國立政治大學九十五學年度研究所博碩士班入學考試命題紙

考試科目	線性代數	所別	應用數學系	考試時間	5月27日 星期六	第一節

Problem. 1. Prove or give an explicit counterexample for each of the following statements.

(a) (5 pts) Two square matrices are similar if they have the same Jordan canonical form.

(b) (5 pts) Let V be a finite-dimensional vector space and $T : V \to V$ a linear operator. V is the direct sum of the null space of T and the range of T.

(c) (5 pts) Let A and B be real $n \times n$ matrices. If A and B are positive definite, then $A + B$ is positive definite.

Problem. 2. Let $\mathbf{H}$ be the vector space of continuous complex-vauled functions defined on $[0, 2\pi]$. Define an inner product on $\mathbf{H}$ by

$$\langle f, g \rangle = \int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

Let $S = \{f_n : f_n(t) = e^{int}, t \in [0, 2\pi], n \in \mathbb{Z}\}$.

(a) (10 pts) Prove S is an orthogonal set in $\mathbf{H}$.

(b) (5 pts) Define a new inner product on $\mathbf{H}$ to make S an orthonormal set in $\mathbf{H}$ with respect to this inner product.

(c) (10 pts) Let $S' = \{f_1, f_2, \ldots, f_7\} \subset S$ and $W = \text{span}(S')$. Let $f(t) = t$, find $f_1(t) \in W$ such that $\|f(t) - f_1(t)\|$ is the smallest, where $\|\cdot\|$ is the norm introduced by the inner product you find in part (b).

Problem. 3. (15 pts) Let W be a subspace of a finite-dimensional inner product space V. Prove

$$V = W \oplus W^{\perp}.$$

Problem. 4. (15 pts) Let A be a real 3×3 matrix. Suppose A has three distinct positive eigenvalues. Show that

$$A^3 - \text{tr}A \cdot A^2 + \det A \cdot I$$

is invertible.

Problem. 5. (15 pts) Let A and B be real $n \times n$ matrices. Suppose $A + B = I$ and $\text{rank}(A) + \text{rank}(B) = n$. Show that both A and B are projections.

Problem. 6. (15 pts) Let A be a complex $m \times n$ matrix. Show that $\text{rank}(A^*A) = \text{rank}(A)$, where $A^* = \overline{A^t}$.

備考	試題隨卷繳交 187

命題委員：　　　　（簽章）　　年　　月　　日

考試科目	分析概論	所別	應數系	考試時間	5月27日 星期六	第二節

1. Let $\Omega = \{x = (x_1, x_2, \cdots, x_n) \in \Re^n \mid x_j > 0, 1 \le j \le n\}$ and $f(x) = -(x_1 x_2 \cdots x_n)^{1/n}$ for $x \in \Omega$. Compute the Hessian $H(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$, $x \in \Omega$ and show that it is positive-semidefinite. (20%)

2. Let $\{E_k\}$ be a sequence of Lebesque measurable sets in $\Re^n$ and $\sum_{k=1}^{\infty} \lambda_n(E_k)$ converge, where λ_n is the Lebesque measure on $\Re^n$. Show that the set $E = \limsup_{k\to\infty} E_k$ is Lebesque measurable and has Lebesque measure zero. (20%)

3. Evaluate the limit

$$\lim_{k\to\infty}\left(\int_{\|x\|\le 1} e^{-\|x\|^{2k}}\right)^{1/k}$$

if exists, where $\| x \|$ denotes the Euclidean norm in $\Re^n$. (20%)

4. State and prove the Lebesque dominated convergence theorem. (20%)

5. Let H be a Hilbert space and M be a closed subspace of H. Show that each $x \in H$ can be written uniquely in the form $x = y + z$ with $y \in M$ and $z \in M^{\perp}$. Moreover, $\| x \|^2 = \| y \|^2 + \| z \|^2$. (20%)

備考	試題隨卷繳交