考試科目	線性代數	所別	應用數學系	考試時間	5月24日 星期六 第一節

1. (20%) Let V be a vector space over $\mathbb{C}$, and let W be an inner product space over $\mathbb{C}$ (i.e. W is an Hermitian inner product space,) with inner product (i.e. Hermitian product) $\langle \cdot, \cdot \rangle$. If $T: V \to W$ is linear, prove that for some fixed scalar r, $\langle \mathbf{x}, \mathbf{y} \rangle' = r\langle T(\mathbf{x}), T(\mathbf{y}) \rangle$ defines an inner product (i.e. Hermitian product) on V if and only if T is one-to-one and $r > 0$.

2. Let $V = P_3(\mathbb{R})$ be the set of all real polynomials with degree at most 3, with the inner product $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$, and consider the subspace $W = P_2(\mathbb{R})$ (the set of all real polynomials with degree at most 2) with the standard ordered basis $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$.
   (a) (6%) Applying the Gram-Schmidt orthogonalization process to β and obtain an orthonormal basis for W.
   (b) (7%) Let $f(x) = x^3 + x$ on $P_3(\mathbb{R})$. Find the orthogonal projection $g(x)$ of $f(x)$ on W.
   (c) (7%) We know $V = W \oplus W^{\perp}$, so $f(x) = \mathbf{w}_1 + \mathbf{w}_2$ for unique $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$. Find $\mathbf{w}_1$ and $\mathbf{w}_2$.

3. (20%) Let A be an $n \times n$ matrix whose characteristic polynomial splits. Prove that A and A^t are similar. (Hint: Show that A and A^t have the same Jordan canonical form.)

4. For an $m \times n$ matrix M, we denote the i-th row by $M_{(i)}$ and j-th column by $M^{(j)}$. Let A be an $m \times n$ matrix and let B be the row echelon form of A. Suppose there are k nonzero rows in B.
   (a) (7%) Show that there are $1 \le j_1 < j_2 < \cdots < j_k \le n$ such that $B^{j_1} = \mathbf{e}_1$, $B^{j_2} = \mathbf{e}_2$, ..., $B^{j_k} = \mathbf{e}_k$, where $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]^T$, $\mathbf{e}_2 = [0, 1, 0, \ldots, 0]^T$, ...,
   (b) (7%) Show that $A^{j_1}, A^{j_2}, \ldots, A^{j_k}$ are linearly independent.
   (c) (6%) Show that $A^{j_1}, A^{j_2}, \ldots, A^{j_k}$ is a basis for the column space of A (the vector space generated by columns of A.)

5. Let A be a Hermitian matrix. That is, A is a real or complex $n \times n$ matrix such that $A^* = \overline{A^T} = A$.
   (a) (10%) Show that each eigenvalue of A is real.
   (b) (10%) Show that A is diagonalizable.

備考	試題隨卷繳交
命題委員：	（簽章） 年 月 日

命題紙使用說明：1.試題將用原件印製，敬請使用黑色墨水正楷書寫或打字（紅色不能製版請勿使用）。
2.書寫時請勿超出格外，以免印製不清。
3.試題由郵寄遞者請以掛號寄出，以免遺失而示慎重。

國立政治大學九十七 學年度研究所博碩士班入學考試命題紙

考試科目	分析概論	所別	應數系	考試時間	5月24日 星期六 第2節

1. (20%) Prove or disprove (by a counter example) the following statements:
   (a) If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, then f is an open mapping.
   (b) If $f: (-1, 1) \to \mathbb{R}$ is an infinitely differentiable function, and $|f^{(n)}(x)| \le 1$ for all $x \in (-1, 1)$ and $n = 0, 1, 2, \ldots$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for all $x \in (-1, 1)$.

2. (20%) Let $\Omega \subset \mathbb{R}^n$ be an open set. Prove that Ω is connected if and only if Ω is pathwise connected.

3. (20%) Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(x) > f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 0$, then $f(x) > 0$ for all $x > 0$.

4. (20%) Let
$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$
Show that the improper integral of $f(x)$ on $(-\infty, \infty)$ exists, but that $f(x)$ is not Lebesgue integrable on $(-\infty, \infty)$.

5. (20%) Let $f(x) = e^{-x^2}$, $x \in \mathbb{R}$.
   (a) Show that $f(x)$ is Lebesgue integrable on $\mathbb{R}$.
   (b) Define
$$\mu(E) = \int_E e^{-x^2} \, d\lambda(x)$$
for all Lebesgue measurable sets in $\mathbb{R}$, where λ is the Lebesgue measure on $\mathbb{R}$. Show that μ is a finite measure on $\mathbb{R}$ and $\mu \ll \lambda$.

備考	試題隨卷繳交

命題委員： （簽章） 年 月 日

命題紙使用說明：1.試題將用原件印製，敬請使用黑色墨水正楷書寫或打字（紅色不能製版請勿使用）。
2.書寫時請勿超出格外，以免印製不清。
3.試題由郵寄遞者請以掛號寄出，以免遺失而示慎重。