國立政治大學 101 學年度碩士班招生考試試題

考試科目	微積分	所別	8111, 8116 應用數學系	考試時間	2月25日(六) 第一節

1. Use Lagrange multipliers to find the points on the ellipsoid

$$x^2 + y^2 + z^2 - 3y + yz - 6z + 5 = 0$$

that are closest and farthest from the xy-plane. (20%)

2. Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist on $\mathbb{R}^2$, but $f(x,y)$ is not continuous on $\mathbb{R}^2$.

(20%)

3. Let C be the ellipse $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$. Compute the line integral

$$\frac{1}{2}\oint_C xdy - ydx.$$ (20%)

4. Show that the sequence $\{a_n\}$ and the series $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ both converge of both diverge. (20%)

5. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and $\lim_{n\to\infty} a_n = A$ exist. Show that

$$\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = A.$$ (20%)

備註	試題隨卷繳交

考試科目	線性代數	所別	8111-8116 應用數學	考試時間	2月25日(六)第二節

1. (20%) Prove or give a counterexample: Let A be a 3×3 real invertible matrix. Then A is diagonalizable. (Justify your answer.)

2. (20%) Let $P_3(\mathbb{R})$ be the collection of all real polynomials of degree at most 3. Let $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the function defined by $T(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$, for all a, b, c, d in $\mathbb{R}$. Show that

   (a) T is a linear transformation.

   (b) Find the rank and nullity of T.

3. (20%) Let

$$A = \begin{bmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{bmatrix}.$$

   Let $T: \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^5$.

   (a) Show that the image of T is spanned by the column vectors of the matrix A.

   (b) Choose some of the column vectors of A to form a basis for the image of T.

4. (20%) (a) What are the criteria for a function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ being an inner product on $\mathbb{R}^n$?

   (b) Let A be an $n \times n$ invertible symmetric matrix. Define $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^t A \mathbf{x},$$

   where $\mathbf{x}, \mathbf{y}$ are column vectors in $\mathbb{R}^n$, and $\mathbf{y}^t$ is the transpose of $\mathbf{y}$. Is $\langle \cdot, \cdot \rangle$ always an inner product? Justify your answer.

5. (20%) Show that the matrices

$$A = \begin{bmatrix} 13 & -2 & 3 \\ 7 & 1 & 1 \\ -38 & 9 & -9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

   are similar by finding the matrix Q and its inverse Q^{-1} such that

$$Q^{-1}AQ = B.$$

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