考試科目	微積分	所別	應用數學系	考試時間	2月26日(六)第1節

1. Let f be a differentiable function such that $xf(x) + f(x^2) = 2$ for all $x > 0$.
   (a) Find $f'(1)$.
   (b) Show that if $f(x_0) = 0$ for some $x_0 > 1$, then there exists $x_1 > x_0$ such that $f(x_1) = 0$. (15%)

2. (a) Let $f(x) = 5\sum_{n=0}^{\infty}(n+1)5^n x^n$. Find the radius of convergence.
   (b) Write the power series for $g(x) = \int_0^x f(t)dt$ and compute the sum of the series.
   (c) Compute the sum of the series for $f(x)$ in (a). (15%)

3. Let

$$f(x) = \begin{cases} \dfrac{1}{x^2}\displaystyle\int_0^{|x|} \sin(t^2)\, dt & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   Is f differentiable at the point $x = 0$? Explain. (15%)

4. Evaluate $\iint_R x^2 - y^2 dxdy$, where R is the region in the plane bounded by the lines $x - y = 0$, $x - y = 1$, $x + y = -2$ and $x + y = 2$. (15%)

5. Given that $\sum a_n$ converges and each $a_n > 0$.
   (a) Prove or disprove $\sum {a_n}^2$ converges.
   (b) Prove or disprove $\sum \sqrt{a_n a_{n+1}}$ converges (15%)

6. Suppose that $f(x,y) = 2x^2 + 3y^2 - 4x - 5$.
   (a) Find all critical points of $f(x,y)$.
   (b) Find the absolute maximum and minimum of $f(x,y)$ on the region $x^2 + y^2 \leq 16$. (15%)

7. Show that $\lim_{x\to 2}(x^2 - x) = 2$ using the $\varepsilon - \delta$ definition of the limit.. (10%)

備註	試題隨卷繳交

考試科目	線性代數	所別	應用數學	考試時間	2月26日(六)第2節

Please show all your work.

1. **(10%)** Let V be a finite-dimensional vector space over a field F, and let $T: V \to V$ be linear. If rank (T) = rank (T^2), prove that $R(T) \cap N(T) = \{0\}$. ($R(T)$ is the range of T and $N(T)$ is the kernel of T.)

2. Compute the specified expression.
   (a) **(10%)** Express A as its LU factorization where
$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}.$$
   (b) **(5%)** Express A in a diagonal form $S\Lambda S^{-1}$ where Λ is a diagonal matrix with eigenvalues of A on its diagonal and
$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$
   (c) **(5%)** Express A as its QR factorization where $A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$.

3. Let $C[0,2\pi]$ denote the inner product space of real continuous functions defined on $[0,2\pi]$. For any $f, g \in [0,2\pi]$, the inner product is defined as
$$< f, g >= \int_0^{2\pi} f(x)g(x)dx.$$
   (a) **(10%)** Let W be a subspace of $C[0,2\pi]$ where $W = \text{span}\{1, \ \sin(x), \ \sin^2(x/2)\}$. Provide an orthonormal basis of W.
   (b) **(5%)** Let $h(x) \in W$ such that $h(x)$ minimizes $\|h(x) - x\|$ where $\|\bullet\|$ is the norm induced by the inner product. Express $h(x)$ as the linear combination of the basis obtained in (a).

4. Let A be a $m \times n$ matrix with rank(A) = n and $m > n$.
   (a) **(10%)** Show that $A^T A$ is a nonsingular matrix.
   (b) **(10%)** Show that $\text{rank}(AA^T) \le \text{rank}(A)$ and deduce that AA^T is singular.

備註	試題隨卷繳交	請注意：背面還有試題。

考試科目	線性代數	所別	應用數學	考試時間	2月26日(六)第2節

5. A square matrix A is called **nilpotent** if there exists a positive integer k such that $A^k = 0$.
   (a) **(5%)** What are the possible eigenvalues of a nilpotent matrix? Justify your answer.
   (b) **(10%)** Show that a nonzero nilpotent matrix is not diagonalizable.

6. **(20%)** Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For any given complex vector x, define $\rho(x) = \dfrac{x^H A x}{x^H x}$. Show that $\max\limits_{x \neq 0} \rho(x) = \lambda_1$ and $\min\limits_{x \neq 0} \rho(x) = \lambda_n$.

備註	試題隨卷繳交