考試科目	線性代數	所別	應用數學系	考試時間	5月28日上午第一節 星期六

I.(25 points) Prove or disprove the following statements:

(i) Let $T \in L(V,V)$ be a linear transformation of a finite-dimensional F-vector space V into itself with $T^2 = T$, then $KerT + ImT = V$

(ii) For arbitrary $T \in L(V,V)$:

$$KerT + ImT \iff KerT = KerT^2 \iff ImT = ImT^2$$

(iii) Let $A \in M_{n\times m}(F)$ and $B \in M_{m\times n}(F)$ with $m < n$, then AB is not invertible.

(iv) Let $A \in M_{n\times m}(F)$. Then AA^T is invertible $\iff rankA = n$.

(v) For arbitrary $A \in M_{n\times m}(F)$, AA^T and A^TA have the same eigenvalues.

II.(20 points) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

(i) Show that $\begin{pmatrix} 1 \\ m \end{pmatrix} \in \mathbb{R}^2$ is an eigenvector to eigenvalue $a + bm \in \mathbb{R} \iff$ $m \in \mathbb{R}$ is a root of the quadratic equation $bx^2 + (a-d)x - c$ with $b \neq 0$.

(ii) Show that if A has two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, then the eigenspace E_{λ_1} = The column space of $A - \lambda_2 I$ and E_{λ_2} = The column space of $A - \lambda_1 I$.

(iii) Show that if $a = d = 0$ and $bc \neq 0$, then A is diagonalizable $\iff bc = y^2$, for some $y \in \mathbb{R}$.

(iv) Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by $T(X) := AX - XA$. Show that T is an $\mathbb{R}$-linear transformation, and calculate the matrix $[T]_{\mathbb{B}}^{\mathbb{B}}$ with respect to the basis

$\mathbb{B} = \{E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

What are *rank* T and *nullity* T, if $A \neq tI_2$?

III.(15 points) Let $A \in M_{n\times n}(\mathbb{R})$.

(i) Give definition of the *adjugate*(the adjoint of A) $adj(A)$ of A, and show that $A\,adj(A) = adj(A)A = det(A)I_n$.

(ii) Use (i) to prove CAYLEY-HAMILTON Theorem.

(iii) Use (i) to show that if $\lambda \neq 0$ is an eigenvalue of A of algebraic multiplicity 1, then the eigenspace E_λ = The column space of $A - \lambda I_n$.

備考	試題隨卷繳交

考試科目	線性代數	所別	應用數學系	考試時間	5月28日上午第一節 星期六

IV.(20 points) (i) Give definition of a row-stochastic(*Markov* matrix) $A \in M_{n\times n}(\mathbb{R})$, and show that all eigenvalues $\lambda \in \mathbb{C}$ of A has absolute values $|\lambda| \leq 1$.
(ii) Does the matrix limit $\lim_{m\to\infty} A^m$ always exist? Explain your answers.

V.(20 points) Find an invertible matrix $P \in M_{4\times 4}(\mathbb{R})$ such that $P^{-1}AP = J(A)$ the Jordan canonical form of the following matrix $A \in M_{4\times 4}(\mathbb{R})$:

$$A = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

備考	試題隨卷繳交

考試科目	分析概論	所 別	應用數學系	考試時期	5 月 28 日 星期六	13:20~15:00

1. Let $\{f_n\}$ be a sequence of measurable functions on $[a,b]$ satisfying the following conditions:
   (a) $\{f_n\}$ is decreasing on $[a,b]$, i.e., $f_n(x) \geq f_{n+1}(x)$ for all $x \in [a,b]$.
   (b) There exists an integrable function g on $[a,b]$ such that $f_n \leq g$ on $[a,b]$, for all $n = 1, 2, \cdots$.
   (c) $f_n \to f$ on $[a,b]$.

   Show that $\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b f\, dx$. (20%)

2. Show that an absolutely continuous function on $[a,b]$ is of bounded variation on $[a,b]$. (20%)

3. Let $S = \{x \in R^n \mid \|x\| \leq 1\}$ and $f : S \to R$ be a nonnegative continuous function. (20%)
   (a) Prove that f has the absolute maximum value on S.
   (b) Let M be the absolute maximum value of f on S. Show that

$$\lim_{k\to\infty} \left(\int_S (f(x))^k dx_1 dx_2 \cdots dx_n \right)^{\frac{1}{k}} = M$$

4. Let $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$, $(x,y) \in R^2 - \{0\}$. Show that ω is a closed 1-form, but not exact on $R^2 - \{0\}$. (20%)

5. Let $X = C[a,b]$ be the space of continuous real-valued function on $[a,b]$. Define, for $f \in X$, (20%)

$$\|f\| = \left(\int_a^b |f(x)|^2\, dx \right)^{\frac{1}{2}}$$

   (a) Show that $(X, \|\cdot\|)$ is a normed linear space.
   (b) Is $(X, \|\cdot\|)$ complete?
   (c) What is the completion of $(X, \|\cdot\|)$?

備 考	試 題 隨 卷 繳 交 173