(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Show that $\begin{pmatrix} 1 \\ m \end{pmatrix} \in \mathbb{R}^2$ is an eigenvec-
r to eigenvalue $a + bm \in \mathbb{R} \Longleftrightarrow m \in \mathbb{R}$ is a root of the quadratic equation
$^2 + (a-d)x - c$ with $b \neq 0$.
) Let $A \in M_{n\times n}(\mathbb{C})$. Show that A is *Hermitian*, i.e. $A = A^* = \bar{A}^T =$ The
mplex conjugate transpose of $A \Longleftrightarrow A^2 = AA^*$.(20 points)

Let $A \in M_{n\times n}(\mathbb{F})$ be an invertible matrix, and J the $n \times n$-matrix all of
ose entries are 1. (i) Prove that $A^{-1}J$ has rank 1 and nullity $n-1$, and it
s two distinct eigenvalues 0 and $s=$ The sum of all the n^2 entries of A^{-1},
d of multiplicities $n-1$ and 1, respectively.
) Show that $det(A+J) = detA.(1+s)$.(15 points)

I. Let $A_n = (a_{ij})_{1\le i,j\le n} \in M_{n\times n}(\mathbb{N})$ defined by

$$a_{ij} = \begin{cases} \text{n} & \text{if } n \text{ divides } ij \\ \text{ij(mod } n) & \text{otherwise.} \end{cases}$$

Determine $det(A_3), det(A_4)$ *and* $det(A_5)$.
) Show that for $n \ge 6$, $det(A_n) = 0$.(Hint: For $n \ge 6$, there are at least
ır distinct integers $1, k, n-k, n-1$ which are relatively prime to n.(15
ints)

. Let $M_1, M_2 \in M_{n\times n}(F)$ with $M_1M_2 = M_2M_1$. (i) Suppose the mini-
ıl polynomials of M_1 and M_2 are of the forms $\chi_{M_1}(t) = (t-\lambda_1)(t-\lambda_2)$,
d $\chi_{M_2}(t) = (t-\alpha_1)(t-\alpha_2)$ with $\alpha_1 \neq \alpha_2$ and $\lambda_1 \neq \lambda_2$, respectively. Show
ıt M_1 and M_2 are simultaneously diagonalizable.
) Suppose M_1 and M_2 have a common eigenvector v, i.e. $M_1 v = \lambda_1 v$ and
$_2 v = \lambda_2 v$. Show that their transposes also have a common eigenvector w
h the same eigenvalues, i.e.$M_1^T w = \lambda_1 w$ and $M_2^T w = \lambda_2 w$.(20 points)

(i) Let $L \subset \mathbb{R}^2$ be the line $y = mx, m \neq 0$, P the projection of $\mathbb{R}^2$ on L,
d R the reflection about L.Write down the expressions $P(a,b), R(a,b)$ for
$b) \in \mathbb{R}^2$.
For $m, n \in \mathbb{N}, m \le n$, let I_n be the identity matrix and $D_m = (d_{ij})_{1\le i,j\le n} \in$
$_{n\times n}(\mathbb{R})$ be an upper triangular matrix defined as follows: The entries
$_m = d_{2,m+1} = d_{3,m+2} = \ldots = d_{n-m+1,n} = \alpha \neq 0$, and all other entries
0. (a) Find the Jordan canonical form of $aI_5 + D_3, a \neq 0$, and determine
minimal polynomial.
Write down the Jordan canonical form of $aI_n + D_m, a \neq 0$ for arbitrary

I. Prove or disprove the following statements

1. The series

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\beta} \cos n^{p}}{n^{\alpha}}$$

converges for every $\alpha > 1$, $\beta \geq 1$ and $p \geq 1$. (10%)

2. Suppose that $f_n(x) = (x^2-1)^n$, $g_n(x) = \frac{1}{2^n n!} f_n^{(n)}(x)$ $(n \in \mathbb{N})$ and $g_0(x) = 1$, then

2a. $f_n(x)$ converges uniformly in $\left[-\sqrt{2}, 0\right]$. (5%)

2b. $g_n(x)$ converges uniformly in $\left[-\sqrt{2}, 0\right]$. (10%)

II. Prove the results

3. If f is continuous differentiable in $\mathbb{R}$ and the integral $\int_1^{\infty} f(t)/t dt$ exists, then for positive a and b we have

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a}. \text{ (10\%)}$$

4. If $T : C[0,1] \to \mathbb{R}$ is defined by $Tf = f(0)$, then $\|T\| = 1$. (10%)

Definition : Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and $S \subset (X, \|\cdot\|_X)$ is an open set in $(X, \|\cdot\|_X)$. Then the map $f : S \to (Y, \|\cdot\|_Y)$ is called differentiable in S, if for every point $s \in S$ there exists a bounded linear map $L_s : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ such that

$$f(s+h) = f(s) + L_s(h) + \varepsilon(h),$$

where $\varepsilon(h)$ satisfies $\lim_{\|h\| \to 0} \frac{\|\varepsilon(h)\|_Y}{\|h\|_X} = 0$.

For a $n \times n$ matrix $A = (a_{ij})_{n \times n}$ we set $\|A\| = \sum_{i,j=1}^{n} |a_{i,j}|$, then $\|\cdot\| : M_n \to \mathbb{R}^+$ is a norm, where M_n is the collection of all $n \times n$ real matrices.

III. Prove the following statements

5. The set $M = \{A \in M_n : \|A - I_n\| < 1\}$ is open in $(M_n, \|\cdot\|)$, where $I_n = (a_{ij})_{n \times n}, a_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. (5%)

6. The function $f(A) = A^2$ is differentiable in M. (20%)

IV. Are the following statements true? If yes, prove them; if no, disprove them.

7. Let $f_n(x)$ be an increasing sequence of measurable functions in (a,b), and let $f = \lim_{n \to \infty} f_n$. Then

$$\int_a^b f(x) dx = \lim_{n \to \infty} \int_a^b f_n(x) dx. \text{ (15\%)}$$

8. The set $(0,1)$ is uncountable. (15%)