國立政治大學 102 學年度碩士班招生考試試題



| 考 試 科 目 | 微積分 | 所　別 | 應用數學系 | 考試時間 | 2 月 24 日(日) 第 一 節 |
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1. Let f be a nonnegative and continuous function on $[a, b]$ such that $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (20%)

2. Let $f(x) = e^{-x^2}$. Find $\lim_{n\to\infty} \left(\int_{-1}^{1} f(x)^n\,dx\right)^{\frac{1}{n}}$. (20%)

3. Evaluate the definite integrals

   (a) $\int_0^{\pi/2} x\sin x\,dx$ (b) $\int_2^4 \frac{1}{(x-1)^2(x+1)}\,dx$. (20%)

4. Let $E = \left\{(x, y) \in \mathbb{R}^2 \middle| \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$, where $a, b > 0$. Evaluate the line integral $\oint -y\,dx + x\,dy$. (20%)

5. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$. (Justify your answer!) (20%)

| 備　　註 | 試題隨卷繳交 |
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# 國立政治大學 102 學年度碩士班招生考試試題



| 考試科目 | 線性代數 | 所別 | 應用數學系 | 考試時間 | 2月24日(日) 第 2 節 |
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**Please show all your work.**

1. Let $T$ be a linear operator on a vector space $V$, let $v$ be a nonzero vector in $V$, and let $W$ be the $T$-cyclic subspace of $V$ generated by $v$. Prove that

   (a) **(5%)** $W$ is $T$-invariant.
   (b) **(10%)** Any $T$-invariant subspace of $V$ containing $v$ also contains $W$.

2. Let $C[-1,1]$ denote the inner product space of real continuous functions defined on $[-1,1]$. For any $f,g \in C[-1.1]$, the inner product is defined as

$$<f,g>=\int_{-1}^{1} f(x)g(x)dx$$

   (a) **(5%)** Show that $u_1(x)=\frac{1}{\sqrt{2}}$ and $u_2(x)=\frac{\sqrt{6}}{2}x$ form an orthonormal set of vectors.

   (b) **(10%)** Let $W$ be the subspace of $C[-1,1]$ spanned by $u_1(x)$ and $u_2(x)$. Find $w(x)\in W$ such that $w(x)$ minimizes $\|w(x)-h(x)\|$ where $\|\cdot\|$ is the norm induced by the inner product and $h(x)=x^{1/3}+x^{2/3}$.

3. **(10%)** Let $A$ be an $n\times n$ matrix and let $B=I-2A+A^2$. Show that if $\lambda=1$ is an eigenvalue of $A$, then the matrix $B$ will be singular.

4. Let $Q$ be a $3\times 3$ orthogonal matrix whose determinant is equal to 1.

   (a) **(10%)** If the eigenvalues of $Q$ are all real and if they are ordered so that $\lambda_1 \ge \lambda_2 \ge \lambda_3$, determine the values of all possible triples of eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$.
   (b) **(10%)** In the case that the eigenvalues $\lambda_2$ and $\lambda_3$ are complex, what are the possible values for $\lambda_1$? Explain.

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5. **(10%)** Let $A=\left[a_{i,j}\right]$ be an $n\times n$ matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. Show that

$$\lambda_j = a_{j,j} + \sum_{i\neq j}(a_{i,i}-\lambda_i) \text{ for } j=1,\ldots,n.$$

6. A linear operator $T$ on a finite-dimensional inner product space with inner product $\langle\bullet,\bullet\rangle$ is called **positive definite** if $T$ is self-adjoint and $\langle T(x),x\rangle>0$ for all $x\neq 0$.

   (a) **(15%)** Let $V$ be a finite-dimensional inner product space with inner product $\langle\bullet,\bullet\rangle$, and let $T$ be a positive definite linear operator on $V$. Prove that $\langle x,y\rangle'=\langle T(x),y\rangle$ for all $x$ and $y$ in $V$ defines another inner product on $V$.

   (b) **(15%)** Prove the converse of (a): Let $V$ be a finite-dimensional inner product space with inner product $\langle\bullet,\bullet\rangle$ and let $\langle\bullet,\bullet\rangle'$ be any other inner product on $V$. Then, there exists a unique linear operator $T$ on $V$ such that $\langle x,y\rangle'=\langle T(x),y\rangle$ for all $x$ and $y$ in $V$.

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國立政治大學 102 學年度碩士班招生考試試題



| 考試科目 | 微積分 | 所別 | 應用數學系 | 考試時間 | 2月24日(日)第一節 |
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