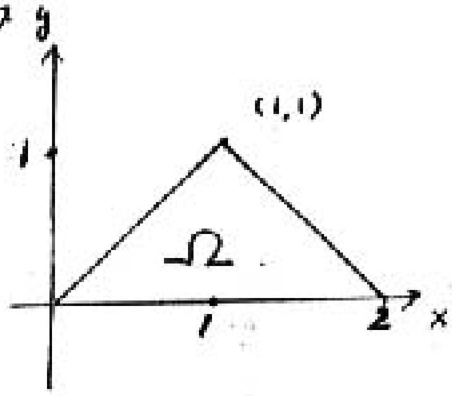
(I)(1)	8%	(II)(1)	8%	(III)(1)	5%	(IV)	10%	(V) 15%
(2)	7%	(2)	3%	(2)	5%			
(3)	10%	(3)	10%	(3)	10%			
		(4)	4%	(4)	5%			

PS. In (v), V is finite-dim.

考試科目	機率統計	所別	應用數學研究所	考試時間	4月28日 星期日	上午第3節

一、設 X 與 Y 的結合機率密度函數為

$$f(x,y)=\begin{cases} k(x+y), & (x,y)\in\Omega \\ 0, & (x,y)\notin\Omega \end{cases}$$



(8分) (1) 求 k 之值

(8分) (2) 分別求 X 與 Y 的邊際密度函數

(9分) (3) 求 $E[X \mid Y=\frac{1}{2}]$

二、設一小時內顧客進入某便利商店的人數 X 服從 Poisson 分配，且一小時平均有 λ 位顧客上門. 現令 T_n 為櫃台服務人員看到有 n 位顧客上門所需等待的時間(以"時"為單位).

(8分) (1) 求 $V(X)$

(8分) (2) 求 T_1 的機率分配.

(9分) (3) 求 T_2 的機率分配

三、考慮線性迴歸模型 $Y_i=\beta x_i+\epsilon_i$, $i=1,2,\cdots,n$, 其中 $\epsilon_1,\epsilon_2,\cdots,\epsilon_n$ 獨立且服從 $N(0,\sigma^2)$. 令 $\hat{\beta}$ 為 β 的最小平方推定.

(8分) (1) 求 $E(\hat{\beta})$, $V(\hat{\beta})$, 以及 $\hat{\beta}$ 的機率分配.

(8分) (2) 令 $e_i=Y_i-\hat{Y}_i$, 其中 $\hat{Y}_i=x_i\hat{\beta}$, $i=1,2,\cdots,n$.

求 $Cov(\hat{\beta},e_i)$, $i=1,2,\cdots,n$.

(9分) (3) 求 β, σ^2 的 MLE

四、假設有三個獨立的常態母体 $N(\mu_i,\sigma^2)$, $i=1,2,3$. 現在分別從每一母体抽取大小為 3, 4, 3 的樣本得下列資料.

母体

1	2	3
4	5	8
6	2	6
5	2	4
	3	

藉著完成下列的變異數分析表，檢定虛無假設 $H_0: \mu_1 = \mu_2 = \mu_3$
而對立假設為 $H_1: \mu_1, \mu_2, \mu_3$ 不完全相同。

變異來源	平方和	自由度	均方	F值
處理	(a)	(d)	(g)	(i)
誤差	(b)	(e)	(h)	
總計	(c)	(f)		

(15分) (1) 列出求 (a), (b), (c) 的計算過程.
(6分) (2) (d), (e), (f), (g), (h), (i) 的答案分別是多少?
(4分) (3) 在顯著水準 $\alpha = 0.05$ 下，你的結論是什麼?

試科目	作業研究	所別	應用數學研究所	考試時間	星期 月 日 上下午第

1. Consider the following problem:

$$
\begin{aligned}
\max \quad & z = 2x_1 - x_2 + x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 = 11 \\
& x_1 - x_2 - x_3 = 7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

(a) Write the dual problem. (5%)

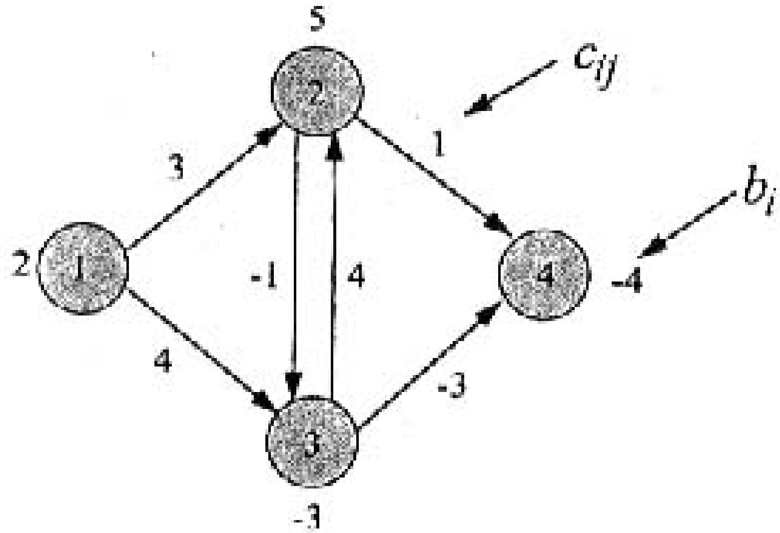
(b) Let $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ and $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. Rewrite the original problem into the following from:

$$
\begin{aligned}
\max \quad & z = z_0 - \sum_{j \in R} (z_j - c_j) x_j \\
\text{s.t.} \quad & \sum_{j \in R} (y_j) x_j + \mathbf{x_B} = \bar{\mathbf{b}}; \ x_j \geq 0, \ \forall j \in R \text{ and } \mathbf{x_B} \geq 0
\end{aligned}
$$

where $z_0 = \mathbf{c_B B^{-1} b}$, $z_j = \mathbf{c_B B^{-1} a}_j$, $y_j = \mathbf{B}^{-1}\mathbf{a}_j$, and R is the current set of the indices of the nonbasic variables. Discuss the improvement of objective value and the change of variable value by observing the new problem. (10%)

(c) Solve this problem by revised simplex method. (10%)

2. Consider the following network flow problem:



(a) Formulate the mathematical model. (5%)

(b) Introducing x_4 as the slack variable and selecting arcs (1, 3), (2, 3), and (2, 4) as basic arcs, please write down the system $\mathbf{Bx} = \mathbf{b}$ and the basic feasible solution respect to this basic. (10%)

(c) Using the solution of (b) as a starting basic, solve the problem by network simplex method. (10%)