國立政治大學 105 學年度碩士班招生考試試題

考試科目	微積分 81111 81161	所別	應用數學系	考試時間	2月28日(日) 第一節

※ Show all your work for full credit. Unjustified answers will receive no credit.

1. Let $f: R \to R$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. Show that:
   (a) $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in R$.
   (b) if, in addition, f is continuous at 0, then f is continuous on R. (20%)

2. Let $ax + by + cz + d = 0$ be a plane in the space R^3. Use Lagrange multipliers to show that the shortest distance from a point (x_0, y_0, z_0) in R^3 to the plane is $\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$. (20%)

3. Let

$$f(x) = \begin{cases} \frac{\sin x}{x}, & 0 < x \le \pi, \\ 1, & x = 0, \end{cases}$$

and R be the region in the plane R^2 bounded by the curve $y = f(x)$, x-axis, and y-axis. Find the volume of the solid generated by revolving the region R about the y-axis.

(20%)

4. Let R be a region in the plane R^2 bounded by a simple closed curve C.

   (a) Show that the area of R is given by $\frac{1}{2}\oint_C xdy - ydx$.

   (b) Use (a) to compute the area of the region bounded by the ellipse $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$.

(20%)

5. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series.
   (a) Is $\sum_{n=1}^{\infty} a_n b_n$ convergent? Justify your answer.
   (b) If, in addition, $a_n \ge 0$ and $b_n \ge 0$ for all $n \in N$, show that $\sum_{n=1}^{\infty} {a_n}^2$ and $\sum_{n=1}^{\infty} a_n b_n$ both converge. (20%)

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

國立政治大學 105 學年度碩士班招生考試試題

考試科目	線性代數 81112、81162	所別	應用數學系	考試時間	2 月 28 日(日) 第二節

Show all your work.

1. (20 pts) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a,b,c) = (a-b, 2c-a)$. Describe $T^{-1}(3,-2)$.

2. (a) (10 pts) Find an orthonormal basis for the subspace spanned by $\vec{x_1} = (2,0,-1,2)$, $\vec{x_2} = (0,1,1,-2)$ and $\vec{x_3} = (3,-1,1,0)$.

   (b) (10 pts) What is the projection of $(2,5,7,-3)$ in this space?

3. Let A denote the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

   (a) (5 pts) Find the eigenvalues of A.

   (b) (5 pts) Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for A.

   (c) (5 pts) Find a 3×3 orthogonal matrix S and 3×3 diagonal matrix D such that $A = SDS^T$.

   (d) (5 pts) For any integer k, write an explicit formula for A^k.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the property that $T\big(T(\vec{x})\big) = T(\vec{x})$ for every vector $\vec{x} \in \mathbb{R}^n$.

   (a) (5 pts) Write V for the range of T. In other words, $V = \{T(\vec{x}) | \vec{x} \in \mathbb{R}^n\}$. If $\vec{x} \in V$, then what is $T(\vec{x})$?

   (b) (5 pts) If $\vec{x} \in \mathbb{R}^n$, then what is $T\big(\vec{x} - T(\vec{x})\big)$?

   (c) (5 pts) Let $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a basis for V. Then we can add some more vectors $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_\ell}$ to get a basis β for $\mathbb{R}^n$. Show that if you replace $\vec{u_1}$ with $\vec{u_1} - T(\vec{u_1})$, then you still have a basis.

   (d) (5 pts) In the same way, we can replace each $\vec{u_i}$ with $\vec{u_i} - T(\vec{u_i})$. What is the matrix of T with respect to the basis $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}, \vec{u_1} - T(\vec{u_1}), \cdots, \vec{u_\ell} - T(\vec{u_\ell})\}$.

5. Show that

   (a) (10 pts) Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then $\det(B) = \det(A)$.

   (b) (10 pts) Let $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$.

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